Vietnam Journal of Mathematics **37**:4(2009) 515-525

Vietnam Journal
of
MATHEMATICS

On Generalized δ-Supplemented Modules

Yahya Talebi and Behnam Talaee

Department of Mathematics, Faculty of Basic Science,
University of Mazandaran, Babolsar, Iran

Received June 9, 2009
Revised August 11, 2009

Abstract. Let M be a module and N, K submodules of M. N is called a generalized supplement of K in M if, $M = N + K$ and $N \cap K \leq \text{Rad}(N)$. A module M is called a generalized supplemented module (briefly a GS-module) if any submodule of M has a generalized supplement in M. Also M is called a generalized amply supplemented (briefly a GAS-module) if whenever $M = A + B$ for submodules A, B of M, then A has a generalized supplement in M contained in B. It is clear that any supplemented module (amply supplemented module) is a GS-module (GAS-module). In this paper we investigate generalizations of these modules. We will show that a module M is Artinian if and only if M is a δ-GAS-module and satisfies DCC on generalized δ-supplment and δ-small submodules.

2000 Mathematics Subject Classification: 16D10, 16D90, 16P70.

Key words: GS-module, GAS-module, δ-GS-module, δ-GAS-module.

1. Introduction

Throughout this article, all rings are associative and have an identity, and all modules are unitary right R-modules.

A submodule L of M is called *small* in M (denoted by $L \ll M$), if for every proper submodule K of M, $L + K \neq M$. A module M is called *hollow*, if every proper submodule of M is small in M. A submodule N of M is called *essential* in M (denoted by $N \trianglelefteq_e M$) if, $N \cap K \neq 0$ for every nonzero submodule K of M. The *singular* submodule of a module M (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid xI = 0;$ for some right ideal $I \trianglelefteq_e R\}$. A module M is called *singular* (*nonsingular*, resp.) if, $Z(M) = M$ ($Z(M) = 0$, resp.). We denote by $\text{End}(M)$ the ring of all endomorphism of a module M. $N \subseteq^{\oplus} M$ means that N is a direct summand of M. We use the notations $\text{Rad}(M)$ for the Jacobson radical of M and $J(R)$ for the Jacobson of a ring R.

A module M is called a $lifting$ module (or said to satisfy D_1), if for every submodule N of M, M has a decomposition $M = A \oplus B$, such that $A \leq N$ and $N \cap B \ll B$.

Let M be a module. Then M is called $\pi - projective$ if for every two submodules X, Y of M, there exists $f \in \text{End}(M)$ with $\text{Im}(f) \leq X$ and $\text{Im}(1 - f) \leq Y$.

For two submodules N and K of M, N is called a *supplement* of K in M if, N is minimal with the property $M = K + N$, equivalently $M = K + N$ and $N \cap L \ll N$. Also N is called a *weak supplement* of K in M if, $M = N + K$ and $N \cap K \ll M$. A module M is called *supplemented* if, every submodule of M has a supplement in M. M is called *amply supplemented* if whenever $M = A + B$ for submodules A, B of M, then A has a supplement in M contained in B. Also M is called *weakly supplemented* if any submodule of M has a weak supplement in M.

By Zhou [9] a submodule L of M is called δ-*small* in M (denoted by $L \ll_\delta M$) if for any submodule N of M with M/N singular, $M = N + L$ implies that $M = N$. The sum of all δ-small submodules of a module M is denoted by $\delta(M)$. By definition of reject it is clear that $\delta(M/\delta(M)) = 0$.

Let K, N be submodules of module M, then N is called a δ-*supplement* of K in M if, $M = N + K$ and $N \cap K \ll_\delta N$. N is called a *weak* δ-*supplement* of K in M if, $M = N + K$ and $N \cap K \ll_\delta M$. A module M is called δ-*supplemented* if every submodule of M has a δ-supplement in M. M is called *amply* δ-*supplemented* if whenever $M = A + B$ for submodules A, B of M, then A has a δ-*supplement* in M contained in B. Also M is called *weakly* $\delta - supplemented$ if every submodule of M has a weak δ-supplement in M.

For two submodules N, K of M, K is called a δ-*cosmall* submodule of N in M if, $K \leq N$ and $N/K \ll_\delta M/K$.

A module M is called δ-*lifting* (or M has δ-D_1) if for any submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll_\delta M_2$.

δ-supplemented modules, δ-lifting modules and some generalizations of them had studied by several authors (see for example [1, 3, 7]).

It is clear that any supplemented (amply supplemented, weakly supplemented, lifting) module is a δ-supplemented (amply δ-supplemented, weakly δ-supplemented, δ-lifting) module.

A module M is called δ-*hollow* if, every proper submodule of M is δ-small in M. We call a homomorphism $f : M \to N$ a δ-small epimorphism if, f is an epimorphism with $\text{Ker}(f) \ll_\delta M$.

It is easy to see that every small submodule of a module M is δ-small in M, so $\text{Rad}(M) \subseteq \delta(M)$ and $\text{Rad}(M) = \delta(M)$ if M is singular. Also any nonsingular semisimple submodule of M is δ-small in M and δ-small submodules of a singular module are small submodules.

Wang and Ding [6] defined *generalized supplemented* modules as generalizations of supplemented modules. They called a submodule N of M a *generalized supplement* submodule of M if there exists a submodule K of M such that $M = N + K$ and $N \cap K \leq \mathrm{Rad}(N)$ and N is called a *weak generalized supplement* of K in M if, $M = N + K$ and $N \cap K \leq \mathrm{Rad}(M)$. A module M is called a *generalized supplemented* module (briefly a GS-module) if every submodule of M has a generalized supplement in M. M is called a *generalized amply supplemented* module (briefly a GAS-module) if whenever $M = A + B$ for submodules A, B of M, then A has a generalized supplement in M contained in B. Also M is called a *generalized weakly supplemented* module (briefly a WGS-module) if every submodule of M has a generalized weak supplement in M.

In this paper we define *generalized* (*amply, weakly*) *δ-supplemented* modules and investigate some properties of these modules.

2. δ-GS Modules and δ-GAS Modules

Definition 2.1. Let M be a module and N, K be submodules of M. N is called a generalized δ-supplement of K in M if, $M = N + K$ and $N \cap K \leq \delta(N)$. A module M is called generalized δ-supplemented (or briefly δ-GS) if every submodule of M has a generalized δ-supplement in M. M is called a generalized amply δ-supplemented module (briefly a δ-GAS module) if whenever $M = A + B$ for submodules A, B of M, then A contains a generalized δ-supplement of B in M.

It is clear that any GS-module (GAS-module) is a δ-GS-module (δ-GAS-module) and δ-supplemented modules are δ-GS.

Lemma 2.2. *Let M be a module and K a δ-supplement submodule of M. Then $K \cap \delta(M) = \delta(K)$.*

Proof. Clearly $\delta(K) \leq K \cap \delta(M)$ by [9, Lemma 1.5]. For the converse suppose that $x \in K \cap \delta(M)$, then clearly Rx is a δ-small submodule of M. Also $Rx \leq K$. Now it suffices to show that $Rx \ll_\delta K$. Let $K = Rx + L$ with K/L singular. Since K is a δ-supplement submodule of M, there exists a submodule K' of M such that $M = K + K'$ and $K \cap K' \ll_\delta K$. Hence $Rx + L + K' = M$ and $M/(K' + L) \cong K/(L + K \cap K')$ is singular, so $M = K' + L$. Therefore $K = L + K \cap K' = L$ as $K \cap K' \ll_\delta K$. ■

Lemma 2.3. *Let M be a module. Then*

(i) *$\delta(M)$ is Artinian if and only if M satisfies DCC on δ-small submodules.*

(ii) *$\delta(M)$ is Noetherian if and only if M satisfies ACC on δ-small submodules.*

Proof. See [7, Theorems 2.3 and 2.5]. ■

Proposition 2.4. *Let M be a δ-GS-module and K a submodule of M with $K \cap \delta(M) = 0$. Then K is semisimple. In particular a δ-GS-module M with $\delta(M) = 0$ is semisimple.*

Proof. Let $K' \leq K$. Since M is a δ-GS-module, there exists a submodule L of M such that $K' + L = M$ and $K' \cap L \leq \delta(L)$. Hence $K = K' + K \cap L$ and $K' \cap K \cap L = K' \cap L \leq K \cap \delta(L) \leq K \cap \delta(M) = 0$. Thus K is semisimple. ■

Proposition 2.5. *Let M be a δ-GAS-module. Then every direct summand of M is δ-GAS.*

Proof. Suppose that $N \subseteq^{\oplus} M$. Write $M = N \oplus N'$. Let $N = A + B$. Then $M = A + (B \oplus N')$. Since M is a δ-GAS-module, there exists $A' \leq A$ such that $M = A' + (B \oplus N')$ and $A' \cap (B \oplus N') \leq \delta(A')$. So $N = A' + B$ and $A' \cap B = A' \cap (B + N) \leq \delta(A')$, as required. ■

Proposition 2.6. *Let M be a δ-GS-module. Then $M = N \oplus L$, where N is semisimple and L is a module with $\delta(L) \trianglelefteq_e L$.*

Proof. By [2, Proposition 5.21], $\delta(M)$ has a complement N such that $N \cap \delta(M) = 0$ and $N \oplus \delta(M) \trianglelefteq_e M$. Since M is a δ-GS-module, there exists $L \leq M$ such that $M = N + L$ and $N \cap L \leq \delta(L)$. Now $N \cap L = N \cap (N \cap L) \leq N \cap \delta(L) \leq N \cap \delta(M) = 0$, so $M = N \oplus L$. By Proposition 2.4, N is semisimple. Also $\delta(M) = \delta(N) \oplus \delta(L) = \delta(L)$. Hence $N \oplus \delta(M) = N \oplus \delta(L) \trianglelefteq_e M = N \oplus L$, and so $\delta(L) \trianglelefteq_e L$ by [5, Proposition 5.20]. ■

Proposition 2.7. *Let M be a module and $A \leq M$. Moreover let $B \leq M$ be a δ-GS-module. If $A + B$ has a generalized δ-supplement in M, then so does A.*

Proof. There exists $X \leq M$ such that $X + (A + B) = M$ and $X \cap (A + B) \leq \delta(X)$. Since B is a δ-GS-module, there exists $Y \leq B$ such that $Y + (X + A) \cap B = B$ and $(X + A) \cap Y \leq \delta(Y)$. Now we show that $X + Y$ is a generalized δ-supplement of A in M. It is clear that $(X + Y) + A = M$. Also $(X + Y) \cap A \leq X \cap (Y + A) + Y \cap (X + A)$, where $X \cap (Y + A) \leq X \cap (B + A) \leq \delta(X)$ and $Y \cap (X + A) \leq \delta(Y)$. So $(X + Y) \cap A \leq \delta(X) + \delta(Y) \leq \delta(X + Y)$ by [9, Lemma 1.5]. This completes the proof. ■

Proposition 2.8. *Let M_1 and M_2 be δ-GS-modules. Then $M = M_1 + M_2$ is again a δ-GS-module.*

Proof. Let X be a submodule of M. Trivially $M_1 + M_2 + X = M$ has a generalized δ-supplement in M and so $M_2 + X$ has a generalized δ-supplement in M (by Proposition 2.7). Thus again by Proposition 2.7, X has a generalized δ-supplement in M, as required. ■

Proposition 2.9. *Let M be a δ-GS-module. Then*

(i) *Every factor module of M is a δ-GS-module.*

(ii) *$M/\delta(M)$ is semisimple.*

Proof. (i) Let M/N be a factor module of M and L a submodule of M containing N. Since M is a δ-GS-module, there exists a submodule K of M such that $L + K = M$ and $L \cap K \leq \delta(K)$. Thus $M/N = L/N + (N+K)/N$ and $L/N \cap (N+K)/K = (N + (L \cap K))/N \leq (N + \delta(L \cap K))/N \leq \delta((N+K)/N)$; that is, $(N+K)/N$ is a generalized δ-supplement of L/N in M/N. Hence M/N is a δ-GS-module.

(ii) Let N be a submodule of M containing $\delta(M)$. Then there exists a submodule K of M such that $M = N + K$ and $N \cap K \leq \delta(K) \leq \delta(M)$. So $M/\delta(M) = N/\delta(M) \oplus (K + \delta(M))/\delta(M)$. Thus $M/\delta(M)$ is semisimple. ■

Corollary 2.10. *Let M be a δ-GS-module. Then every finitely M-generated module is a δ-GS-module.*

Proof. By Proposition 2.8, every finite sum of δ-GS-modules is δ-GS-module and by Proposition 2.9, every factor module of a δ-GS-module is δ-GS-module. These complete the proof. ■

Let M be a module and $N \leq M$. N is said to has *generalized ample δ-supplements* in M if for every submodule K of M with $M = N + K$, N has a generalized δ-supplement contained in K.

Proposition 2.11. *Let M be a module such that $M = M_1 + M_2$ for submodules M_1, M_2 of M. If M_1, M_2 have generalized ample δ-supplements in M, then $M_1 \cap M_2$ also has generalized ample δ-supplements in M.*

Proof. Let $X \leq M$ and $M_1 \cap M_2 + X = M$. Then $M_1 = M_1 \cap M_2 + X \cap M_1$ and $M_2 = M_1 \cap M_2 + X \cap M_2$, so $M = M_1 + X \cap M_2$ and $M = M_2 + X \cap M_1$. Since M_1, M_2 have generalized ample δ-supplements in M, there exist $X'_2 \leq X \cap M_2$ and $X'_1 \leq X \cap M_1$ such that $M_1 + X'_2 = M$ and $M_1 \cap X'_2 \leq \delta(X'_2)$, and $M_2 + X'_1 = M$ and $M_2 \cap X'_1 \leq \delta(X'_1)$. Therefore $X'_1 + X'_2 \leq X$ and $M_1 = M_1 \cap M_2 + X'_1$ and $M_2 = M_1 \cap M_2 + X'_2$. Thus $(M_1 \cap M_2) + (X'_1 + X'_2) = M$ and $(M_1 \cap M_2) \cap (X'_1 + X'_2) = (M_2 \cap X'_1) + (M_1 \cap X'_2) \leq \delta(X'_1 + X'_2)$. Hence $X'_1 + X'_2$ is a generalized δ-supplement of $M_1 \cap M_2$ contained in X. ■

Proposition 2.12. *Let M be a module and $U \leq M$. Then the following statements are equivalent:*

(i) *There is a decomposition $M = X \oplus X'$ with $X \leq U$ and $X' \cap U \leq \delta(X')$.*

(ii) *There is an idempotent $e \in \mathrm{End}(M)$ with $e(M) \leq U$ and $(1-e)U \leq \delta((1-e)M)$.*

(iii) *There is a direct summand X of M with $X \leq U$ and $U/X \leq \delta(M/X)$.*

(iv) *U has a generalized δ-supplement V in M, such that $V \cap U$ is a direct summand of U.*

Proof. (i)$\Longrightarrow$(ii) For decomposition $M = X \oplus X'$, there is an idempotent $e \in \mathrm{End}(M)$ with $e(M) = X$ and $(1-e)M = X'$. Since $X \leq U$, we have $(1-e)U = U \cap (1-e)M \leq \delta((1-e)M)$.

(ii)$\Longrightarrow$(iii) Take $X = e(M)$.

(iii)$\Longrightarrow$(i) Let $M = X \oplus X'$ and $U/X \leq \delta(M/X)$, then $U = X \oplus (X' \cap U)$. Since $X' \cap U \cong U/X \leq \delta(M/X)$, we have $X' \cap U \leq \delta(X')$.

(i)$\Longrightarrow$(iv) Take $V = X'$.

(iv)$\Longrightarrow$(i) Let V be a generalized δ-supplement of U in M and $U = X \oplus (V \cap U)$. Then $M = U + V = X + (V \cap U) + V = X + V$ and $U \cap V \leq \delta(V)$. Also $X \cap V = (X \cap U) \cap V = X \cap (U \cap V) = 0$; that is, X is a direct summand of M. ■

We say a module M has property δ-P^* if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \leq \delta(M/K)$.

It is clear that any module with δ-P^* is a δ-GS-module.

It is not difficult to see that a module M has δ-P^* if and only if for any submodule N of M, there exists a decomposition $M = K \oplus K'$ for M such that $K \leq N$ and $N \cap K' \leq \delta(K')$. Hence every δ-lifting module has δ-P^*.

Lemma 2.13. *Let M be a module and A, B submodules of M such that B is a generalized δ-supplement of A in M. If M/A is a singular simple module, then $A \cap B = \delta(B)$ and hence $B/\delta(B)$ is a singular simple module.*

Proof. We have $M = A + B$ and $A \cap B \leq \delta(B)$. Since $B/A \cap B \cong M/A$ is singular simple, $\delta(B) \leq A \cap B$. Therefore $A \cap B = \delta(B)$. ■

Proposition 2.14. *Let M be a module. If every submodule of M is a δ-GS-module, then M is a δ-GAS-module.*

Proof. Let L, N be submodules of M such that $M = N + L$. Thus there exists $H \leq L$ such that $(L \cap N) + H = L$ and $(L \cap N) \cap H = N \cap H \leq \delta(H)$. Hence $H + N \geq L$ and so $M = H + N$. ■

Corollary 2.15. *Let R be a ring. Then the following statements are equivalent:*

(i) *Every R-module is a δ-GAS-module.*

(ii) *Every R-module is a δ-GS-module.*

Proof. It is clear. ■

Proposition 2.16. *Let M be a π-projective δ-GS-module. Then M is a δ-GAS-module.*

Proof. Suppose that A, B are submodules of M such that $M = A + B$. Then there exists an endomorphism e of M such that $e(M) \leq A$ and $(1-e)M \leq B$. Note that $(1-e)A \leq A$. Let C be a generalized δ-supplement of A in M. Then $M = e(M) + (1-e)(A + C) \leq A + (1-e)C$. So $M = A + (1-e)C$. Also $(1-e)C$ is contained in B. Moreover $A \cap (1-e)C = (1-e)(A \cap C)$, hence $A \cap (1-e)C \leq (1-e)(\delta(C)) \leq \delta((1-e)C)$ as $A \cap C \leq \delta(C)$; that is, $(1-e)C$ is a generalized δ-supplement of A in M. Thus M is a δ-GAS-module. ■

Theorem 2.17. *Let M be a module. Then M is Artinian if and only if M is a δ-GAS-module and satisfies DCC on generalized δ-supplemented submodules and on δ-small submodules.*

Proof. Since every Artinian module is amply supplemented, the necessity is clear.

For the converse, suppose that M is a δ-GAS-module and satisfies DCC on generalized δ-supplemented submodules and on δ-small submodules. Then $\delta(M)$ is Artinian (Lemma 2.3). Thus it suffices to show that $M/\delta(M)$ is Artinian. First we show that $M/\delta(M)$ is semisimple. For this let N be any submodule of M containing $\delta(M)$, then there exists a submodule K of M such that $M = N + K$ and $N \cap K \leq \delta(K) \leq \delta(M)$. Thus $M/\delta(M) = N/\delta(M) \oplus (K + \delta(M))/\delta(M)$ and so $M/\delta(M)$ is semisimple.

Now it remains to show that $M/\delta(M)$ is Noetherian. For this suppose that $\delta(M) \leq N_1 \leq N_2 \leq \ldots$ is an ascending chain of submodules of M. Since M is δ-GAS-module, there exists a descending chain of submodules $K_1 \geq K_2 \geq \ldots$ such that each K_i is a generalized δ-supplement of N_i in M ($i = 1, 2, \ldots$). By hypothesis there exists a natural number n, such that $K_n = K_{n+1} = \ldots$. Moreover we have $M/\delta(M) = N_i/\delta(M) \oplus (K_i + \delta(M))/\delta(M)$ for all $i \geq n$. It follows that $N_n = N_{n+1} = \ldots$, so $M/\delta(M)$ is Noetherian. ■

Corollary 2.18. *Let M be a finitely generated δ-GAS-module. Then M is Artinian if and only if M satisfies DCC on δ-small submodules.*

Proof. The necessity is clear.

Conversely since M is a δ-GAS-module, $M/\delta(M)$ is semisimple by the proof of Theorem 2.17. Hence $M/\delta(M)$ is Artinian. Since M satisfies DCC on δ-small submodules, by Lemma 2.3, $\delta(M)$ is Artinian. Therefore M is Artinian. ■

Corollary 2.19. *Let R be a ring such that R_R is a δ-GAS-module. Then R is a right Artinian ring if and only if R satisfies DCC on δ-small right ideals.*

Proof. It is clear. ■

Example 2.20. Let M denote the $\mathbb{Z}$-module $\mathbb{Z}$. Then M is finitely generated and satisfies DCC on δ-small submodules. But M is not Artinian. Note that $\mathbb{Z}_{\mathbb{Z}}$ is not δ-GAS. It follows that the condition δ-GAS in Corollary 2.18 is necessary.

Proposition 2.21. *Let M be a module with ACC on δ-small submodules. Then the following hold:*

(i) *If M is a δ-GAS-module, then M is amply δ-supplemented.*

(ii) *M has δ-P^* if and only if M is δ-lifting.*

Proof. (i) Let $M = A + B$, then there exists $C \leq B$ such that $M = A + C$ and $A \cap C \leq \delta(C)$. Since M satisfies ACC on δ-small submodules, by Lemma 2.3, $\delta(C)$ is Noetherian, and hence $\delta(C)$ is finitely generated. It is not difficult to see that $\delta(C) \ll_\delta C$. Thus M is amply δ-supplemented.

(ii) Clearly any δ-lifting module has δ-P^*.

For the converse, it suffices to show that every factor module of M satisfies ACC on δ-small submodules.

Let A be any submodule of M and $B_1/A \leq B_2/A \leq \ldots$ an ascending chain of δ-small submodules of M/A. Since M has δ-P^*, M is a δ-GS-module and so M is δ-supplemented.

Let C be a δ-supplement of A in M. Then $M/A = (A+C)/A \cong C/(A\cap C)$. Since $B_i/A \ll_\delta M/A$, $B_i/A \cong D_i/(A\cap C) \ll_\delta C/(A\cap C)$ for some $D_i \leq C$ ($i = 1, 2, \ldots$). We prove that $D_i \ll_\delta M$. Let $D_i + E = M$ such that M/E is singular. Then $D_i/(A\cap C) + (E + (A\cap C))/(A\cap C) = M/(A\cap C)$. Hence $E + A\cap C = M$ and so $M = E$.

Now since $B_1 \leq B_2 \leq \ldots$, we have $D_1 \leq D_2 \leq \ldots$ and so there exists $n \in \mathbb{N}$ such that $D_k = D_{k+1}$ for all $k \geq n$. Therefore $B_k/A = B_{k+1}/A$ for all $k \geq n$; that is, M/A satisfies ACC on δ-small submodules. ■

Corollary 2.22. *Let M be a module and $\delta(M)$ be Noetherian. If M is a δ-GS (δ-GAS)-module, then M is δ-supplemented (amply δ-supplemented).*

Proof. Since $\delta(M)$ is Noetherian, it is δ-small in M. Now if M is δ-GS (δ-GAS), then from the proof of Proposition 2.21, M is δ-supplemented (amply δ-supplemented). ■

Example 2.23. (i) Let $R = \mathbb{Z}_8$. Then $M = R \oplus 2R/4R$ is a δ-supplemented module and so is a δ-GS-module (see [3, Example 2.16]).

(ii) The $\mathbb{Z}$-module $\mathbb{Z}$ is not a δ-GS-module (note that $\delta(\mathbb{Z}) = 0$).

3. δ-WGS-modules

Definition 3.1. Let M be a module and N, K submodules of M. N is called a *generalized weak δ-supplement* of K in M if, $M = N + K$ and $N \cap K \leq \delta(M)$. A module M is called *generalized weakly δ-supplemented* (or briefly δ-WGS) if every submodule of M has a generalized weak δ-supplement in M.

It is clear that by definition any GWS-module (weakly δ-supplemented module) is a δ-WGS-module.

Proposition 3.2. *Let M be a δ-WGS-module. Then*

(i) *Every δ-supplement submodule of M is a δ-WGS-module.*

(ii) *Every factor module of M is a δ-WGS-module.*

Proof. (i) Let K be a δ-supplement submodule of M and N a submodule of K. Since M is δ-WGS, there exists $L \leq M$ such that $M = N + L$ and $N\cap L \leq \delta(M)$. Hence $K = N + K\cap L$ and $N\cap(K\cap L) = N\cap L = K\cap(N\cap L) \leq K\cap\delta(M) = \delta(K)$ by Lemma 2.2. So K is δ-WGS.

(ii) Let N be a submodule of M and L/N any submodule of M/N. Therefore there exists $K \leq M$ such that $L + K = M$ and $K \cap L \leq \delta(M)$. Hence $M/N = L/N + (K + N)/N$. Let $\pi : M \longrightarrow M/N$ denote the natural epimorphism, then $L/N \cap (K + N)/N = (L \cap (K + N))/N = (N + (K \cap L))/N = \pi(L \cap K)$. Since $K \cap L \leq \delta(M)$, $\pi(L \cap K) \leq \pi(\delta(M)) \leq \delta(M/N)$. This completes the proof. ■

Proposition 3.3. *Let M be a finitely generated module. Then M is a δ-WGS-module if and only if M is weakly δ-supplemented.*

Proof. The necessity is clear.

For the converse, let N be a submodule of M, then there exists a submodule L of M such that $M = N + L$ and $N \cap L \leq \delta(M)$. Since M is finitely generated, $\delta(M)$ is δ-small in M and so M is a weakly δ-supplemented module. ■

Example 3.4. The module $M = \mathbb{Z}_{\mathbb{Z}}$ is a finitely generated module. Since M is not weakly δ-supplemented, it is not a δ-WGS-module by Proposition 3.3.

Lemma 3.5. *Let M be a module and K, M_1 submodules of M. Moreover suppose that M_1 is a δ-WGS-module. If $M_1 + K$ has a generalized weak δ-supplement in M, then so does K.*

Proof. Let N be a generalized weak δ-supplement of $M_1 + K$ in M; i.e, $M_1 + K + N = M$ and $N \cap (M_1 + K) \leq \delta(M)$. Since M_1 is a δ-WGS-module, there exists a submodule L of M_1 such that $M_1 \cap (N + K) + L = M_1$ and $L \cap (N + K) \leq \delta(M_1)$. Hence we have $M = K + N + L$ and $K \cap (N + L) \leq (K + M_1) \cap N + L \cap (N + K) \leq \delta(M)$; that is, $N + L$ is a generalized weak δ-supplement of K in M. ■

Proposition 3.6. *Let $M = M_1 + M_2$. If M_1 and M_2 are δ-WGS-modules, then M is a δ-WGS-module.*

Proof. Let N be a submodule of M. We have $M_1 + M_2 + N = M$ and so $M_1 + M_2 + N$ has a generalized weak δ-supplement in M. Hence $M_2 + N$ has a generalized weak δ-supplement in M (by Lemma 3.5). Again by Lemma 3.5, N has a generalized weak δ-supplement in M. So M is δ-WGS. ■

Theorem 3.7. *Let M be a module with $\delta(M) \ll_\delta M$ and $M/\delta(M)$ singular. Then the following statements are equivalent:*

(i) *M is a δ-WGS-module.*

(ii) *$M/\delta(M)$ is semisimple.*

(iii) *There is a decomposition $M = M_1 \oplus M_2$ such that M_1 is semisimple, $\delta(M) \trianglelefteq_e M_2$ and $M_2/\delta(M)$ is semisimple.*

Proof. (i)$\Longrightarrow$(ii) Let L be a submodule of M containing $\delta(M)$. Then there exists $N \leq M$ such that $M = N + L$ and $N \cap L \leq \delta(M)$. Thus $M/\delta(M) = L/\delta(M) \oplus (N + \delta(M))/\delta(M)$; that is, $M/\delta(M)$ is semisimple.

(ii)$\Longrightarrow$(i) Let N be a submodule of M. There exists a submodule L of M containing $\delta(M)$, such that $M/\delta(M) = (N + \delta(M))/\delta(M) \oplus L/\delta(M)$. Hence $M = N + \delta(M) + L = N + L$, as $\delta(M) \ll_\delta M$. Moreover $N \cap L \leq \delta(M)$. Therefore M is δ-WGS.

(ii)$\Longrightarrow$(iii) Let M_1 be a complement of $\delta(M)$ in M. Then $M_1 \cong (M_1 \oplus \delta(M))/\delta(M)$ is a direct summand of $M/\delta(M)$, and hence M_1 is semisimple. So there exists a semisimple module $M_2/\delta(M)$ such that $M/\delta(M) = (M_1 + \delta(M))/\delta(M) \oplus M_2/\delta(M)$. Therefore $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$. This implies that $M = M_1 \oplus M_2$. Since M_1 is a complement of $\delta(M)$, $M_1 \oplus \delta(M) \trianglelefteq_e M = M_1 \oplus M_2$ and so $\delta(M) \trianglelefteq_e M_2$ by [5, Proposition 5.20].

(iii)$\Longrightarrow$(ii) It is clear. ■

Example 3.8. Let M denote the $\mathbb{Z}$-module $\mathbb{Z}/12\mathbb{Z}$. Then $\delta(M) = 6\mathbb{Z}/12\mathbb{Z}$ is δ-small in M. Moreover $M/\delta(M) \cong \mathbb{Z}/6\mathbb{Z}$ is a singular semisimple module. So by Theorem 3.7, M is δ-WGS. Moreover let $M_1 = 4\mathbb{Z}/12\mathbb{Z}$ and $M_2 = 3\mathbb{Z}/12\mathbb{Z}$, then $M = M_1 \oplus M_2$ where M_1 is semisimple, $\delta(M) \trianglelefteq_e M_2$ and $M_2/\delta(M)$ is semisimple.

Recall that a ring R is called *semilocal* if $R/J(R)$ is semisimple. Here we say a ring R, *δ-semilocal* if $R/\delta(R)$ is semisimple, where $\delta(R) = \delta(R_R)$ denotes the sum of all δ-small right ideals of R.

Theorem 3.9. *Let R be a ring and consider the following statements:*

(i) *R is δ-semilocal.*

(ii) *Every R-module M with $\delta(M) \ll_\delta M$ and $M/\delta(M)$ singular, is a δ-WGS-module.*

(iii) *Every finitely generated R-module M with $M/\delta(M)$ singular, is a δ-WGS-module.*

(iv) *Every cyclic R-module M with $M/\delta(M)$ singular is a δ-WGS-module.*

Then (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv) *hold. Moreover if $R/\delta(R)$ is singular then* (iv)$\Longrightarrow$(i) *holds.*

Proof. (i)$\Longrightarrow$(ii) For module M there is a set Λ and an epimorphism $f : R^\Lambda \longrightarrow M$ such that $f(\delta(R^\Lambda)) \leq \delta(M)$. Then $R^\Lambda/\delta(R^\Lambda) \cong (R/\delta(R))^\Lambda$ and hence we obtain an epimorphism $g : R^\Lambda/\delta(R^\Lambda) \longrightarrow M/\delta(M)$. Thus $M/\delta(M)$ is semisimple and so M is a δ-WGS-module, by Theorem 3.7.

(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv) are clear.

(iv)$\Longrightarrow$(i) It is clear that if R_R is a δ-WGS-module, then R is δ-semilocal. Now if $R/\delta(R)$ is singular, then R_R is δ-WGS by (iv), and so R is δ-semilocal. ■

References

1. K. Al-Takhman, Cofinitely δ-supplemented and cofinitely δ-semiperfect modules, *Inter. J. Algebra.* **12** (2007), 601–613.

2. F. Anderson and K. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, New York, 1992.
3. M. T. Kosan, δ-Lifting and δ-Supplemented Modules, *Alg. Coll.* **14** (1) (2007), 53–60.
4. M. T. Kosan and A. Harmanci, Generalization of coatomic modules, *CEJM.* **3** (2) (2005), 273–281.
5. S. H. Mohamed and B. J. Muller, *Continuous and Discrete Modules*, Cambridge, UK: Cambridge Univ. Press, 1990.
6. Y. Wang and N. Ding, Generalized supplemented modules, *Taiwanese J. Math.* **10** (6) (2006), 1589–1601.
7. Y. Wang, δ-small submodules and δ-supplemented modules, *Inter. J. Math. and Math. sciences*, Hindawi publishing corporation, (2007), Article ID 58132, 8 pages.
8. R. Wisbauer, *Foundations of Modules and Ring Theory*, Gordon and Breakch, philadephia, 1991.
9. Y. Zhou, Generalizations of perfect, semiperfect, and semiregular rings, *Alg. Coll.* **7** (3) (2000), 305–318.