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Central Limit Theorems for Mixing Arrays II

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Abstract. This paper, continuing [9], gives more C.L.T.s for ℓ'-mixing random variables and the likes, which improve those of Withers [10], Dvoretzky [2], Serfling [8] and Mcleish [6], created for ℓ-mixing, α-mixing, β-mixing, martingale and martingale-like random variables.

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1. Introduction

Recall [9] that any triangular array of real random variables $(X_{n,k})$, $k = 1, 2, \dots, k_n$, $n = 1, 2, \dots$, is ℓ'-mixing if for all real t

$$\ell'_n(c,t) := \max_{\substack{a,b \\ a+b+c \le k_n}} \left| \mathrm{Cov} \left\{ \exp\left(it \sum_{j=1}^{a} X_{n,j} \right),\ \exp\left(-it \sum_{j=a+c}^{a+c+b} X_{n,j} \right) \right\} \right| \to 0, \tag{1}$$

as c and $n \to \infty$ such that $c < k_n$, where $\mathrm{Cov}\,(X,Y) = EX\overline{Y} - EX\overline{EY}$ is the covariance of complex r.v.s. X, Y. All ℓ-mixing arrays [10], consequently all α-, β-, φ-, and ρ-mixing ones, are ℓ'-mixing [9].

In this paper we apply the results obtained in [9] to give several C.L.T.s which improve known C.L.T.s of mixing and martingale types. Firstly we improve a part of Theorem 2.3 of Withers [10] written for ℓ-mixing random variables. Then the assumption of being α-mixing in Theorem 5.1 of Dvoretzky [2] is replaced by ℓ'-mixing, making it applicable to any array mixing in ℓ-, α-, β-, φ-, ρ-mixing senses. These two ℓ'-mixing C.L.T.s are written in Sec. 2.

In Sec. 3 the ℓ'-mixing notion is modified, in order to improve martingale C.L.T.s and the likes. Theorem 5.3 of Dvoretzky [2], consequently Theorem 4.1 of Serfling [8], is improved by replacing its dependence assumptions which use conditional expectations in their formulation, by an ℓ'-mixing type one. For known martingale C.L.T.s [1, 2, 6, 8] we replace the martingale or martingale-like property by a 'parameterized covariance' condition, which need no conditional expectations in its expression. Although our moment conditions in the latter case are not as weak as in [2], but still are well-known normalization and Lindeberg conditions.

Because the ℓ'-mixing notion is derived from the method of characteristic functions, any C.L.T. which uses characteristic functions in its proof, is expected to be extendible to this type of mixing. The results in this paper can be served as examples of that fact.

The proofs are given in Sec. 4, which use only elementary computations in textbook level. This paper, combined with [9], is made self-contained for easy reading.

2. Central Limit Theorems for ℓ'-mixing Arrays

Denote [9]

$$S_{n,a,b} := \sum_{k=a+1}^{b} X_{n,k}, \quad S_{n,b} := S_{n,0,b} \quad \text{and} \quad S_n := S_{n,k_n}.$$

Recall that [9], for any given set of partitions $0 = m_{n,0} < m_{n,1} < \ldots < m_{n,v_n} = k_n$ of the interval of integers $[1, k_n]$, $n = 1, 2, \ldots$, we can associate the array $(Y_{n,k})$, *an array of partial sums of* $(X_{n,k})$, defined by $Y_{n,k} := S_{n,m_{n,k-1},m_{n,k}}$, $k = 1, 2, \ldots, v_n$. Then $S_n(Y) := \sum_{k=1}^{v_n} Y_{n,k} = S_n$, hence the limit laws of them are the same, if existed.

Especially, for a given sequence (v_n) let $m_n = [k_n/v_n]$ and $m_{n,k} = m_n k$ for $0 < k < v_n$ ($[a]$ denotes the integer part of a real number a). Then all the elements $Y_{n,k}$, except the last one, of the associated array $(Y_{n,k}) = (Y_{n,k}(v_n))$, are the sums of exactly m_n $X_{n,i}$. And for the number of $X_{n,i}$ in the last sum Y_{n,v_n} we have the estimation $m_n \le k_n - m_n(v_n - 1) \le m_n + v_n$, which follows from $0 \le k_n/v_n - [k_n/v_n] < 1$. Call $(Y_{n,k}(v_n))$ as *the array of* v_n *partial sums of equal size of* $(X_{n,k})$.

Theorem 2.1. *For any* ℓ'*-mixing array of zero-mean r.v.s.* $(X_{n,k})$, $k \le k_n \to \infty$, *and for each positive integer* v *denote by* $(Y_{n,k}(v))$ *the array of* v *partial sums of equal size of* $(X_{n,k})$. *Suppose that*

$$A_v := \limsup_n \left| \sum_{k=1}^{v} E\, Y_{n,k}^2(v) - 1 \right| \underset{v\to\infty}{\longrightarrow} 0, \tag{2}$$

$$C_v(\varepsilon) := \limsup_n \sum_{k=1}^{v} E\, Y_{n,k}^2(v) I\left(|Y_{n,k}(v)| > \varepsilon\right) \underset{v\to\infty}{\longrightarrow} 0 \text{ for every } \varepsilon > 0, \tag{3}$$

$$\max_a E\,|S_{n,a,a+c}| \underset{n}{\longrightarrow} 0, \quad \text{for all } c > 0. \tag{4}$$

Then $S_n \xrightarrow{D} N(0,1)$.

Note that condition (4) is weak as implied by

$$\max_k \; E\, X_{n,k}^2 \underset{n}{\longrightarrow} 0,$$

which in turn follows from the Lindeberg condition ((2.4b) in [9]).

To continue, denote $\bar{X} := X/\sqrt{EX^2}$ the normalization of X, for any zero-mean random variable X such that $0 < EX^2 < \infty$, and $\bar{X} := 0$ if $EX^2 = 0$ otherwise.

Corollary 2.2. *Let* $(X_{n,k})$, $k \le k_n \to \infty$, *be any* ℓ'*-mixing array of zero-mean r.v.s. Suppose that*

$$\limsup_{n, b \le k_n} \max_a E\, \bar{S}_{n,a,a+b}^2 I\left(|\bar{S}_{n,a,a+b}| > q\right) \underset{q\to\infty}{\longrightarrow} 0, \tag{5}$$

and that there exists a function $d(x) : [0,\epsilon) \to [0,\infty)$ *such that* $d(0) = 0$, $d(x)/x \to 0$ *as* $x \to 0$, *and*

$$\limsup_n \max_a \left| E\, S_{n,a,a+a_n}^2 - \frac{a_n}{k_n} \right| \le d\left(\lim_n \frac{a_n}{k_n}\right), \tag{6}$$

for every sequence of positive integers (a_n) *such that* $\lim_n a_n/k_n < \epsilon$ *exists. Then* $S_n \xrightarrow{D} N(0,1)$.

Remark 2.3. Condition (6) imposes a weak stationary upon the $X_{n,k}$'s. It holds if there exists a sequence (V_n) such that $V_n \to \infty$ and a slowly varying function ([3, 7]) $h(x)$ such that $V_n = h(n)n$ and

$$\max_a \left| \frac{V_{k_n}}{V_b} E\, S_{n,a,a+b}^2 - 1 \right| \longrightarrow 0,$$

as n and $b \to \infty$ ($b \le k_n$). Since in this case for any sequence of positive integers (a_n) such that $0 < \lim a_n/k_n < 1$ exists we have $(V_{k_n}/V_{a_n})/(k_n/a_n) \to 1$ (see [7, Theorem 1.1]), and $V_{a_n}/V_{k_n} \to 0$ if $a_n/k_n \to 0$ (see Lemma 4.3). Hence $d(x)$ can be taken as zero on $[0,1)$ because

$$\left| E\, S_{n,a,a+a_n}^2 - \frac{a_n}{k_n} \right| \le \frac{V_{a_n}}{V_{k_n}} \left| \frac{V_{k_n}}{V_{a_n}} E\, S_{n,a,a+a_n}^2 - 1 \right| + \left| \frac{V_{a_n}}{V_{k_n}} - \frac{a_n}{k_n} \right|.$$

The sequences of zero-mean random variables X_k satisfying $E\left(\sum_{a+1}^{a+b} X_k\right)^2 /b \to A$ uniformly in a as $b \to \infty$, satisfy (6) with the array $X_{n,k} := X_k/\sqrt{nA}$, $1 \le k \le n$, for which the above formula holds with $V_n = n$.

The condition (6) is implied by a more obvious condition: for large enough n and every a, b

$$\left| E\, S^2_{n,a,a+b} \;-\; \frac{b}{k_n} \right| \;\le\; d\left(\frac{b}{k_n}\right),$$

where $d(x)$ is as in (6) but continuous on $[0, \epsilon)$.

This corollary weakens the conditions in part b) of Theorem 2.3 of [10], which improves other results on α-mixing and β-mixing r.v.s. of Longnecker and Serfling [5], and Yoshihara [11].

Next, we give another characterization of the asymptotical independence. It is the relation (10) below.

Theorem 2.4. *For any array $(X_{n,k})$, $k \le k_n \to \infty$, of zero-mean r.v.s. suppose there exists $(Y_{n,k})$, $k \le v_n$, an array of partial sums of $(X_{n,k})$, such that*

$$\text{either} \quad \sum_{k\ \text{even}} E\, Y^2_{n,k} \underset{n}{\longrightarrow} 0 \qquad \text{or} \qquad E\,(\sum_{k\ \text{even}} Y_{n,k})^2 \underset{n}{\longrightarrow} 0, \tag{7}$$

$$\sum_{k\ \text{odd}} E\, Y^2_{n,k} \underset{n}{\longrightarrow} 1, \tag{8}$$

$$\sum_{k\ \text{odd}} E\, Y^2_{n,k} I\,(|Y_{n,k}| > \varepsilon) \underset{n}{\longrightarrow} 0 \qquad \text{for every } \varepsilon > 0. \tag{9}$$

Then $S_n \xrightarrow{D} N(0,1)$ if for all real t

$$\sum_{k=1}^{v_n} \left| \mathrm{Cov}\left\{ \exp\,(\,it \sum_{j\in(0,k)_{2k}} Y_{n,j}),\ \exp\,(-it\, Y_{n,k}) \right\} \right| \underset{n}{\longrightarrow} 0, \tag{10}$$

where I_{2k} denotes the set of even (odd) integers in I if k is even (odd), and if the second condition of (7) holds, the first sum of (10) can be taken with only odd k's.

More precisely, if both conditions of (7) as well as (8), (9) hold, then $S_n \xrightarrow{D} N(0,1)$ if and only if for all real t

$$\sum_{k=1}^{v_n} \prod_{s\in(k,v_n]_{2k}} E\exp(itY_{n,s})\mathrm{Cov}\left\{ \exp(it \sum_{j\in(0,k)_{2k}} Y_{n,j}),\ \exp\,(-it\, Y_{n,k}) \right\} \underset{n}{\longrightarrow} 0. \tag{11}$$

This theorem extends Theorem 5.1 of Dvoretzky [2] for α-mixing arrays. It also gives a part of Lemma 3.1 of Withers [10] and the related result of Ibragimov and Linnik [4]. Otherwise, it applies to arrays mixing in known sense, since (10) is implied by $v_n K_n(d_n) \to 0$ as $n \to \infty$, where $d_n = \min_k(m_{n,k} - m_{n,k-1})$ and $K_n(d_n)$ can be $\alpha_n(d_n), \beta_n(d_n), \ell_n(d_n), \varphi_n(d_n)$, or $\rho_n(d_n)$ [10]. All the last terms exceed

$$\max_k \left| E \exp\big(it \sum_{j \in (0,k)_{2k}} Y_{n,j} \big) \{ \exp(it\, Y_{n,k}) - E \exp(it\, Y_{n,k}) \} \right| .$$

Since Theorem 5.2 of Dvoretzky [2] is a direct consequence of Theorem 5.1, we can derive easily a similar consequence of Theorem 2.4 which implies it.

3. C.L.T.s for Dependent Increment Arrays

Beside the arrays, in which 'distant' random variables depend weakly on each other, as discussed in Sec. 2 and [9], there are ones in which weak dependence appears even between 'neighboring' partial sums of the random variables. These arrays are treated in Serfling [8], where the measure of dependence uses conditional expectations. We give here a different approach.

Similar to (1), let us call an array $(X_{n,k})$, $k \le k_n$, *weakly correlated increment array* if for all real t

$$L_n(t) = \max_{a,b: a<b \le k_n} |\mathrm{Cov}\{ \exp(it\, S_{n,a}), \exp(-it S_{n,a,b})\}| \underset{n}{\longrightarrow} 0. \qquad (12)$$

Although the measures of dependence $\ell'_n(c,t)$ and $L_n(t)$ are logically different, the results in [9] apply well to weakly correlated increment arrays.

Theorem 3.1. *For any weakly correlated increment array of zero-mean r.v.s.* $(X_{n,k})$*,* $k \le k_n \to \infty$*, suppose that the conditions* (3.1b) *and* (3.1c) *in* [9]*, or instead* (3.2a), (3.2b), (3.2c), (3.2d) *in* [9]*, hold, then* $S_n \xrightarrow{D} N(0,1)$*.*

Next, we give an extension of Theorem 5.3 of Dvoretzky [2], which improved Theorem 4.1 of Serfling [8], by the help of Corollary 2 of [9]. The assumptions on conditional expectations, characterizing the dependence, are replaced here by the likes of $L_n(t)$ (see (15), (16)).

Theorem 3.2. *Let* $(X_{n,k})$*,* $k \le k_n \to \infty$*, be any array of zero-mean r.v.s. Denote* $T_{n,a,a+b} := S_{n,a,a+b}/\sqrt{b}$*. If for all real* t

$$E\, T^2_{n,a,a+b} \longrightarrow 1, \qquad (13)$$

$$b^{-\gamma} E\, |T_{n,a,a+b}|^{2+\beta} \longrightarrow 0, \qquad (14)$$

$$b^{\theta} \mathrm{Cov}\{ \exp(it\, S_{n,a}/\sqrt{k_n}), T_{n,a,a+b}\} \longrightarrow 0, \qquad (15)$$

$$\mathrm{Cov}\{ \exp(it\, S_{n,a}/\sqrt{k_n}), T^2_{n,a,a+b}\} \longrightarrow 0, \qquad (16)$$

as $n \to \infty$, $b \to \infty$, $a + b \le k_n$, *uniformly in* a, *then* $S_n/\sqrt{k_n} \xrightarrow{D} N(0,1)$, *provided the positive* γ, β, θ *satisfy*

$$\gamma \quad \le \quad \beta\theta.$$

Another type of characterization of dependence between neighboring random variables is the martingale property. Some martingale C.L.T.'s are created in [1, 2, 6, 8]. We add two notes.

Suppose for an array $(X_{n,k})$ that

$$E_{\,S_{n,k-1}}(X_{n,k}) := E\,(X_{n,k}|\ \sigma(S_{n,k-1})) = 0, \text{ a.s.,} \tag{17}$$

for all n and $k \le k_n$. A stronger form of (17) is that $S_{n,k}$ forms a martingale for each n, i.e.

$$E_{\,n,k-1}X_{n,k} \ := \ E\,(X_{n,k}|\ \sigma(X_{n,1},\ldots,X_{n,k-1})) \ = \ 0, \text{ a.s.}$$

As a direct consequence of Corollary 1 of [9] we have the following result.

Corollary 3.3. *If for any array of zero-mean r.v.s.* $(X_{n,k})$, $k \le k_n \to \infty$, *the conditions* (2.4a), (2.4b) *in* [9] *and* (17) *hold, then* $S_n \xrightarrow{D} N(0,1)$ *if and only if for all real* t

$$\sum_{k=1}^{k_n} \prod_{j=k+1}^{k_n} E \exp\,(\,itX_{n,j}\,)\mathrm{Cov}\,\{\,\exp\,(\,it\,S_{n,k-1}), X_{n,k}^2\} \underset{n}{\longrightarrow} 0. \tag{18}$$

This corollary cannot be implied by martingale C.L.T.s in [1, 2, 6], because of the necessity of condition (18). Condition (2.4a) in [9] in this case is the same as $E\,S_n^2 \to 1$, since $ES_n^2 = EX_{n,k_n}^2 + ES_{n,k_n-1}^2 = \cdots = \sum_{k=1}^{k_n} EX_{n,k}^2$. In the case $X_{n,k} = X_k/\sqrt{E\,S_n^2}$, as supposed in Brown [1], (2.4a) becomes trivial. Note that condition (18) is implied by, hence can be substituted by,

$$\sum_{k=1}^{k_n} \left|E\,\exp\,(\,it\,S_{n,k-1})(X_{n,k}^2 - E\,X_{n,k}^2)\right| \ \underset{n}{\longrightarrow} \ 0 \qquad \text{for all real } t, \tag{19}$$

or by

$$\sum_{k=1}^{k_n} E\,\left|E_{\,S_{n,k-1}}(X_{n,k}^2 - E\,X_{n,k}^2)\right| \ \underset{n}{\longrightarrow} \ 0,$$

or by

$$\sum_{k=1}^{k_n} E\,\left|E_{\,n,k-1}(X_{n,k}^2 - E\,X_{n,k}^2)\right| \ \underset{n}{\longrightarrow} \ 0. \tag{20}$$

The corollary with (20) instead of (18) is a consequence of the results in [1, 6].

Also note that Corollary 1 in [9], with (19) and

$$\sum_{k=1}^{k_n} |E \exp(it S_{n,k-1}) X_{n,k}| \underset{n}{\longrightarrow} 0 \quad \text{for all real } t, \tag{21}$$

instead of (2.5) and (2.6) in [9], is applicable for martingale-like sequences, since (21) follows from

$$E \sum_{k=1}^{k_n} |E_{\,n,k-1} X_{n,k}| \underset{n}{\longrightarrow} 0.$$

Corollary 3.4. *For any array of zero-mean r.v.s.* $(X_{n,k})$, $k \le k_n \to \infty$, *if the conditions* (2.4a), (2.4b) *in* [9], *and* (21) *hold, then* $S_n \overset{D}{\to} N(0,1)$ *if and only if* (18) *holds, consequently if* (19) *holds.*

The practical numerical computations of the terms in (19) and (20) seem to have the same degree of difficulty. In this aspect, the conditional expectations have no more preference than 't parameterized covariance' terms. Besides, the condition (19) is nearly sharp, as showed in the corollaries above.

4. Proofs

For brevity, below we shall use the notations defined in [9] without recalling.

Lemma 4.1. *For any independent zero-mean r.v.s.* $X_1, X_2, \ldots, X_n$

$$\begin{aligned} &\left| E \exp\left(it \sum_1^n X_k \right) - \exp\left(\frac{-t^2}{2} \right) \right| \\ &\le \left(\frac{A}{2} + C(\varepsilon) \right) t^2 + \varepsilon (A+1)|t|^3 + \left(C(\varepsilon) + \varepsilon^2 \right) (A+1) \frac{t^4}{8} \end{aligned} \tag{22}$$

for all real t *and* $\varepsilon > 0$, *where*

$$A := \left| \sum_1^n EX_k^2 \; - \; 1 \right|,$$

$$C(\varepsilon) := \sum_1^n EX_k^2 I\left(|X_k| > \varepsilon\right).$$

Proof. The left-hand side of (22) equals

$$\Bigg| \left\{ \exp\left(\frac{-t^2}{2} \sum_{k=1}^n EX_k^2 \right) - \exp\left(\frac{-t^2}{2} \right) \right\} +$$

$$+ \sum_{k=1}^n E \exp\left(it \sum_{j=1}^{k-1} X_j - \frac{t^2}{2} \sum_{j=k+1}^n EX_j^2 \right) \left\{ E \exp(itX_k) - \exp\left(\frac{-t^2}{2} EX_k^2 \right) \right\} \Bigg|$$

$$\leq \frac{t^2}{2}A \; + \; \sum_{k=1}^{n} \Big| E\exp(itX_k) \; - \; \exp\Big(\frac{-t^2}{2}EX_k^2\Big)\Big|,$$

since by the Mean Value Theorem for any $0 \leq b < a$ there exist $b \leq c \leq a$ such that $|\exp(-a) - \exp(-b)| = |(a-b)\exp(-c)(-1)| \leq |a-b|$. Hence (22) follows, since the term $|E\exp(itX_k) \, - \, \exp(-t^2 EX_k^2/2)|$ is bounded by the sum of

$$\left| E\,\exp\,(\,it\,X_k) \; - \; 1 \; + \; \frac{t^2}{2} E\,X_k^2 \right| \;\; \leq \;\; t^2 E\,X_k^2 I(|X_k| > \varepsilon) \; + \; \varepsilon |t|^3 E\,X_k^2,$$

using the inequalities in the proof of Corollary 1 in [9], and

$$\left| E\,\exp\,\Big(\frac{-t^2}{2} E\,X_k^2\Big) \; - \; 1 \; + \; \frac{t^2}{2} E\,X_k^2 \right| \;\; \leq \;\; \frac{t^4}{8} E\,X_k^2 \big\{ E\,X_k^2 I\,(|X_k| > \varepsilon) + \varepsilon^2 \big\},$$

as $|\exp(-x) - 1 + x| \leq x^2/2$ for all $x \geq 0$ by the Taylor's formula. ■

Proof of Theorem 2.1. Fix a value of t. By (22), (2) and (3) for any $\varepsilon > 0$ there are a large enough $v = v_\varepsilon$ and N_ε such that for all $n \geq N_\varepsilon$ and for $(Y_{n,k}) = (Y_{n,k}(v))$, the array of v partial sums of equal size of $(X_{n,k})$,

$$\begin{aligned} \varepsilon > & \left| E\exp\left(it \sum_{k=1}^{v} Y_{n,k}^* \right) - \exp\left(\frac{-t^2}{2} \right) \right| = \\ & > \left| E\exp\left(itS_n \right) - \exp\left(\frac{-t^2}{2} \right) - (S_1 - S_2) - S_2 \right|, \end{aligned} \tag{23}$$

where $S_1 \, = \, E\exp\,(\,it\,S_n) - E\exp\,(\,it\,\sum_1^v Y_{n,k}^*)$ is the term in (2.2) in [9] for $(Y_{n,k})$ and S_2 is the term in (2.8b) in [9] for $(Y_{n,k})$, with $d_{n,k} = c > 0$.

Similar to the proof of Proposition 2.2 in [9], by (4) and putting $m = [k_n/v_\varepsilon]$

$$|S_1 - S_2| \;\; \leq \;\; 2\sum_{k=2}^{v_\varepsilon} E\, \left| S_{n,(k-1)m-c,(k-1)m} \right| \;\; \leq \;\; 2v_\varepsilon \max_a E\, |S_{n,a,a+c}| \;\; \underset{n}{\longrightarrow} 0.$$

For S_2 we have

$$|S_2| \;\; \leq \;\; v_\varepsilon \sup_{2 \leq k \leq v_\varepsilon} \left| \mathrm{Cov}\, \{\, \exp\,(\,it\,S_{n,(k-1)m-c}), \exp\,(-itY_{n,k}) \} \right| \leq v_\varepsilon \ell_n'(c,t).$$

Since $(X_{n,k})$ is ℓ'-mixing, there exists an M such that for any $c > M$ and all $n > M$ both sides of the last inequality are less than ε. So by choosing a $c > M$, for the remainder term in (23) we have $|\exp(itS_n) - \exp(-t^2/2)| < 3\varepsilon$ for n large enough, that is, it tends to zero as $n \to \infty$. Since t can be arbitrarily chosen we obtain the conclusion. ■

Proof of Corollary 2.2. Since $\max_a ES_{n,a,a+c}^2 \leq \max_a |ES_{n,a,a+c}^2 - c/k_n| + c/k_n$ and $k_n \to \infty$, the condition (6) implies (4). To show (2) and (3) fix a value of $v > 1/\epsilon$. Let $(Y_{n,k})$ be the array of v partial sums of equal size of $(X_{n,k})$. Set

$m_n = [k_n/v]$ and $m'_n = k_n - m_n(v-1)$. Then, since $m_n \leq m'_n < m_n + v$, both m_n/k_n and m'_n/k_n tend to $1/v$ as $n \to \infty$. So (6), applying to these m_n, m'_n and $Y_{n,k}$, implies

$$\begin{aligned} A_v &\leq \limsup_n \left\{ \sum_{k=1}^{v-1} \max_a \left| E\,S^2_{n,a,a+m_n} - \frac{m_n}{k_n} \right| + \max_a \left| E\,S^2_{n,a,a+m'_n} - \frac{m'_n}{k_n} \right| \right\} \\ &< vd(1/v), \end{aligned}$$

by which (2) follows. To obtain (3), note that the sum in it does not exceed

$$\left(\sum_{k=1}^{v} E\,Y^2_{n,k} \right) \max_k \; E\,\overline{Y}^2_{n,k} I \left(|\overline{Y}_{n,k}| > \varepsilon / \sqrt{E\,Y^2_{n,k}} \right).$$

Therefore, since by (6) $E\,Y^2_{n,k} \leq |E\,Y^2_{n,k} - a_n/k_n| + a_n/k_n < d(1/v) + 2/v$ for all large enough n, where a_n is either m_n or m'_n depending on either $k < v$ or $k = v$, respectively,

$$C_v(\varepsilon) \quad \leq \quad (A_v + 1) \limsup_n \; \max_k \; E\,\overline{Y}^2_{n,k} I \left(|\overline{Y}_{n,k}| > \varepsilon / \sqrt{2/v + d(1/v)} \right).$$

Hence by (5) $C_v(\varepsilon) \to 0$ as $v \to \infty$. ∎

Because the conditions (2.2) and (2.1) in [9] are equivalent we have the following lemma.

Lemma 4.2. *If* $\sum_{k=1}^{k_n} E\,X^2_{n,k} \to 0$ *then* $S_n \xrightarrow{P} 0$ *if and only if for all real* t

$$\sum_{k=1}^{k_n} \Pi_k(X,t)\,\mathrm{Cov}\{ \exp(it\,S_{n,k-1}), \exp(-itX_{n,k}) \} \underset{n}{\longrightarrow} 0.$$

Proof of Theorem 2.4. If the second part of (7) holds then $\sum_{k \text{ even}} Y_{n,k} \xrightarrow{P} 0$. In the case the first condition of (7) holds, by Lemma 4.2 and condition (10), applied to $(Y_{n,k})$, k even, $\sum_{k \text{ even}} Y_{n,k} \xrightarrow{P} 0$, too. In addition, if both conditions of (7) hold then by Lemma 4.2, (11) holds for $(Y_{n,k})$, k even, meaning that the summations in (11) can be taken with even values only. Consequently, the conditions (11) and (11) for $(Y_{n,k})$, k odd, are equivalent. Then Proposition 2.1 of [9], applied to $(Y_{n,k})$, k odd, give us the conclusions, using the conditions (2.2), (2.3) in [9], (8) and (9). ∎

Proof of Theorem 3.1. The proof, for the case when (3.1b) and (3.1c) in [9] hold, is similar to, and simpler than, that of Theorem 3.1 of [9]. Here Proposition 2.2 of [9] is applied to $d_{n,k} = 0$. So conditions (2.8a) and (3.1a) in [9] hold trivially.

Since the conditions of Theorem 3.2 of [9] imply those of Theorem 3.1 of [9], as showed in the proof, here they also lead to the conclusion. ∎

Proof of Theorem 3.2. Let us apply Corollary 2 of [9], with $d_{n,k} = 0$, to the array $(X'_{n,k}) = (X_{n,k}/\sqrt{k_n})$ and the following array of its partial sums $(Y'_{n,k})$. For any $0 < \delta < 1$ set $m_n = [k_n^\delta]$, $m_{n,k} = m_n k$ and $v_n = [k_n/m_n]$. Then $Y'_{n,k} = T_{n,m_n(k-1),m_n k}\sqrt{m_n/k_n}$ for $k < v_n$, and $Y'_{n,v_n} = T_{n,m_n(v_n-1),k_n}\sqrt{V_n/k_n}$ where $V_n := k_n - (v_n - 1)m_n$. We need to show that $(Y'_{n,k})$ satisfies (2.4a), (2.4b) and (2.11) in [9], with $d_{n,k} = 0$.

Since $k_n = (v_n - 1)m_n + V_n$ and

$$\sum_{k=1}^{v_n} E\,{Y'}_{n,k}^2 - 1 = \sum_{1}^{v_n-1}\left(E\,T_{n,k}^2 - 1\right)\frac{m_n}{k_n} + \left(E\,T_{n,v_n}^2 - 1\right)\frac{V_n}{k_n},$$

the condition (13) implies (2.4a) in [9] for $(Y'_{n,k})$.

To continue, note that for large enough n $k_n^\delta/2 \le k_n^\delta - 1 \le m_n \le k_n^\delta$, $v_n \le 2k_n^{1-\delta}$ and $k_n^\delta/2 \le m_n \le V_n \le 2m_n \le 2k_n^\delta$, where $m_n \le V_n \le 2m_n$ because $0 \le m_n(k_n/m_n - [k_n/m_n]) \le m_n$. Hence for any real number a

$$\max(m_n^a, V_n^a) \le 2^{|a|}k_n^{\delta a}. \tag{24}$$

For large enough n by (24) we have

$$\begin{aligned}
&\sum_{k=1}^{v_n} E\,{Y'}_{n,k}^2 I\left(|Y'_{n,k}| > \varepsilon\right)\\
&\le \frac{1}{\varepsilon^\beta}\sum_{k=1}^{v_n} E\,|Y'_{n,k}|^{2+\beta}\\
&\le \frac{v_n}{\varepsilon^\beta}\max_{a<v_n, b=m_n,V_n} E\,\left|S'_{n,a,a+b}\right|^{2+\beta}\\
&\le C v_n\left(\frac{k_n^\delta}{k_n}\right)^{\frac{2+\beta}{2}} k_n^{\delta\gamma}\max_{a<v_n, b=m_n,V_n}\left\{b^{-\gamma}E|T_{n,a,a+b}|^{2+\beta}\right\},
\end{aligned}$$

where $C = 2^{(2+\beta)/2+\gamma}/\varepsilon^\beta$. Hence (2.4b) in [9] follows from (14) if

$$\gamma \le \frac{\beta}{2}\frac{1-\delta}{\delta}. \tag{25}$$

For checking (2.11) in [9] we have, by (24),

$$\begin{aligned}
&\sum_{k=1}^{v_n}\left|\mathrm{Cov}\left\{\exp\left(itS'_{n,m_n(k-1)}\right), Y'_{n,k}\right\}\right|\\
&\le v_n\max_k\left|\mathrm{Cov}\left\{\exp\left(itS'_{n,m_n(k-1)}\right), Y'_{n,k}\right\}\right|\\
&\le 2^{1/2+\theta}v_n k_n^{\delta/2-\delta\theta-1/2}\max_{a<v_n, b=m_n,V_n}\left(b^\theta\left|\mathrm{Cov}\left\{\exp\left(itS'_{n,a}\right), T_{n,a,a+b}\right\}\right|\right),
\end{aligned}$$

the right-hand side of which tends to zero by (15) if

$$\theta \geq \frac{1-\delta}{2\delta}. \tag{26}$$

Also by (16)

$$\begin{aligned}
&\sum_{k=1}^{v_n} \Big|\mathrm{Cov}\left\{\exp\left(itS'_{n,m_n(k-1)}\right), {Y'}^2_{n,k}\right\}\Big| \\
&\leq v_n \max_k \Big|\mathrm{Cov}\left\{\exp\left(it\,S'_{n,m_n(k-1)}\right), {Y'}^2_{n,k}\right\}\Big| \\
&\leq 2v_n k_n^{\delta-1} \max_{a<v_n, b=m_n, V_n} \left|\mathrm{Cov}\left\{\exp\left(itS'_{n,a}\right), T^2_{n,a,a+b}\right\}\right| \underset{n}{\to} 0.
\end{aligned}$$

Since $\gamma \leq \beta\theta$ we can find δ satisfying both inequalities (25) and (26), hence all the conditions of Corollary 2 in [9] are satisfied for $(Y'_{n,k})$. ■

Lemma 4.3. *For any slowly varying function $h(x) : (0,\infty) \to (0,\infty)$ and two sequences of positive numbers (a_n), (b_n) such that $b_n \to \infty$, suppose that $0 < \lim a_n/b_n < \infty$ exists then $h(a_n)/h(b_n) \to 1$. Otherwise if $a_n/b_n \to 0$ and the set $\{a_n; n = 1, 2 \dots\} \cap [0, a]$ is finite for every $a < \infty$ then $a_n h(a_n)/(b_n h(b_n)) \to 0$.*

Proof. By [3, Corollary VIII.9 (p. 274)] $h(x) = a(x)\exp(\int_1^x \epsilon(y)/y\,dy)$ where $\epsilon(x) \to 0$ and $a(x) \to c$, $0 < c < \infty$, as $x \to \infty$.

So, if $0 < \lim a_n/b_n < \infty$ then $a_n \to \infty$ and $h(a_n)/h(b_n) \to 1$ in the case $|\int_{b_n}^{a_n} \epsilon(y)/y\,dy| \to 0$. But for large enough n the term in the last convergence is bounded by $|\int_{b_n}^{a_n} \varepsilon/y\,dy| = \varepsilon|\ln a_n/b_n| \leq \varepsilon(|\ln\lim a_n/b_n| + \varepsilon)$, for any $\varepsilon > 0$.

Otherwise when the left conditions hold, choose a so large $A > c$ that $|\epsilon(x)| < 1/2$ and $|a(x)| < A$ for $x \geq A$, and $A^{-1} < \min_n a_n$. Then, denoting $A_n = \max(a_n, A)$, for all large enough n such that $b_n > A_n$ we have

$$\Big|\int_{b_n}^{a_n} \frac{\epsilon(y)}{y}dy\Big| \leq \int_{a_n}^{A_n} \frac{|\epsilon(y)|}{y}dy + \int_{A_n}^{b_n} \frac{1}{2y}dy \leq \int_{A^{-1}}^{A} \frac{|\epsilon(y)|}{y}dy + \frac{1}{2}\ln\frac{b_n}{a_n}.$$

Hence

$$\frac{a_n h(a_n)}{b_n h(b_n)} \leq \frac{a_n a(a_n) C}{b_n a(b_n)} \sqrt{\frac{b_n}{a_n}} \longrightarrow 0,$$

where $C = \exp(\int_{A^{-1}}^{A} |\epsilon(y)|/y\,dy) < \infty$, noting that $\max_n a(a_n) < \infty$, since the set $\{a_n\} \cap [0, A]$ is finite by the assumptions, and $|a(a_n)| < A$ on the set $\{a_n > A\}$. ■

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