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Duality Equation and Efficiency Conditions in a Vector-optimization Problem

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Abstract. In this article we extend the duality scheme in [4] for a vector-optimization problem that appears in the equilibrium of a co-operative economy. According to the duality scheme, the dual problem is also a vector-optimization problem. Moreover, we can obtain a duality equation that represents a necessary and sufficient condition for the weak efficiency and a sufficient condition for the efficiency.

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1. Introduction

In [4] a dual problem of a linear maximization over a convex set is a problem of maximizing a Leontief function over a convex set in the dual space. The objective function of the dual problem is the conjugate of the objective function in the primal problem, and the dual feasible set is the conjugate of the primal feasible set. A primal feasible solution and a dual feasible solution are primal optimal and dual optimal, respectively, if and only if they satisfy the duality equation. In this article we extend this duality scheme for a vector-optimization problem that appears in the equilibrium of a co-operative economy. According to the duality scheme, the dual problem is a vector-optimization problem. Moreover, we can obtain a duality equation that characterizes the set of primal and dual weakly efficient solutions. By the duality equation we can see that the dual vector objective function value at a dual weakly efficient solution gives a scalarization weight vector of the primal problem, and conversely, the primal vector objective function value at a primal weakly efficient solution gives a scalarization weight vector of the dual problem. So, there is a duality correspondence between the primal weakly efficient set and the dual weakly efficient set. Furthermore, a primal weakly efficient solution corresponding to a positive dual weakly efficient solution must be efficient, and a dual weakly efficient solution corresponding to a nonzero primal weakly efficient solution must be efficient.

The article consists of 5 sections. After the introduction Sec. 2 presents some relevant known results. Sec. 3 is devoted to a vector-optimization problem and its dual, and in Sec. 4, we study a duality equation together with efficiency conditions. Finally, Sec. 5 draws several concluding remarks.

2. Preliminaries

To ease the readers we recall in this section some relevant known results. Let c be a positive ℓ-dimensional vector : $c \in R^\ell,\ c > 0$. We define a linear function f on R^ℓ_+ :

$$f(y) \ = \ c^T y \quad \forall y \in R^\ell_+,$$

where $c^T y$ stands for the inner product between c and y, and define a Leontief function f^* on R^ℓ_+ :

$$f^*(v) \ = \ \min \left\{ \frac{v_j}{c_j} \ : \ j = 1, 2, \ldots, \ell \right\} \quad \forall v \in R^\ell_+.$$

The functions f and f^* are the conjugate of each other in the sense that

$$\begin{aligned} f^*(v) &= \frac{1}{\sup\{f(y) : \ v^T y \leq 1, \ y \geq 0\}} \quad \forall v \in R^\ell_+, \\ f(y) &= \frac{1}{\sup\{f^*(v) : \ v^T y \leq 1, \ v \geq 0\}} \quad \forall y \in R^\ell_+, \end{aligned} \tag{1}$$

(cf.[2, 4]). Since this conjugacy maintains the quasi-convexity, but it does not maintain the convexity in general (cf.[1]), sometimes it is called quasi-conjugacy. However, in this article without confusion we obmit the term "quasi" for the simplicity.

Lemma 2.1. *For any $y \in R^\ell_+$ and $v \in R^\ell_+$ if $v^T y \leq \alpha$ where $\alpha \geq 0$, then $f(y) f^*(v) \leq \alpha$.*

Proof. If $\alpha = 0$, then either there is $i \in \{1, 2, \ldots, \ell\}$ such that $v_i = 0$, hence $f^*(v) = 0$ and consequently $f(y)f^*(v) = 0$, or $v > 0$, hence $y = 0$ and consequently $f(0)f^*(v) = 0$. If $\alpha > 0$, then by setting $v' = \frac{1}{\alpha}\ v$ we have $v'^T y \leq 1$, hence $f(y)f^*(v') \leq 1$ (by (1)), and consequently

$$1 \ \geq \ f(y) f^* \left(\frac{1}{\alpha}\ v \right) \ = \ \frac{1}{\alpha}\ f(y) f^*(v),$$

or equivalently $\alpha \geq f(y) f^*(v)$. ∎

Let Y be a bounded closed convex set with nonempty interior in R_+^ℓ such that Y satisfies the free disposal condition :

$$y \in Y,\ 0 \leq y' \leq y \Rightarrow y' \in Y.$$

Let V be the conjugate of Y :

$$V = \{v \geq 0 :\ v^T y \leq 1\ \forall y \in Y\}.$$

Then, V is a bounded closed convex set with the nonempty interior contained in R_+^ℓ and satisfies the free disposal condition :

$$v \in V,\ 0 \leq v' \leq v \Rightarrow v' \in V.$$

Moreover, Y is also the conjugate of V (cf.[4]) :

$$Y = \{y \geq 0 :\ v^T y \leq 1\ \forall v \in V\}.$$

Now we consider the following the primal problem

$$\max f(y),\ \text{ s.t. } y \in Y.$$

The dual of this problem is

$$\max f^*(v),\ \text{ s.t. } v \in V.$$

For this primal-dual pair we know that $y \in Y$ is optimal to the primal problem and $v \in V$ is optimal to the dual problem if and only if $f(y)f^*(v) = 1$ (cf.[4]). In the next section we extend this duality to a vector-optimization problem that appears in an equilibrium of a co-operative economy.

3. A vector-optimization problem and its dual

In an economy we are dealing with m companies. An activity of the i-th company ($i \in \{1, 2, \ldots, m\}$) is characterized by a nonnegative n_i-dimensional vector x^i : $x^i \in R_+^{n_i}$, where n_i is a positive integer. The production of the i-th comany at the activity vector x^i is $f_i(x^i)$ where f_i is a linear function on $R_+^{n_i}$:

$$f_i(x^i) = c^{i^T} x^i,$$

where $c^i \in R_+^{n_i}$ and $c^i > 0$. Set $n = n_1 + n_2 + \ldots + n_m$. An n-dimensional vector $x = (x^1, x^2, \ldots, x^m)$, where x^i is an activity vector of the i-th company ($i = 1, 2, \ldots, m$), is called a production strategy (or in brief, strategy). A strategy x is called feasible if $x \in X$ and it satisfies the free disposal condition, where X is a bounded closed convex set with the nonempty interior in R_+^n. Let us consider the following vector-optimization problem

$$\begin{aligned} f_i(x^i) \ &\rightarrow \ \max \quad i = 1, 2, \ldots, m, \\ \text{s.t. } x \ &= \ (x^1, x^2, \ldots, x^m) \ \in \ X. \end{aligned} \tag{2}$$

Regarding to the economy we are interested in the co-operative concept of (weak) Pareto equilibrium, or (weak) Pareto efficiency. If we define

$$\overline{f}_i(x) \ = \ f_i(x^i) \quad \forall x = (x^1, x^2, \ldots, x^m) \in R^n$$

for any $i = 1, 2, \ldots, m$, then an alternative formulation of the problem (2) is

$$\overline{f}_i(x) \ \rightarrow \ \max \quad i = 1, 2, \ldots, m, \quad \text{s.t. } x \ \in \ X. \tag{3}$$

For duality we define the following problem

$$\begin{aligned} f_i^*(u^i) \ &\rightarrow \ \max \quad i = 1, 2, \ldots, m, \\ \text{s.t. } u \ &= \ (u^1, u^2, \ldots, u^m) \ \in \ U, \end{aligned} \tag{4}$$

where

$$f_i^*(u^i) \ = \ \min \left\{ \frac{u_j^i}{c_j^i} \ : \ j = 1, 2, \ldots, n_i \right\} \quad i = 1, 2, \ldots, m,$$

and U is the conjugate of X :

$$U \ = \ \{u \in R_+^n : \ u^T x \leq 1 \ \forall x \in X\}.$$

The vector $u^i \in R_+^{n_i}$ stands for the dual activity of the i-th company, and the vector $u = (u^1, u^2, \ldots, u^m) \in R_+^n$ stands for the dual production strategy (in brief, dual strategy). The function f_i^* is a Leontief dual production function of the dual activities of the i-th company. If we define

$$\overline{f}_i^*(u) \ = \ f_i^*(u^i) \quad \forall u = (u^1, u^2, \ldots, u^m) \in R_+^n$$

for any $i = 1, 2, \ldots, m$, then an alternative formulation of the dual problem (4) is

$$\overline{f}_i^*(u) \ \rightarrow \ \max \quad i = 1, 2, \ldots, m, \quad \text{s.t. } u \ \in \ U. \tag{5}$$

If $m = 1$, then the above duality is exactly the duality presented in the previous section.

4. Duality equation and efficiency conditions

First we present an inequality that is about the weak duality between the primal vector-optimization problem (3) and its dual vector-optimization problem (5).

Theorem 4.1. *For any $x \in X$ and $u \in U$ we have*

$$\sum_{i=1}^{m} \overline{f}_i(x) \overline{f}_i^*(u) \ \leq \ 1. \tag{6}$$

Proof. Let $x \in X$ and $u \in U$. Since U is the conjugate of X we have

$$u^T x \ = \ \sum_{i=1}^{m} u^{i^T} x^i \ \leq \ 1.$$

By setting $\alpha_i = u^{i^T} x^i$ for any $i = 1, 2, \ldots, m$, we have $\alpha_i \geq 0$ for any $i = 1, 2, \ldots, m$ and

$$\sum_{i=1}^{m} \alpha_i \ \leq \ 1.$$

Moreover, by Lemma 2.1

$$\begin{aligned}\overline{f}_i(x)\overline{f}_i^*(u) &= f_i(x^i) f_i^*(u^i) \\ &\leq \alpha_i.\end{aligned}$$

Therefore, we obtain (6). ■

On the basis of the weak duality we consider the following equation

$$\sum_{i=1}^{m} \overline{f}_i(x)\overline{f}_i^*(u) \ = \ 1. \tag{7}$$

This equation is called the duality equation that is about the question "is the duality a zero-gap duality?".

Theorem 4.2. *Let $x \in X$ and $u \in U$. If x and u satisfy the duality equation (7), then x is primal weakly Pareto efficient and u is dual weakly Pareto efficient.*

Proof. Let $x \in X$ and $u \in U$ such that (x, u) satisfies the duality equation (7). From (7) it follows that

$$\exists i \ \in \ \{1, 2, \ldots, m\} : \ \overline{f}_i^*(u) \ > \ 0,$$

and together with Theorem 4.1 it follows that

$$\sum_{i=1}^{m} \overline{f}_i^*(u)\overline{f}_i(x) \ = \ \max\left\{\sum_{i=1}^{m} \overline{f}_i^*(u)\overline{f}_i(x') : \ x' \in X\right\},$$

where $(\overline{f}_1^*(u), \overline{f}_2^*(u), \ldots, \overline{f}_m^*(u))$ is a nonzero scalarization weight vector of the problem (3). Consequently, x is primal weakly Pareto efficient. Quite similarly, u is dual weakly Pareto efficient. ■

Theorem 4.3. *Let $x \in X$ and $u \in U$ such that (x, u) satisfies the duality equation (7).*

(i) *If $x^i \neq 0$ for any $i = 1, 2, \ldots, m$, then u is dual Pareto efficient;*

(ii) *If $u > 0$, then x is primal Pareto efficient.*

Proof. First we observe that

$$\begin{aligned}\overline{f}_i(x) &> 0 \Leftrightarrow x^i \neq 0 \quad \forall i = 1, 2, \ldots, m,\\ \overline{f}_i^*(u) &> 0 \Leftrightarrow u^i > 0 \quad \forall i = 1, 2, \ldots, m.\end{aligned}$$

Then, by the quite similar arguments as in the proof of Theorem 4.2 we can derive (i) and (ii). ■

Now we are going to show that the duality is a zero-gap duality. This comment will be clarified in the following theorem.

Theorem 4.4. *If x is primal weakly Pareto efficient, then there is $u \in U$ such that (x, u) satisfies the duality equation (7). Similarly, if u is dual weakly Pareto efficient, then there is $x \in X$ such that (x, u) satisfies the duality equation (7).*

Proof. Suppose that x is primal weakly Pareto efficient. Then, $X - R^n_+$ has no intersection with the following open convex set

$$\{z \in R^n : \ \overline{f}_i(z) > \overline{f}_i(x) \ i = 1, 2, \ldots, m\}.$$

Alternatively, this open convex set can be represented as follows

$$\{z \in R^n : \ {c^i}^T z^i > \overline{f}_i(x) \ i = 1, 2, \ldots, m\}.$$

By the separation theorem, there is $v \in R^n$ and $\alpha \in R$ such that

$$\begin{aligned}& v^T z \leq \alpha \quad \forall z \in X - R^n_+, && (8)\\ & v^T z > \alpha \quad \forall z : \ {c^i}^T z^i > \overline{f}_i(x) \ i = 1, 2, \ldots, m.\end{aligned}$$

From (8) it follows that $v \geq 0$. Moreover, since X has a nonempty interior, 0 belongs to the interior of the closed convex set $X - R^m_+$. This together with (8) implies $\alpha > 0$. Setting $u = \frac{1}{\alpha} \ v$ we have

$$u^T z \leq 1 \quad \forall z \in X, \tag{9}$$

$$u^T z > 1 \quad \forall z : \ {c^i}^T z^i > \overline{f}_i(x) \ i = 1, 2, \ldots, m. \tag{10}$$

From (9) it follows that $u \in U$. By Farkas Lemma, from (10) it follows that there are $\mu_i \geq 0 \ i = 1, 2, \ldots, m$ such that

$$u = (\mu_1 c^1, \mu_2 c^2, \ldots, \mu_m c^m), \tag{11}$$

$$\sum_{i=1}^m \mu_i \overline{f}_i(x) \geq 1. \tag{12}$$

From (11) it follows that

$$\overline{f}_i^*(u) = \overline{f}_i^*(\mu_i c^i) = \mu_i \quad \forall i = 1, 2, \ldots, m.$$

This together with (12) implies

$$\sum_{i=1}^{m} \overline{f}_i(x)\overline{f}_i^*(u) \ \geq \ 1.$$

By Theorem 4.1, a consequence of the above inequality is

$$\sum_{i=1}^{m} \overline{f}_i(x)\overline{f}_i^*(u) \ = \ 1,$$

i.e., the duality equation (7) holds at (x, u).

Now suppose that u is dual weakly Pareto efficient. Then, $U - R^n_+$ has no intersection with the following open convex set

$$\{z : \ \overline{f}_i^*(z) > \overline{f}_i^*(u) \ i = 1, 2, \ldots, m\}.$$

Alternatively, this open convex set can be represented as follows

$$\{z : \ z^i > \overline{f}_i^*(u)c^i \ i = 1, 2, \ldots, m\}.$$

By the separation theorem, there is $y \in R^n$ and $\alpha \in R$ such that

$$\begin{aligned} z^T y \leq \alpha, \quad & \forall z \in U - R^n_+, \\ z^T y > \alpha, \quad & \forall z : \ z^i > \overline{f}_i^*(u)c^i \ i = 1, 2, \ldots, m. \end{aligned} \tag{13}$$

From (13) it follows that $y \geq 0$. Since U has a nonempty interior, 0 belongs to the interior of the closed convex set $U - R^n_+$. This together with (13) implies $\alpha > 0$. Seting $x = \frac{1}{\alpha}\, y$, we have

$$z^T x \leq 1, \quad \forall z \in U, \tag{14}$$
$$z^T x > 1, \quad \forall z : \ z^i > \overline{f}_i^*(u)c^i \ i = 1, 2, \ldots, m. \tag{15}$$

From (14) it follows that $x \in X$. By Farkas Lemma, from (15) it follows that

$$\sum_{i=1}^{m} \overline{f}_i^*(u){c^i}^T x^i \ \geq \ 1,$$

or equivalently,

$$\sum_{i=1}^{m} \overline{f}_i(x)\overline{f}_i^*(u) \ \geq \ 1.$$

By Theorem 4.1, a consequence of the above inequality is

$$\sum_{i=1}^{m} \overline{f}_i(x)\overline{f}_i^*(u) \ = \ 1,$$

i.e., the duality equation (7) holds at (x, u). ■

As an immediate consequence of Theorem 4.2, Theorem 4.3 and Theorem 4.4 we have the following corollary.

Corollary 4.5. *We have*

(i) *A primal feasible strategy x is primal weakly Pareto efficient if and only if there is a dual feasible strategy u such that the duality equation (7) holds at (x, u). Moreover, given a feasible strategy x, if there is a positive dual feasible strategy u such that the duality equation (7) holds at (x, u), then x is primal Pareto efficient.*

(ii) *A dual feasible strategy u is dual weakly Pareto efficient if and only if there is a primal feasible strategy x such that the duality equation (7) holds at (x, u). Moreover, given a dual feasible strategy u, if there is a primal feasible strategy x such that $x^i \neq 0$ for any $i = 1, 2, \ldots, m$ and the duality equation (7) holds at (x, u), then u is dual Pareto efficient.*

5. Conclusions

It is well known that by Lagrange duality the dual problem of a linear vector-optimization problem is a linear multiple-constraint optimization problem (cf.[3, 5]). Since this dual problem is not a vector-optimization problem, the above mentioned duality is not involutory. In this article, we have extended the duality scheme in [4] for a vector-optimization problem. The dual problem by this duality scheme is also a vector-optimization problem. Since the functions f_i $i = 1, 2, \ldots, m$ are the conjugate of the functions f_i^* $i = 1, 2, \ldots, m$, respectively, and X is the conjugate of U, the primal problem is also the dual of the dual problem. Thus, the presented duality is involutory. Moreover, we have obtained the duality equation that characterizes the primal-dual weakly Pareto efficient solutions. By the duality equation there is a correspondence between the primal Pareto efficient set and the dual Pareto efficient set. So, we can study the primal efficiency via dual efficiency, and vice versa.

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