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Quasi-duality Property of Generalized Macaulay-Northcott Modules

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Abstract. Let R be a ring and $(S, \leq)$ a strictly totally ordered monoid which is also artinian and finitely generated. Then we show that M is an artinian quasi-duality left R-module if and only if the module $[M^{S,\leq}]$ consisting of generalized inverse polynomials over M is a quasi-duality left $[[R^{S,\leq}]]$-module.

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1. Introduction and Preliminary

This paper is motivated by [2] in which it was proved that M is an artinian quasi-duality left R-module if and only if the Macaulay-Northcott module $M[x^{-1}]$ over $R[[x]]$ is quasi-duality, and by a series of works about generalized Macaulay-Northcott modules, developed by Zhongkui Liu in [3-6]. We will show that, under some additional conditions, the module $[M^{S,\leq}]$ consisting of generalized inverse polynomials over M is a quasi-duality left $[[R^{S,\leq}]]$-module if and only if M is an artinian quasi-duality left R-module. Our result will give more examples of quasi-duality modules.

All rings considered here are associative with identity. Any concept and notion not defined here can be found in [3-6, 11-13].

Let $(S,\leq)$ be an ordered set. Recall that $(S,\leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S,\leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is due to [11-13].

Let $(S,\leq)$ be a strictly ordered monoid (that is, $(S,\leq)$ is an ordered monoid satisfying the condition that, if $s,s',t\in S$ and $s<s'$, then $s+t<s'+t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f:S\to R$ such that $\text{supp}(f)=\{s\in S\mid f(s)\neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s\in S$ and $f,g\in[[R^{S,\leq}]]$, let $X_s(f,g)=\{(u,v)\in S\times S\mid u+v=s, f(u)\neq 0, g(v)\neq 0\}$. It follows from [11, 4.1] that $X_s(f,g)$ is finite. This fact allows to define the operation of convolution:

$$(fg)(s)=\sum_{(u,v)\in X_s(f,g)} f(u)g(v).$$

Clearly $\text{supp}(fg)\subseteq\text{supp}(f)+\text{supp}(g)$, thus by [11, 3.4] $\text{supp}(fg)$ is artinian and narrow, hence $fg\in[[R^{S,\leq}]]$. With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes a ring, which is called the ring of generalized power series. The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in R and exponents in S.

For example, if $S=\mathbb{N}\cup\{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N}\cup\{0\},\leq}]]\cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]]\cong R[S]$, the monoid ring of S over R. Further examples are given in [11].

Let $(S,\leq)$ be a strictly totally ordered monoid which is also artinian. If M is a left R-module, we let $[M^{S,\leq}]$ be the set of all maps $\phi:S\to M$ such that the set $\text{supp}(\phi)=\{s\in S\mid \phi(s)\neq 0\}$ is finite. Now $[M^{S,\leq}]$ can be turned into a left $[[R^{S,\leq}]]$-module. The addition in $[M^{S,\leq}]$ is componentwise and the scalar multiplication is defined as follows:

$$(f\phi)(s)=\sum_{t\in S} f(t)\phi(s+t),\qquad \text{for every}\ \ s\in S,$$

where $f\in[[R^{S,\leq}]]$ and $\phi\in[M^{S,\leq}]$. It was proved in [3] that $f\phi$ belongs to $[M^{S,\leq}]$ and $[M^{S,\leq}]$ is a left $[[R^{S,\leq}]]$-module, which we call the generalized Macaulay-Northcott module. The elements of $[M^{S,\leq}]$ are called generalized inverse polynomials with coefficients in M and exponents in S. Similarly, if M is a right R-module, then $[M^{S,\leq}]$ is a right $[[R^{S,\leq}]]$-module.

For example, if $S=\mathbb{N}\cup\{0\}$ and $\leq$ is the usual order, then $[M^{\mathbb{N}\cup\{0\},\leq}]\cong M[x^{-1}]$, the usual left $R[[x]]$-module introduced in [7, 8], which is called the Macaulay-Northcott module in [9, 10]. If S is the multiplicative monoid $(\mathbb{N},\cdot)$, endowed with the usual order $\leq$, then $[[R^{(\mathbb{N},\cdot),\leq}]]$ is the ring of arithmetical functions with values in R, endowed with the Dirichlet convolution:

$$(fg)(n) = \sum_{d|n} f(d)g(n/d), \quad \text{for each } n \geq 1.$$

If M is a left R-module, then the left $[[R^{(\mathbb{N},\cdot),\leq}]]$-module $[M^{(\mathbb{N},\cdot),\leq}]$ is the set $\{\sum_{i=1}^{n} m_i x^{-i} \mid m_i \in M, i = 1, 2, \ldots, n, n \in \mathbb{N}\}$ with scalar multiplication as below:

$$\Big(\sum_{i\geq 1} r_i x^i\Big)\Big(\sum_{j\geq 1} m_j x^{-j}\Big) = \sum_{j\geq 1}\Big(\sum_{i\geq 1} r_i m_{i\cdot j}\Big)x^{-j},$$

where $\sum_{i\geq 1} r_i x^i \in [[R^{(\mathbb{N},\cdot),\leq}]]$ and $\sum_{j\geq 1} m_j x^{-j} \in [M^{(\mathbb{N},\cdot),\leq}]$. Note that in particular,

$$(rx^i)(mx^{-j}) = \begin{cases} rmx^{-j/i}, & i \mid j, \\ 0, & i \nmid j. \end{cases}$$

2. Main Results

We shall henceforth assume that $(S, \leq)$ is a strictly totally ordered monoid which is also artinian in this section. Then, by [5], for any $s \in S$, we have $0 \leq s$. This result will be often used throughout the rest of this paper.

Let M be a left R-module. A family $\{m_i, M_i\}_{i\in I}$ (where $m_i \in M$ and $M_i \leq M, i \in I$) is called *solvable* if there exists an $m \in M$ such that $m - m_i \in M_i$ for all $i \in I$; it is called *finitely solvable* if $\{m_i, M_i\}_{i\in F}$ is solvable for any finite subset $F \subseteq I$, and the module M is called *linearly compact* in the case where any finitely solvable family of M is solvable. Let $s_1, \ldots, s_n \in S$, we denote by $\langle s_1, \ldots, s_n\rangle$ the set of all elements $\sum_{i=1}^{n} k_i s_i$ (with k_i integer, $k_i \geq 0$). A monoid S is called *finitely generated* if there exists a finite subset $\{s_1, \ldots, s_n\}$ such that $S = \langle s_1, \ldots, s_n\rangle$.

Lemma 2.1. *Let R be a ring, M a left R-module and S a finitely generated monoid. Then the following conditions are equivalent:*

(1) *M is an artinian left R-module.*

(2) *$[M^{S,\leq}]$ is an artinian left $[[R^{S,\leq}]]$-module.*

(3) *$[M^{S,\leq}]$ is a linearly compact left $[[R^{S,\leq}]]$-module.*

Proof. (1)$\Longrightarrow$(2) follows from [14, Corollary 5].
(2)$\Longrightarrow$(3). Each artinian module is linearly compact.
(3) $\Longrightarrow$ (1). See [6, Proposition 2.5] ∎

Before stating the next result we explain the notions involved.

Let $m \in M$. Define a mapping $\phi_{0,m} \in [M^{S,\leq}]$ via:

$$\phi_{0,m}(0) = m, \quad \phi_{0,m}(s) = 0,\ 0 \neq s \in S.$$

Let T be a ring with identity. For any $s \in S$, $r \in T$, define $d_r^s \in [[T^{S,\leq}]]$ as follows:

$$d_r^s(s) = r, \quad d_r^s(x) = 0, \ \forall s \neq x \in S.$$

Denote $c_r = d_r^0, \ e_s = d_1^s$.

For every $0 \neq \phi \in [M^{S,\leq}]$, we denote by $\sigma(\phi)$ the maximal element in $\mathrm{supp}(\phi)$.

Lemma 2.2. *Let T be a ring with identity and M a right T-module. Then* $\mathrm{Soc}([M^{S,\leq}]_{[[T^{S,\leq}]]}) = \{\phi_{0,m} \mid m \in \mathrm{Soc}(M_T)\}$.

Proof. Let $0 \neq \varphi \in \mathrm{Soc}([M^{S,\leq}]_{[[T^{S,\leq}]]})$ be such that $\varphi[[T^{S,\leq}]]$ is a simple right $[[T^{S,\leq}]]$-module. Assume that $\sigma(\varphi) = s$. If $s > 0$, then

$$(\varphi e_s)(0) = \sum_{x \in S} \varphi(x) e_s(x) = \varphi(s) \neq 0.$$

Thus $0 \neq \varphi e_s \in \varphi[[T^{S,\leq}]]$, and so $\varphi[[T^{S,\leq}]] = \varphi e_s[[T^{S,\leq}]]$. Hence $\varphi = \varphi e_s f$ for some $f \in [[T^{S,\leq}]]$. Then

$$\varphi(s) = (\varphi e_s f)(s) = \sum_{x \in S} \varphi(x+s)(e_s f)(x) = \varphi(s) e_s(0) f(0) = 0,$$

a contradiction. Therefore, $s = 0$, and so $\varphi = \phi_{0,m}$ for some $m \in M$. For any $f \in [[T^{S,\leq}]]$, $\phi_{0,m} f = \phi_{0,mf(0)} = \phi_{0,m} f(0)$, so $\varphi[[T^{S,\leq}]] = \phi_{0,m}[[T^{S,\leq}]] = \phi_{0,m} T$. Thus $\phi_{0,m} T$ is a simple right T-module. Since $\phi_{0,m} T \cong mT$, it follows that mT is a simple right T-module. Hence, $m \in \mathrm{Soc}(M_T)$, and so $\varphi \in \{\phi_{0,m} \mid m \in \mathrm{Soc}(M_T)\}$, which implies that $\mathrm{Soc}([M^{S,\leq}]_{[[T^{S,\leq}]]}) \subseteq \{\phi_{0,m} \mid m \in \mathrm{Soc}(M_T)\}$. The other inclusion is directly verified. ■

Lemma 2.3. *Let T be a ring and $W = \{f \in [[T^{S,\leq}]] \mid f(0) = 0\}$. Then W is an ideal of $[[T^{S,\leq}]]$, and $W \subseteq J([[T^{S,\leq}]])$.*

Proof. Let $f \in W$, $g \in [[T^{S,\leq}]]$. Then

$$(gf)(0) = \sum_{(u,v) \in X_0(g,f)} g(u) f(v) = g(0) f(0) = 0.$$

This means that $gf \in W$. Similarly, $fg \in W$. Now it is easy to see that W is an ideal of $[[T^{S,\leq}]]$.

Let $f \in W$. Then $(c_1 - f)(0) = 1$. Thus by [12, 2.3], $(c_1 - f) \in U([[T^{S,\leq}]])$. Hence $f \in J([[T^{S,\leq}]])$, which implies that $W \subseteq J([[T^{S,\leq}]])$. ■

Let T be a ring with identity and M a right T-module. If S is a finitely generated monoid and T is a right noetherian ring, in [5, Theorem 6], it was proved that $[M^{S,\leq}]$ is an injective right $[[T^{S,\leq}]]$-module if and only if M is an injective right T-module. Using this fact, we can prove

Lemma 2.4. *Let S be a finitely generated monoid and T a right noetherian ring. Then M_T is an injective cogenerator if and only if $[M^{S,\leq}]_{[[T^{S,\leq}]]}$ is an injective cogenerator.*

Proof. $\Longrightarrow$) By [5, Theorem 6], $[M^{S,\leq}]$ is an injective right $[[T^{S,\leq}]]$-module. Let N be a simple right $[[T^{S,\leq}]]$-module, by [1, Proposition 18.15], it suffices to prove that $\mathrm{Hom}_{[[T^{S,\leq}]]}(N, [M^{S,\leq}]) \neq 0$. Since N is a simple right $[[T^{S,\leq}]]$-module, $NJ([[T^{S,\leq}]]) = 0$. For any $f \in [[T^{S,\leq}]]$, if $f(0) = 0$, then $f \in J([[T^{S,\leq}]])$ by Lemma 2.3. Thus $Nf = 0$. Hence $Nf = Nc_{f(0)} \triangleq Nf(0)$. This means that N as a right $[[T^{S,\leq}]]$-module coincides with N as a right T-module. Hence N is a simple right T-module. Since M_T is an injective cogenerator, there exists a nonzero right T-homomorphism $N \longrightarrow M$. Let $f \in [[T^{S,\leq}]]$, define $Mf = Mf(0)$. Then M is a right $[[T^{S,\leq}]]$-module. Thus $N \longrightarrow M$ is also a nonzero right $[[T^{S,\leq}]]$-homomorphism. Since $M \subseteq [M^{S,\leq}](m \longmapsto \phi_{0,m})$, there exists a nonzero right $[[T^{S,\leq}]]$-homomorphism $N \longrightarrow [M^{S,\leq}]$.

$\Longleftarrow$) By [5, Theorem 6], M is an injective right T-module. Suppose that N is a simple right T-module. Let $G = \{\phi_{0,n} \mid n \in N\}$. Then G is a simple right $[[T^{S,\leq}]]$-module. Hence, by [1, Proposition 18.15], there exists a $0 \neq \alpha \in \mathrm{Hom}_{[[T^{S,\leq}]]}(G, [M^{S,\leq}])$. Define $\beta : N \longrightarrow M : n \longmapsto \alpha(\phi_{0,n})(0)$, $\forall\, n \in N$. Then it is easy to see that β is a right T-homomorphism. Since $\alpha \neq 0$, there exists an $n \in N$ such that $\alpha(\phi_{0,n}) \neq 0$. By [4, Lemma 2.1], $\sigma(\alpha(\phi_{0,n})) \leq \sigma(\phi_{0,n}) = 0$. Thus $\sigma(\alpha(\phi_{0,n})) = 0$. Hence $\beta(n) = \alpha(\phi_{0,n})(0) \neq 0$. Thus $\beta \neq 0$. Hence, by [1, Proposition 18.15], M_T is an injective cogenerator. ■

According to [2], a module ${}_RM$ is called *quasi-duality* if ${}_RM$ is quasi-injective, finitely cogenerated, linearly compact and cogenerated all its factor modules. Also, by [2], if ${}_RM$ is a quasi-duality module, then $A = \mathrm{End}({}_RM)$ is right linearly compact and M_A is an injective cogenerator with $\mathrm{Soc}({}_RM) = \mathrm{Soc}(M_A)$ essential in M_A. Moreover, R operates densely in $B(= \mathrm{End}(M_A))$ on M, and ${}_BM$ is also a quasi-duality module. The following result appeared in [2, Theorem 2.1].

Lemma 2.5. *Let A be a right linearly compact ring and M_A an injective cogenerator with $\mathrm{Soc}(M_A)$ essential in M_A. Then*

(1) *${}_BM$ is a quasi-duality module, where $B = \mathrm{End}(M_A)$; and*

(2) *If R operates densely in B on M, then ${}_RM$ is a quasi-duality module.*

Lemma 2.6. ([2, Lemma 4.1]) *Let ${}_RM$ be a quasi-duality module with $A = \mathrm{End}({}_RM)$. If I is an ideal of R with $\mathrm{End}({}_{R/I}r_M(I)) \cong A/r_A(r_M(I))$, then $r_M(I)$ is a quasi-duality module over R/I.*

Let M be a left R-module, $A = \mathrm{End}({}_RM)$ and $B = \mathrm{End}(M_A)$. Then by the construction of $[M^{S,\leq}]$, we obtain a bimodule ${}_{[[B^{S,\leq}]]}[M^{S,\leq}]_{[[A^{S,\leq}]]}$. Moreover, by [3, Lemma 3.1, Lemma 3.3], we may identify $[[A^{S,\leq}]] = \mathrm{End}({}_{[[R^{S,\leq}]]}[M^{S,\leq}])$ and $[[B^{S,\leq}]] = \mathrm{End}([M^{S,\leq}]_{[[A^{S,\leq}]]})$. With these facts, we can prove

Theorem 2.7. *Let S be a finitely generated monoid and M a left R-module. Then the following conditions are equivalent:*

(1) *M is an artinian quasi-duality left R-module.*

(2) *$[M^{S,\leq}]$ is a quasi-duality left $[[R^{S,\leq}]]$-module.*

Proof. (1) $\Longrightarrow$ (2). Since M is an artinian quasi-duality left R-module, A is right noetherian and right linearly compact, so $[[A^{S,\leq}]]$ is right linearly compact by [6, Proposition 2.5] and $[M^{S,\leq}]_{[[A^{S,\leq}]]}$ is an injective cogenerator by Lemma 2.4. Let $0 \neq \varphi \in [M^{S,\leq}]_{[[A^{S,\leq}]]}$. Assume that $\sigma(\varphi) = s$. Since $\mathrm{Soc}(M_A)$ is essential in M_A, there exists an $a \in A$ such that $0 \neq \varphi(s)a \in \mathrm{Soc}(M_A)$. Thus, by Lemma 2.2, $0 \neq \phi_{0,\varphi(s)a} \in \mathrm{Soc}([M^{S,\leq}]_{[[A^{S,\leq}]]})$. It is easy to see that $\phi_{0,\varphi(s)a} = \varphi d_a^s$. Hence $0 \neq \varphi d_a^s \in \mathrm{Soc}([M^{S,\leq}]_{[[A^{S,\leq}]]})$. This means that $\mathrm{Soc}([M^{S,\leq}]_{[[A^{S,\leq}]]})$ is essential in $[M^{S,\leq}]_{[[A^{S,\leq}]]}$. By Lemma 2.5 (1), ${}_{[[B^{S,\leq}]]}[M^{S,\leq}]$ is a quasi-duality module. To show that $[M^{S,\leq}]$ is a quasi-duality left $[[R^{S,\leq}]]$-module, by Lemma 2.5 (2), it suffices to show that $[[R^{S,\leq}]]$ operates densely in $[[B^{S,\leq}]]$ on $[M^{S,\leq}]$. Let $\varphi_1, \varphi_2, \ldots, \varphi_n \in [M^{S,\leq}]$ and $f \in [[B^{S,\leq}]]$. Set $X = \cup_{i=1}^{n}\{\varphi_i(s) \mid s \in \mathrm{supp}(\varphi_i)\}$. Then X is a finite set of M. Assume that $X = \{m_1, \ldots, m_k\}$. For each $s \in \mathrm{supp}(f)$, since R operates densely in B on $[M^{S,\leq}]$, we have $r_s \in R$ such that $r_s m_i = f(s) m_i$ for all i. Define $g : S \longrightarrow R$ as follows:

$$g(x) = \begin{cases} r_x, \; x \in \mathrm{supp}(f), \\ 0, \quad \text{otherwise}. \end{cases}$$

Then $g \in [[R^{S,\leq}]]$. For any $x \in S$ and any $i = 1, 2, \ldots, n$,

$$\begin{aligned}(f\varphi_i)(x) &= \sum_{y\in S} f(y)\varphi_i(x+y) = \sum_{y\in \mathrm{supp}(f)} f(y)\varphi_i(x+y) \\ &= \sum_{y\in \mathrm{supp}(f)} r_y\varphi_i(x+y) \\ &= \sum_{y\in \mathrm{supp}(f)} g(y)\varphi_i(x+y) = \sum_{y\in S} g(y)\varphi_i(x+y) \\ &= (g\varphi_i)(x).\end{aligned}$$

This means that $f\varphi_i = g\varphi_i$ for all i, and which implies that $[[R^{S,\leq}]]$ operates densely in $[[B^{S,\leq}]]$ on $[M^{S,\leq}]$. Now the result follows.

(2) $\Longrightarrow$ (1). Since $[M^{S,\leq}]$ is a quasi-duality left $[[R^{S,\leq}]]$-module, $[M^{S,\leq}]$ is a linearly compact left $[[R^{S,\leq}]]$-module, and so M is an artinian left R-module by Lemma 2.1.

Let

$$I = \{f \in [[R^{S,\leq}]] \mid f(0) = 0\}.$$

Then by Lemma 2.3, I is an ideal of $[[R^{S,\leq}]]$. Define a mapping $\alpha : R \longrightarrow [[R^{S,\leq}]]/I$ via

$$\alpha(a) = c_a + I, \quad \forall a \in R.$$

Then it is easy to see that α is a homomorphism of rings. For any $f \in [[R^{S,\leq}]]$, $f + I = c_{f(0)} + I = \alpha(f(0))$, which implies that α is an epimorphism. Clearly α is a monomorphism. Thus there is an isomorphism of rings $R \cong [[R^{S,\leq}]]/I$.

Let $G = \{\phi_{0,m} \mid m \in M\}$. Then it is easy to see that $G \cong M$ as an R-module. For any $f \in I$, any $m \in M$ and any $s \in S$,

$$(f\phi_{0,m})(s) = \sum_{y \in S} f(y)\phi_{0,m}(s+y) = f(0)\phi_{0,m}(s) = 0,$$

which implies that $G \subseteq r_{[M^{S,\leq}]}(I)$. Conversely, for any $0 \neq \varphi \in r_{[M^{S,\leq}]}(I)$, let $\sigma(\varphi) = s$. Assume that $s > 0$. Then $e_s \in I$ and so

$$0 = (e_s\varphi)(0) = \sum_{y \in S} e_s(y)\varphi(y) = \varphi(s),$$

a contradiction. Thus $s = 0$. Hence $\varphi \in G$. Therefore $G = r_{[M^{S,\leq}]}(I)$.

Note that

$$\begin{aligned} \mathrm{End}({}_{[[R^{S,\leq}]]/I}r_{[M^{S,\leq}]}(I)) &\cong \mathrm{End}({}_RG) \cong \mathrm{End}({}_RM) \\ &= A \cong [[A^{S,\leq}]]/\{f \in [[A^{S,\leq}]] \mid f(0) = 0\} \\ &= [[A^{S,\leq}]]/r_{[[A^{S,\leq}]]}\big(r_{[M^{S,\leq}]}(W)\big). \end{aligned}$$

Hence, by Lemma 2.6, $r_{[M^{S,\leq}]}(I)$ is a quasi-duality left R-module. Thus M is a quasi-duality left R-module. ■

Corollary 2.8. *Let S be a finitely generated torsion-free and cancellative monoid, and $(S, \leq)$ be artinian and narrow. Then M is an artinian quasi-duality left R-module if and only if $[M^{S,\leq}]$ is a quasi-duality left $[[R^{S,\leq}]]$-module.*

Proof. If S is torsion-free and cancellative, then by [11, 3.3], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$, that is, for any $s, t \in S$, $s \leq t$ implies that $s \leq' t$. Since $(S, \leq)$ is artinian and narrow, by [11, 2.5] it follows that $(S, \leq')$ is artinian and narrow. Thus by Theorem 2.7, $[M^{S,\leq'}]$ is a quasi-duality left $[[R^{S,\leq'}]]$-module if and only if M is an artinian quasi-duality left R-module.

On the other hand, since $(S, \leq)$ is narrow, by [11, 4.4], $[[R^{S,\leq}]] = [[R^{S,\leq'}]]$. Clearly $[M^{S,\leq}] = [M^{S,\leq'}]$. Now the result follows. ■

The following corollaries will give more examples of quasi-duality modules.

Any submonoid of the additive monoid $\mathbb{N} \cup \{0\}$ is called a *numerical* monoid. It is well-known that any numerical monoid is finitely generated [11, 1.3]. Thus, we have

Corollary 2.9. *Let S be a numerical monoid and $\leq$ the usual natural order of $\mathbb{N} \cup \{0\}$. Then M is an artinian quasi-duality left R-module if and only if $[M^{S,\leq}]$ is a quasi-duality left $[[R^{S,\leq}]]$-module.*

Corollary 2.10. *Let $(S_1, \leq_1), \ldots, (S_n, \leq_n)$ be strictly totally ordered monoids which are also artinian and finitely generated. Denote by $(lex \leq)$ and $(revlex \leq)$ the lexicographic order and the reverse lexicographic order, respectively, on the monoid $S_1 \times \cdots \times S_n$. Then the following conditions are equivalent:*

(1) *M is an artinian quasi-duality left R-module.*
(2) *$[M^{S_1\times\cdots\times S_n,(lex \leq)}]$ is a quasi-duality left $[[R^{S_1\times\cdots\times S_n,(lex \leq)}]]$-module.*
(3) *$[M^{S_1\times\cdots\times S_n,(revlex \leq)}]$ is a quasi-duality left $[[R^{S_1\times\cdots\times S_n,(revlex \leq)}]]$-module.*

Proof. It is easy to see that $(S_1 \times \cdots \times S_n, (lex \leq))$ is a strictly totally ordered monoid which is also artinian and finitely generated. Thus, by Theorem 2.7, $[M^{S_1\times\cdots\times S_n,(lex \leq)}]$ is a quasi-duality left $[[R^{S_1\times\cdots\times S_n,(lex \leq)}]]$-module if and only if M is an artinian quasi-duality left R-module.

The proof of (1)$\Longleftrightarrow$(3) is similar. ■

Corollary 2.11. *Let $x_1, \ldots, x_n$ be n commuting indeterminates over R and M a left R-module. Then the following conditions are equivalent:*

(1) *M is an artinian quasi-duality left R-module.*
(2) *$M[x_1^{-1}, \ldots, x_n^{-1}]$ is a quasi-duality left $R[[x_1, \ldots, x_n]]$-module.*

Proof. Take $S_1 = \cdots = S_n = \mathbb{N}$ with $\leq_i, i = 1, \ldots, n$, to be the usual order of $\mathbb{N}$ in Corollary 2.10. Then the result follows from [11, Example 3]. ■

Let $p_1, p_2, \ldots, p_n$ be prime numbers. Set

$$N(p_1, p_2, \ldots, p_n) = \{p_1^{m_1} p_2^{m_2} \ldots p_n^{m_n} \mid m_1, m_2, \ldots, m_n \in \mathbb{N} \cup \{0\}\}.$$

Then $N(p_1, p_2, \ldots, p_n)$ is a submonid of $(\mathbb{N}, \cdot)$. Let $\leq$ be the usual natural order. Then by Theorem 2.7, we have

Corollary 2.12. *Let M be a left R-module. Then M is an artinian quasi-duality left R-module if and only if $[M^{N(p_1,p_2,\ldots,p_n),\leq}]$ is a quasi-duality left $[[R^{N(p_1,p_2,\ldots,p_n),\leq}]]$-module.*

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References

1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1992.
2. K. R. Fuller and W. M. Xue, On quasi-duality modules, *Comm. Algebra* **25** (4) (2000), 1919-1937.

3. Z. Liu and H. Cheng, Quasi-duality for the rings of generalized power series, *Comm. Algebra* **28** (3) (2000), 1175-1188.
4. Z. Liu and Y. Fan, Co-Hopfian modules of generalized inverse polynomials, *Acta Math. Sinica* English Series, **17** (3) (2001), 431-436.
5. Z. Liu, Injectivity of modules of generalized inverse polynomials, *Comm. Algebra* **29** (2) (2001), 583-592.
6. Z. Liu, Morita duality for the rings of generalized power series, *Acta Math. Sinica, English Series* **18** (2) (2002), 245-252.
7. A. S. McKerrow, On the injective dimension of modules of power series, *Quart. J. Math. Oxford*, **25** (1) (1974), 359-368.
8. D. G. Northcott, Injective envelopes and inverse polynomials, *London Math. Soc.* **8** (2) (1974), 290-296.
9. S. Park, The Macaulay-Northcott functor, *Arch. Math.* **63** (1994), 225-230.
10. S. Park, Inverse polynomials and injective covers, *Comm. Algebra* **21** (12) (1993), 4599-4613.
11. P. Ribenboim, Noetherian rings of generalized power series, *J. Pure Appl. Algebra* **79** (1992), 293-312.
12. P. Ribenboim, Rings of generalized power series II: units and zero-divisors, *J. Algebra* **168** (1994), 71-89.
13. P. Ribenboim, Semisimple rings and von Neumann regular rings of generalized power series, *J. Algebra* **198** (1997), 327-338.
14. R. Y. Zhao and Z. Liu, Artiness of generalized Macaulay-Northcott modules, *Comm. Algebra* **37** (2) (2009), 525-531.