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Some Continuous Properties of Norm in Orlicz-Lorentz Spaces*

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Abstract. In this paper we investigate some continuous properties of norm in Orlicz-Lorentz spaces.

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1. Introduction

Orlicz-Lorentz spaces as a generalization of Orlicz spaces L_φ and Lorentz spaces Λ_ω have been studied by many authors (we refer to [1- 9] for basic properties of Orlicz-Lorentz spaces as well to references therein). In this paper we give some continuous properties of norm in Orlicz-Lorentz spaces. Let us first recall some notations of Orlicz-Lorentz spaces:

Let $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ be a measure space with the complete and σ-finite measure μ, $L^0(\mu)$ be a space of all μ-equivalent classes of Σ-measurable functions on Ω with topology of convergence in measure on μ-finite sets.

A Banach space $(E, \|.\|_E)$ is called the Banach function space on (Ω, μ) if it is a subspace of $L^0(\mu)$, such that there exists a function $h \in E$ such that $h > 0$ a.e. on Ω, and if $f \in L^0(\mu)$, $g \in E$ and $|f| \le |g|$ a.e. on Ω then $f \in E$ and $\|f\|_E \le \|g\|_E$. Moreover, if the unit ball $B_E = \{f \in E : \ \|f\|_E \le 1\}$ is closed on

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$L^0(\mu)$ then we say that E has the Fatou property. A Banach function space E is said to be symmetric if for every $f \in L^0(\mu)$ and $g \in E$ such that $\mu_f = \mu_g$ then $f \in E$ and $\|f\|_E = \|g\|_E$, where for any $h \in L^0(\mu)$, μ_h denotes the distribution of h, defined by $\mu_h(t) = \mu\left(\{x \in \Omega : \ |h(x)| \geq t\}\right), \ t \geq 0$.

Let E be a Banach function space on (Ω, μ) then the Köthe dual space $E^{'}$ of E is a Banach function space, which can be identified with the space of all functionals possessing an integral representation; that is,

$$E^{'} = \left\{ g \in L^0(\mu) : \ \|g\|_{E'} = \sup_{\|f\|_E \leq 1} \int_{\Omega} |fg| d\mu < \infty \right\}.$$

Let $\varphi : [0, \infty) \to [0, \infty)$ be an Orlicz function (i.e, it is a convex function, takes value zero only at zero) and $\omega : (0, \infty) \to (0, \infty)$ be a weight function (i.e., it is a non-increasing function and locally integrable and $\int_0^\infty \omega dm = \infty$). The Orlicz-Lorentz space $\Lambda^{\Omega}_{\varphi,\omega}$ on (Ω, μ) is the set of all functions $f(x) \in L^0(\mu)$ such that

$$\int_0^\infty \varphi(\lambda f^*(x))\omega(x) dm < \infty$$

for some $\lambda > 0$, where f^* is the non-increasing rearrangement of f defined by

$$f^*(x) = \inf\{\lambda > 0 : \ \mu_f(\lambda) \leq x\},$$

with $x > 0$ (by convention, $\inf \emptyset = \infty$).

It is easy to check that $\Lambda^{\Omega}_{\varphi,\omega}$ is a symmetric function space with the Fatou property, equipped with the Luxemburg norm

$$\|f\|_{\Lambda^{\Omega}_{\varphi,\omega}} = \inf \left\{ \lambda > 0 : \ \int_0^\infty \varphi\left(\frac{f^*(x)}{\lambda}\right) \omega(x) dm \leq 1 \right\},$$

Note that: If $\omega \equiv 1$ then $\Lambda^{\Omega}_{\varphi,\omega}$ is the Orlicz space L_φ, if $\varphi(t) = t$ then $\Lambda^{\Omega}_{\varphi,\omega}$ is the Lorentz space Λ_ω.

Denote by φ_* the Young conjugate function of φ, that is

$$\varphi_*(t) = \sup\{st - \varphi(s) | s \geq 0\}, \quad t \geq 0.$$

Let φ be an Orlicz function, we define

$$I(f) = \int_0^\infty \varphi_*\left(\frac{f^*(x)}{\omega(x)}\right) \omega(x) dm$$

for any $f(x) \in L^0(\mu)$, we will denote by $M^{\Omega}_{\varphi_*,\omega}$ the space defined by

$$M^{\Omega}_{\varphi_*,\omega} = \left\{ f(x) \in L^0(\mu): \ \ I\left(\frac{f}{\lambda}\right) < \infty \ \text{ with some } \ \ \lambda > 0 \right\}.$$

In the space $M^{\Omega}_{\varphi_*,\omega}$ we define a monotone and homogeneous functional

$$\|f\|_{M^{\Omega}_{\varphi_*,\omega}} = \inf\left\{ \lambda > 0: \ \ I\left(\frac{f}{\lambda}\right) \le 1 \right\}.$$

Recall that φ satisfies Δ_2 condition (we write, $\varphi \in \Delta_2$) if there exists $C > 0$ such that $\varphi(2t) \le C\varphi(t) \ \ \forall t > 0$ and φ is a N-function if $\lim\limits_{t\to 0} \varphi(t)/t = 0$ and $\lim\limits_{t\to+\infty} \varphi(t)/t = +\infty$. Put

$$S(t) = \int\limits_0^t \omega(s)ds, \qquad t > 0,$$

we say that the weight function ω is regular if there is a constant $K > 1$ independent of t such that $S(2t) \ge KS(t)$ for any $t > 0$. In what follows we will write $f \asymp g$ for nonnegative functions f and g whenever $C_1 f \le g \le C_2 f$ for some $C_j > 0, j = 1, 2$.

There were proved in [4] the following results on the dual space of $(\Lambda^I_{\varphi,\omega})$: Let ω be a weight function and $\varphi(t) = t$ or φ be an N-function, $I = (0, \infty)$. Then the following assertions hold:

i) If ω is regular, then $(\Lambda^I_{\varphi,\omega})' = M^I_{\varphi_*,\omega}$ and $\|.\|_{(\Lambda^I_{\varphi,\omega})'} \asymp \|.\|_{M^I_{\varphi_*,\omega}}$.

ii) If $\varphi \in \Delta_2$ and $(\Lambda^I_{\varphi,\omega})' = M^I_{\varphi_*,\omega}$, then ω is regular.

If $1 \le p < \infty$ then it is known the following continuous property of the norm in $L_p(\mathbb{R}^n)$: $\|f(t + .) - f(.)\|_p \to 0$ as $t \to 0$, and $\|f(t.) - f(.)\|_p \to 0$ as $t \to 1$ for any $f \in L_p(\mathbb{R}^n)$. For Orlicz spaces L_φ, these results hold when $\varphi \in \Delta_2$. In this paper we generalize these results to Orlicz-Lorentz spaces.

2. Main Results

For $t = (t_1, \ldots, t_n), x = (x_1, \ldots, x_n)$, we denote $tx = (t_1x_1, \ldots, t_nx_n)$ and $|x| = \left(\sum\limits_{i=1}^n x_i^2\right)^{1/2}$.

Theorem 2.1. *Let $f \in \Lambda^{\mathbb{R}^n}_{\varphi,\omega}$ and $g \in M^{\mathbb{R}^n}_{\varphi_*,\omega}$. Then*

$$\lim_{t\to 1} \int\limits_{\mathbb{R}^n} (f(tx) - f(x))g(x)dm = 0. \tag{1}$$

Proof. Since $f \in \Lambda^{\mathbb{R}^n}_{\varphi,\omega}$, there exists $a > 0$ such that

$$\int_0^\infty \varphi\left(\frac{f^*(x)}{a}\right)\omega(x)dm < \infty.$$

Since $g \in M^{\mathbb{R}^n}_{\varphi_*,\omega}$, there is $b > 0$ such that

$$\int_0^\infty \varphi_*\left(\frac{g^*(x)}{b\omega(x)}\right)\omega(x)dm < \infty.$$

Therefore,

$$\int_0^\infty f^*(x)g^*(x)dm$$
$$\leq ab\left(\int_0^\infty \varphi(\frac{f^*(x)}{a})\omega(x)dm + \int_0^\infty \varphi_*(\frac{g^*(x)}{b\omega(x)})\omega(x)dm\right) < \infty. \quad (2)$$

We first prove (1) for characteristic functions of measurable sets A. There are two cases, that is

Case 1. $m(A) < \infty$. Since $f(x) = \chi_A(x)$, we have $f^*(x) = \chi_{(0,m(A))}(x)$. We denote by tA the set $\{tx : x \in A\}$. Then it follows from $m(A) < \infty$ that $\lim\limits_{t\to 1} m(A\Delta(tA)) = 0$. From (2) we have

$$\int_0^{m(A)} g^*(x)dx < \infty.$$

Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\int\limits_0^\delta g^*(x)dm < \epsilon$, and then there is $t_0 > 0$ such that $m(A\Delta(tA)) < \delta$ for all $|t-1| < t_0$. Put $C_t := A\Delta(tA)$ then $m(C_t) < \delta$ for all $|t-1| < t_0$. So, for $|t-1| < t_0$:

$$\begin{aligned}\left|\int_{\mathbb{R}^n}(\chi_A(tx) - \chi_A(x))g(x)dm\right| &\leq \int_{\mathbb{R}^n}|(\chi_A(tx) - \chi_A(x))g(x)|\,dm \\ &= \int_{\mathbb{R}^n}|\chi_{C_t}(x)|\,.\,|g(x)|\,dm \\ &\leq \int_0^{m(C_t)} g^*(x)dm < \epsilon.\end{aligned}$$

This implies

$$\lim_{t\to 1}\int_{\mathbb{R}^n}(\chi_A(tx)-\chi_A(x))g(x)dm = 0.$$

Case 2. $m(A)=\infty$. Then $f^*(x)\equiv 1\ \forall x\in(0,\infty)$ and it follows from (2) that $g^*(x)$ is integrable on $(0,\infty)$, thus $g(x)\in L^1(m)$. Denote $g_t(x) := g(\frac{x}{t})$. Then

$$\lim_{t\to 1}\|g-g_t\|_{L^1(m)} = 0. \tag{3}$$

We see that

$$\begin{aligned}&\int_{\mathbb{R}^n}(\chi_A(tx)-\chi_A(x))g(x)dm\\ &=\frac{1}{t}\int_{\mathbb{R}^n}\chi_A(x)g_t(x)dm-\int_{\mathbb{R}^n}\chi_A(x)g(x)dm\\ &=\frac{1}{t}\int_{\mathbb{R}^n}\chi_A(x)(g_t(x)-g(x))dm+\left(\frac{1}{t}-1\right)\int_{\mathbb{R}^n}\chi_A(x)g(x)dm.\end{aligned}$$

Therefore,

$$\left|\int_{\mathbb{R}^n}(\chi_A(tx)-\chi_A(x))g(x)dm\right| \le \frac{1}{|t|}\int_{\mathbb{R}^n}|g_t(x)-g(x)|\,dm+\left|\frac{1}{t}-1\right|\int_{\mathbb{R}^n}|g(x)|\,dm.$$

So, it follows from (3) that

$$\lim_{t\to 1}\int_{\mathbb{R}^n}(\chi_A(tx)-\chi_A(x))g(x)dm = 0.$$

From the linearity of integral, (1) is true for all simple functions $f\in\Lambda^{\mathbb{R}^n}_{\varphi,\omega}$.

Now, to complete the proof, we need only to prove (1) for nonnegative measurable $f\in\Lambda^{\mathbb{R}^n}_{\varphi,\omega}$ and $f(x)<\infty$ a.e. We consider the following sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$:

$$f_n(x)=\sum_{k=0}^{n.2^n-1}\frac{k}{2^n}\chi_{A_{n,k}}(x)+n\chi_{A_n}(x),$$

where

$$\begin{cases}A_{n,k} &= \left\{x:\ \frac{k}{2^n}\le f(x)<\frac{k+1}{2^n}\right\},\\ A_n &= \{x:\ f(x)\ge n\}.\end{cases}$$

It is easy to see that $f_n(x)\uparrow f(x)$ a.e., and $\lim_{n\to\infty}m(A_n)=0$. Therefore, given $\epsilon>0$ and $\delta>0$, there exists n_0 such that $1/2^n<\epsilon$ and $m(A_n)<\delta$ for all $n\ge n_0$. Then $\{x: f(x)-f_n(x)\ge\epsilon\}\subset A_n$ for all $n\ge n_0$. Hence, for $n\ge n_0$ we have $m(\{x: |f(x)-f_n(x)|\ge\epsilon\})\le m(A_n)<\delta$, which gives $f_n\xrightarrow{m} f$. So,

$(f_n - f)^*(x) \to 0$. By the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_0^\infty (f_n - f)^*(x) g^*(x) dm = 0.$$

On the other hand, we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} (f(tx) - f(x)) g(x) dm \Big| \\
&= \Big| \int_{\mathbb{R}^n} (f(tx) - f_n(tx)) g(x) dm + \int_{\mathbb{R}^n} (f_n(tx) - f_n(x)) g(x) dm + \\
&\quad + \int_{\mathbb{R}^n} (f_n(x) - f(x)) g(x) dm \Big| \\
&\le 2 \int_0^\infty (f_n - f)^*(x) g^*(x) dm + \Big| \int_{\mathbb{R}^n} (f_n(tx) - f_n(x)) g(x) dm \Big|.
\end{aligned}$$

Letting $t \to 1$, we get

$$\limsup_{t\to 1} \Big| \int_{\mathbb{R}^n} (f(tx) - f(x)) g(x) dm \Big| \le 2 \int_0^\infty (f_n - f)^*(x) g(x) dm \quad \forall n \in \mathbb{N}.$$

Hence,

$$\limsup_{t\to 1} \Big| \int_{\mathbb{R}^n} (f(tx) - f(x)) g(x) dm \Big| \le 2 \lim_{n\to\infty} \int_0^\infty (f_n - f)^*(x) g^*(x) dm = 0.$$

This gives

$$\lim_{t\to 1} \int_{\mathbb{R}^n} (f(tx) - f(x)) g(x) dm = 0.$$

The proof is complete. ■

Theorem 2.2. *Let $f \in \Lambda_{\varphi,\omega}^{\mathbb{R}^n}$ and $\varphi \in \Delta_2$. Then*

$$\lim_{t\to 0} \|f(t + .) - f(.)\|_{\Lambda_{\varphi,\omega}^{\mathbb{R}^n}} = 0. \tag{4}$$

Proof. We first prove (4) for characteristic functions of measurable sets A. Indeed, since $\chi_A(x) \in \Lambda_{\varphi,\omega}^{\mathbb{R}^n}$, we have $m(A) < +\infty$. Since $f(x) = \chi_A(x)$, $f^*(x) = \chi_{(0,m(A))}(x)$. We denote by $t + A$ the set $\{t + x : x \in A\}$. Then it follows from $m(A) < +\infty$ that $\lim_{t\to 0} m(A \Delta (t + A)) = 0$. For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^\delta \omega(x)dm < \epsilon,$$

and then there is $t_0 > 0$ such that $m(A\Delta(t + A)) < \delta$ for all $|t| < t_0$. Put $C_t := A\Delta(t + A)$ then $m(C_t) < \delta$ for all $|t| < t_0$. So, for $|t| < t_0$:

$$\left|\int_0^\infty \varphi\big(f(t+.) - f(.)\big)^*(x)\omega(x)dm\right| = \int_0^\infty \left|\varphi\big(\chi_{(0,m(C_t))}(x)\big)\omega(x)\right|dm$$
$$\leq \varphi(1)\int_0^\delta \omega(x)dm < \varphi(1)\epsilon.$$

This implies

$$\lim_{t\to 0}\int_0^\infty \varphi\big((f(t+.) - f(.)\big)^*(x)\omega(x)dm = 0.$$

Then it follows from $\varphi \in \Delta_2$ that $\lim_{t\to 0} \|f(t+.) - f(.)\|_{\Lambda^{\mathbb{R}^n}_{\varphi,\omega}} = 0$.

Because $\Lambda^{\mathbb{R}^n}_{\varphi,\omega}$ is a Banach space, (4) is true for all simple functions $f \in \Lambda^{\mathbb{R}^n}_{\varphi,\omega}$.

Now, to complete the proof, we only have to show (4) for nonnegative measurable functions $f \in \Lambda^{\mathbb{R}^n}_{\varphi,\omega}$ and $f(x) < \infty$ a.e. We consider the sequence of functions $\{f_n(x)\}_{n=1}^\infty$ as follows

$$f_n(x) = \sum_{k=0}^{n.2^n-1} \frac{k}{2^n}\chi_{A_{n,k}}(x) + n\chi_{A_n}(x),$$

where

$$\begin{cases} A_{n,k} & = \{x : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\}, \\ A_n & = \{x : f(x) \geq n\}. \end{cases}$$

Then it is easy to see that $f_n(x) \uparrow f(x)$ a.e. and $\lim_{n\to\infty} m(A_n) = 0$. Therefore, given $\epsilon > 0$ and $\delta > 0$, there exists $N \in \mathbb{N}$ such that $1/2^n < \epsilon$ and $m(A_n) < \delta$ for all $n \geq N$. Then $\{x : f(x) - f_n(x) \geq \epsilon\} \subset A_n$ for all $n \geq N$. Hence, for $n \geq N$ we get $m(\{x : |f(x) - f_n(x)| \geq \epsilon\}) \leq m(A_n) < \delta$, which gives $f_n \overset{m}{\to} f$. So, $(f_n - f)^*(x) \to 0$. By the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_0^\infty \varphi(f_n - f)^*(x)\omega(x)dm = 0.$$

Therefore, for $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\int_0^\infty \varphi(f_{n_0} - f)^*(x)\omega(x)dm < \epsilon.$$

Hence,

$$\int_0^\infty \varphi(f_{n_0}(t+.) - f(t+.))^*(x)\omega(x)dm < \epsilon. \tag{5}$$

Because f_{n_0} is a simple function, there exists $t_0 > 0$ such that for $|t| < t_0$

$$\int_0^\infty \varphi(f_{n_0}(t+.) - f_{n_0})^*(x)\omega(x)dm < \epsilon. \tag{6}$$

We see that

$$(f+g+h)^*(x) \le f^*\left(\frac{x}{3}\right) + g^*\left(\frac{x}{3}\right) + h^*\left(\frac{x}{3}\right)$$

and there exists a number C such that

$$\varphi(a+b+c) \le C(\varphi(a)+\varphi(b)+\varphi(c)) \quad \forall a,b,c>0 \quad (\text{because } \varphi \in \Delta_2).$$

Therefore, by (5)-(6), we have for $|t| < t_0$

$$\begin{aligned}
&\int_0^\infty \varphi(f(t+.) - f)^*(x)\omega(x)dm \\
&\le C\Big(\int_0^\infty \varphi(f_{n_0} - f)^*\left(\frac{x}{3}\right)\omega(x)dm + \int_0^\infty \varphi(f_{n_0}(t+.) - f(t+.))^*\left(\frac{x}{3}\right)\omega(x)dm + \\
&\quad + \int_0^\infty \varphi(f_{n_0}(t+.) - f_{n_0})^*\left(\frac{x}{3}\right)\omega(x)dm\Big) \\
&\le C\Big(\int_0^\infty \varphi(f_{n_0} - f)^*\left(\frac{x}{3}\right)\omega\left(\frac{x}{3}\right)dm + \int_0^\infty \varphi(f_{n_0}(t+.) - f(t+.))^*\left(\frac{x}{3}\right)\omega\left(\frac{x}{3}\right)dm \\
&\quad + \int_0^\infty \varphi(f_{n_0}(t+.) - f_{n_0})^*\left(\frac{x}{3}\right)\omega\left(\frac{x}{3}\right)dm\Big) < 9C\epsilon.
\end{aligned}$$

Therefore,

$$\lim_{t\to 0}\int_0^\infty \varphi(f(t+.) - f)^*(x)\omega(x)dm = 0.$$

Then it follows from $\varphi \in \Delta_2$ that

$$\lim_{t\to 0}\|f(t+.) - f(.)\|_{\Lambda^{\mathbb{R}^n}_{\varphi,\omega}} = 0.$$

The proof is complete. ■

Corollary 2.3. *Let* $f \in \Lambda^{\mathbb{R}^n}_{\varphi,\omega}$ *and* $\varphi \in \Delta_2$. *Then the modulus of smoothness* $\omega_\Phi(\delta, f) := \sup_{0<|h|\le\delta} \|f(h+.) - f(.)\|_{\Lambda^{\mathbb{R}^n}_{\varphi,\omega}} \to 0$ *as* $\delta \to 0$.

Remark 2.4. The condition $\varphi \in \Delta_2$ in Theorem 2.2 is essential.

Proof. Indeed, let $n = 1, \varphi(x) = 2^x$ if $x \ge 2$ and $\varphi(x) = 2x$ if $0 \le x < 2$, and let $\omega(x) \equiv 1$. Then $\varphi(\notin \Delta_2)$ is an Orlicz function. We put

$$f(x) = \begin{cases} \log_2\left(\frac{1}{2^x-1}\right) & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Then it is easy to check that

$$\int_0^\infty \varphi\left(\frac{f^*(x)}{2}\right)\omega(x)dm < \infty.$$

So, $f \in \Lambda^{\mathbb{R}}_{\varphi,\omega}$, and we see that for $t > 0$

$$f(x) - f(t+x) = \log_2 \frac{2^{t+x}-1}{2^x-1} \ge 2, \quad \forall x \in \left(0, \frac{t}{3}\right).$$

Therefore, for $t \ge 0$ we have

$$\begin{aligned} \int_0^\infty \varphi(f - f(t+.))^*(x)\omega(x)dm &\ge \int_0^\infty \varphi(f(x) - f(t+x))dm \\ &\ge \int_0^{t/3} \frac{2^{t+x}-1}{2^x-1}dm \\ &\ge \int_0^{t/3} \frac{2^t-1}{2^x-1}dm = \infty. \end{aligned}$$

Hence, for $t > 0$

$$\|f - f(t+.)\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \ge 1,$$

which implies

$$\lim_{t\to 0} \|f - f(t+.)\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \ne 0.$$

The proof is complete. ∎

Next, we have the following theorem.

Theorem 2.5. *Let* $f \in \Lambda^{\mathbb{R}^n}_{\varphi,\omega}$. *Assume that* $\varphi \in \Delta_2$, *then*

$$\lim_{t\to 1} \|f(t.) - f(.)\|_{\Lambda^{\mathbb{R}^n}_{\varphi,\omega}} = 0. \tag{7}$$

Proof. We first prove (7) whenever $f(x) = \chi_A(x)$ is the characteristic function of a measurable set A. Indeed, since $f(x) = \chi_A(x) \in \Lambda_{\varphi,\omega}^{\mathbb{R}^n}$, we have $m(A) < \infty$ and $f^*(x) = \chi_{(0,m(A))}(x)$. We denote by tA the set $\{tx : x \in A\}$. Then it follows from $m(A) < \infty$ that $\lim\limits_{t\to 1} m(A\Delta(tA)) = 0$. For an arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^\delta \omega(x)dm < \epsilon,$$

and then there is $t_0 > 0$ such that $m(A\Delta(tA)) < \delta$ for all $|t-1| < t_0$. Put $C_t := A\Delta(tA)$ then $m(C_t) < \delta$ for all $|t-1| < t_0$. So, for $|t-1| < t_0$:

$$\begin{aligned}\Big|\int_0^\infty \varphi\big((f(t.) - f(.))^*(x)\big)\omega(x)dm\Big| &= \int_0^\infty \Big|\varphi\big(\chi_{(0,m(C_t))}(x)\big)\omega(x)\Big|dm \\ &\leq \varphi(1)\int_0^\delta \omega(x)dm < \varphi(1)\epsilon.\end{aligned}$$

That is

$$\lim_{t\to 1}\int_0^\infty \varphi\big((f(t.) - f(.))^*(x)\big)\omega(x)dm = 0.$$

Then it follows from $\varphi \in \Delta_2$ that $\lim\limits_{t\to 1} \|(f(t.) - f(.)\|_{\Lambda_{\varphi,\omega}^{\mathbb{R}^n}} = 0$.

Since the Orlicz-Lorentz space $\Lambda_{\varphi,\omega}^{\mathbb{R}^n}$ is a Banach space, (7) is true for all simple functions $f \in \Lambda_{\varphi,\omega}^{\mathbb{R}^n}$.

Now, to complete the proof, we only prove (7) for $f \in \Lambda_{\varphi,\omega}^{\mathbb{R}^n}$ being the non-negative, measurable function and $f(x) < \infty$ a.e. We consider the sequence of functions $\{f_n(x)\}_{n=1}^\infty$ as follows

$$f_n(x) = \sum_{k=0}^{n.2^n-1} \frac{k}{2^n}\chi_{A_{n,k}}(x) + n\chi_{A_n}(x),$$

where

$$\begin{cases} A_{n,k} &= \left\{x : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\right\}, \\ A_n &= \left\{x : f(x) \geq n\right\}. \end{cases}$$

Then it is easy to see that $f_n(x) \uparrow f(x)$ a.e., and $\lim\limits_{n\to\infty} m(A_n) = 0$. Therefore, given $\epsilon > 0$ and $\delta > 0$, there exists $N \in \mathbb{N}$ such that $1/2^n < \epsilon$ and $m(A_n) < \delta$ for all $n \geq N$. Then $\{x : f(x) - f_n(x) \geq \epsilon\} \subset A_n$ for all $n \geq N$. Hence, for $n \geq N$:

$$m(\{x : |f(x) - f_n(x)| \geq \epsilon\}) \leq m(A_n) < \delta,$$

which gives $f_n \xrightarrow{m} f$. So $(f_n - f)^*(x) \to 0$. By the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_0^\infty \varphi(f_n - f)^*(x)\omega(x)dm = 0.$$

Therefore, for $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\int_0^\infty \varphi(f_{n_0} - f)^*(x)\omega(x)dm < \epsilon.$$

Hence,

$$\int_0^\infty \varphi(f_{n_0}(t.) - f(t.))^*(x)\omega(x)dm < \epsilon. \tag{8}$$

Since f_{n_0} is a simple function, there exists $t_0 > 0$ such that for $|t-1| < t_0$:

$$\int_0^\infty \varphi(f_{n_0}(t.) - f_{n_0})^*(x)\omega(x)dm < \epsilon. \tag{9}$$

We see that

$$(f+g+h)^*(x) \le f^*\left(\frac{x}{3}\right) + g^*\left(\frac{x}{3}\right) + h^*\left(\frac{x}{3}\right)$$

and there exists a number C such that

$$\varphi(a+b+c) \le C(\varphi(a)+\varphi(b)+\varphi(c)) \quad \forall a,b,c>0 \quad (\text{because } \varphi \in \Delta_2).$$

Therefore, from (8) and (9), we have for $|t-1| < t_0$:

$$\begin{aligned}
&\int_0^\infty \varphi(f(t.) - f)^*(x)\omega(x)dm \\
&< C\Big(\int_0^\infty \varphi(f_{n_0} - f)^*\left(\frac{x}{3}\right)\omega(x)dm + \int_0^\infty \varphi(f_{n_0}(t.) - f(t.))^*\left(\frac{x}{3}\right)\omega(x)dm \\
&\quad + \int_I \varphi(f_{n_0}(t.) - f_{n_0})^*\left(\frac{x}{3}\right)\omega(x)dm\Big) \\
&\le C\Big(\int_0^\infty \varphi(f_{n_0} - f)^*\left(\frac{x}{3}\right)\omega\left(\frac{x}{3}\right)dm + \int_0^\infty \varphi(f_{n_0}(t.) - f(t.))^*\left(\frac{x}{3}\right)\omega\left(\frac{x}{3}\right)dm \\
&\quad + \int_0^\infty \varphi(f_{n_0}(t.) - f_{n_0})^*\left(\frac{x}{3}\right)\omega\left(\frac{x}{3}\right)dm\Big) < 9C\epsilon.
\end{aligned}$$

Hence,

$$\lim_{t\to 1}\int_0^\infty \varphi(f(t.)-f)^*(x)\omega(x)dm = 0.$$

Then it follows from $\varphi \in \Delta_2$ that

$$\lim_{t\to 1}\|f(t\ .)-f(.)\|_{\Lambda^{\mathbb{R}^n}_{\varphi,\omega}} = 0.$$

The proof is complete. ∎

Similarly as above, we have the following:

Remark 2.6. The condition $\varphi \in \Delta_2$ in Theorem 2.5 is essential.

Remark 2.7. Theorems 2.1, 2.2, 2.5 still hold when we take $f \in \Lambda^{R^n_+}_{\varphi,\omega}$, $g \in M^{R^n_+}_{\varphi_*,\omega}$ (in Theorem 2.2, $t \geq 0$ is required), and the condition $\varphi \in \Delta_2$ is also essential.

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