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A New Method to Estimate the Stable Degrees of Some Previous Stability Theorems Based on the Stability of the Differential Equations for Characteristic Functions

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Abstract. We reconsider two previous stability problems in [2, 3] and give other estimations of the stable degrees of these problems with the following metric:

$$\lambda(F_X; F_Y) = \min_{T>0} \max \left\{ \sup_{|t|\leq T} |\varphi_X(t) - \varphi_Y(t)|; \frac{1}{T} \right\}$$

for two distribution functions $F_X; F_Y$ of two random variables X, Y and their corresponding characteristic functions $\varphi_X(t)$ and $\varphi_Y(t)$, eventhough when the condition in Theorem 2.1 in [1] is not satisfied.

1. Introduction

In 1968, Hoang Huu Nhu considered the stability of the characterization of normal distribution function, when he changed the condition of independence of two statistics by the $\varepsilon-$ independence. In more detail, we have the stability theorem as follows.

Theorem 1.1 (See [2]). *Suppose that* $(X_1, X_2, ..., X_n)$ *is a simple sample from distribution function* $F(x)$ *so that* $E|X_j|^{2(1+\delta)} < +\infty (0 < \delta \le 1)$. *If the statistics* $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ *and* $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ *are* $\varepsilon-$ *independent then*

$$\sup_{x\in R} |F(x) - \phi(x)| \le c_1 \frac{1}{\sqrt{\ln \frac{1}{\varepsilon}}}, \tag{1}$$

where $\phi(x)$ *is a normal distribution function and* c_1 *is a constant independent of* ε.

In the proof of this theorem, if we call $f(t)$ to be the characteristic function with the corresponding distribution function $F(x)$ then from the condition $\overline{X}$ and S^2 being independent we shall have the differential equation

$$f''(t)f^{n-1}(t) - f^{n-2}(t)[f'(t)]^2 + f^n(t) = 0, \tag{2}$$

where $f(0) = 1; f'(0) = 0$.

Changing the independence condition of $\overline{X}$ and S^2 by the $\varepsilon-$ independence of $\overline{X}$ and S^2(see [2]), we have the following differential equation

$$f''(t)f^{n-1}(t) - f^{n-2}(t)[f'(t)]^2 + f^n(t) = R_n(t), \tag{3}$$

where $R_n(0) = 0; \overline{R_n(-t)} = R_n(t); |R_n(t)| \le \varepsilon (\forall t)$.

In 1970,(see [3]), Kagan considered the stability of the characterization theorem based on the condition of admissibility of $\overline{X}$ as an estimator of a shift parameter where $F(x)$ satisfies the condition

$$\int x dF(x) = 0;\ 0 < \int x^2 dF(x) = \sigma^2 < +\infty. \tag{4}$$

From his result, we also know that if $\overline{X}$ is an $\varepsilon-$ admissibility estimator then the characteristic $f(t)$ of the corresponding distribution function $F(x)$ which is symmetric $(f(-t) = f(t))$ and satisfies the differential equation

$$f''(t)f(t) - [f'(t)]^2 + \sigma^2 [f(t)]^2 = R_n(t), \tag{5}$$

where $R_n(0) = 0; \overline{R(-t)} = R(t); |R_n(t)| \le c_2\sigma^2\varepsilon$ (c_2 is a constant independent of ε) and we have

Theorem 1.2 (See [3]). *Let* $(X_1, X_2, \cdots, X_n)$ *be a random sample of size* $n \ge 3$ *from the population* $F(x)$ *where* $F(x - \theta)$ *is symmetric and has zero mean and finite variance* σ^2. *If* $\overline{X}$ *is a* $(\frac{\sigma^2\varepsilon^2}{n})-$ *admissibility estimator of* θ, *then we have*

$$\sup_{x\in\mathbb{R}^1} |F(x) - \phi(x)| \le c_3(\log\frac{1}{\varepsilon})^{-\frac{1}{2}}, \tag{6}$$

where $\phi(x)$ is a normal distribution function and c_3 is a constant independent of ε.

Using some recent results we may get two new stability theorems with shorter proofs. The similar situation could be done, in reconsidering the previous stability problems.

2. Stability theorems

Theorem 2.1 (With the hypothesis of Theorem 1.1). *If $\overline{X}$ and S^2 are $\varepsilon-$independent, then*

$$\lambda(F;\phi) \le c_4 \max\{\varepsilon^{1-\delta}; \frac{1}{\delta\ln\frac{1}{\varepsilon}}\}, 0 < \delta < 1, \tag{7}$$

where $\phi(x)$ is a normal distribution function and c_4 is a constant independent of ε, δ is a real number such that $0 < \delta < 1$.

Theorem 2.2. *With the hypothesis of Theorem 1.2 (for all $t \in \mathbb{R}^1$) we have*

$$\lambda(F;\phi) \le c_5 \max\{\varepsilon^{1-\delta}; \frac{1}{\delta\ln\frac{1}{\varepsilon}}\}, \quad 0 < \delta < 1, \tag{8}$$

where $\phi(x)$ is a normal distribution function and c_5 is a constant independent of ε, δ is a real number such that $0 < \delta < 1$.

To prove the theorems, we can see the following problems of differential equation.

Let us consider the following two differential equations:

$$A[f(t), f'(t), ..., f^{(n)}(t)] = 0 \tag{9}$$

and

$$A[f_\varepsilon(t), f'_\varepsilon(t), ..., f_\varepsilon^{(n)}(t)] = R(t) \tag{10}$$

where $R_n(0) = 0; \overline{R_n(-t)} = R_n(t); |R_n(t)| \le \varepsilon$ for some small enough number ε.

We know that, if the function $A(.)$ (in (9) and (10)) satisfies the condition $\dfrac{\partial A}{\partial f^{(n)}} \neq 0$ then we can represent $f^{(n)}(t)$ in the form

$$f^{(n)}(t) = g[f(t), f'(t), ..., f^{(n-1)}(t)] \tag{11}$$

and if $f_\varepsilon(t)$ is a solution of the equation (10) then we can represent $f_\varepsilon^{(n)}(t)$ in the form

$$f_\varepsilon^{(n)}(t) = g[f_\varepsilon(t), f'_\varepsilon(t), ..., f_\varepsilon^{(n-1)}(t)] + a(t), \tag{12}$$

where $a(t)$ depends on $R(t)$ and $|a(t)| \leq \varepsilon$ for all $t \in \mathbb{R}^n$.

Lemma 2.3. *If the function $g(x_1, x_2, ..., x_n)$ is continuously differentiable and satisfies the Lipschitz's condition, that means there exists a positive constant N such that*

$$|g(x_1, x_2, ..., x_n) - g(y_1, y_2, ..., y_n)| \leq \sum_{k=0}^{n} N|x_i - y_i|, \tag{13}$$

then, for every small enough positive number ε, there exists a positive number $T = T(\varepsilon)(T(\varepsilon) \to +\infty$ as $\varepsilon \to 0)$ and a positive number $\delta(0 < \delta < 1)$ such that

$$|f(t) - f_\varepsilon(t)| < c.\varepsilon^{1-\delta}, \text{ for all } t, |t| \leq T(\varepsilon),$$

where c is a constant independent of ε.

Proof. The proof is similar to that of Theorem 2.1 in [1], but we do not need the boundedness of $f(t)$ and $f^{(k)}(t)$ for all $k = 1, 2, ..., n$ and for all $t \in \mathbb{R}^1$. We notice that in the proof of Theorem 2.1 in [1], actually, we only need the Lipschitz condition of the function $g(x_1, x_2, ..., x_n)$.

At first, we consider $t > 0$ (the case $t \leq 0$ is carried out similarly). Putting $x_1 = f(t), x_2 = f'(t), ..., x_n = f^{(n-1)}(t)$ then the differential equation (11) can be written in the form

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ ... \\ \dfrac{dx_n}{dt} = g(x_1, x_2, ..., x_n). \end{cases} \tag{14}$$

Let us denote $X = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} -n & 1 & 1 \dots & 1 \\ 1 & -n & 1 \dots & 1 \\ \vdots & \vdots & \vdots \ \vdots & \vdots \\ 1 & 1 & 1 \dots & -n \end{pmatrix},$$

$$G(X) = \begin{pmatrix} nx_1 - x_3 - \cdots - x_n \\ -x_1 + nx_2 - \cdots - x_n \\ \vdots \\ -x_1 - x_2 - \cdots + nx_{n-1} \\ -x_1 - x_2 - \cdots - x_{n-1} + nx_n + g(x_1, ..., x_n) \end{pmatrix}.$$

Then the differential equation (14) reduces to the equation

$$\frac{dX}{dt} = AX + G(X). \tag{15}$$

In a similar way, the differential equation (12) can be rewritten as follows:

$$\frac{dY}{dt} = AY + G(Y) + a(t), \tag{16}$$

where $Y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n$, $y_1 = f_\varepsilon(t)$, $y_2 = f'_\varepsilon(t)$, ..., $y_n = f_\varepsilon^{(n-1)}(t)$ and $a(t)$ depends on $R(t)$, $|a(t)| \leq \varepsilon$ for all $t \in \mathbb{R}^1$.

Since $g(x_1, x_2, ..., x_n)$ is continuous in variables and satisfies the Lipschitz's condition, there exists a positive constant k such that

$$||G(X) - G(Y)|| \leq k||X - Y||, \text{ for all } X, Y \in \mathbb{R}^n \tag{17}$$

where $||.||$ denotes the norm in $\mathbb{R}^n$.
On the other hand, we have

$$\det(A - \lambda E) = (\lambda + 1)(\lambda + n + 1)^{n-1}.$$

So, the eigenvalues of matrix A are

$$\lambda_1 = -1; \lambda_2 = -(n+1) = \lambda_3 = \cdots = \lambda_n.$$

We see that the eigenvalues of the matrix A have negative real parts. According to a result of [4],

$$||e^{At}|| \leq \beta e^{-\alpha t}, \text{ where } e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}\cdot \tag{18}$$

From (15) and (16) we get

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-s)}G[X(s)]ds,$$
$$Y(t) = e^{At}Y(0) + \int_0^t e^{A(t-s)}G[Y(s)]ds + \int_0^t e^{A(t-s)}a(s)ds\cdot$$

Since $X(0) = Y(0)$,

$$||X(t) - Y(t)|| \leq \int_0^t ||e^{A(t-s)}||.||G[X(s)] - G[Y(s)]||ds + \int_0^t ||e^{A(t-s)}||.||a(s)||ds.$$

Using the estimations (17) and (18) we have

$$||X(t)-Y(t)|| \leq \beta e^{-\alpha t}\int_0^t ke^{\alpha s}||X(s)-Y(s)||ds + \beta e^{-\alpha t}\varepsilon\int_0^t e^{\alpha s}ds.$$

Hence

$$||X(t)-Y(t)||e^{\alpha t} \leq \beta\varepsilon\int_0^t e^{\alpha s}ds + \int_0^t \beta k||X(s)-Y(s)||e^{\alpha s}ds.$$

If we put $u(t) = ||X(t)-Y(t)||e^{\alpha t}, f(t) = \beta\varepsilon\int_0^t e^{\alpha s}ds, k(t,s) = k\beta$ and $t_0 = 0$ then according to a result of [4] we also know that if the following inequality holds:

$$u(t) \leq f(t) + \int_{t_0}^t k(t,s)u(s)ds.$$

then $u(t) \leq h(t)$ for all $t, t_0 \leq t \leq t_0+T$, where $h(t)$ is a solution of the equation

$$h(t) = g(t) + \int_{t_0}^t k(t,s)h(s)ds.$$

That means $u(t) \leq h(t)$, where $f(t)$ is a solution of the equation

$$f(t) = g(t) + \int_0^t \beta kf(s)ds.$$

Therefore, we have

$$\begin{aligned}
f(t) &= e^{\int_0^t \beta k ds}[g(0) + \int_0^t g'(s)e^{\int_0^s \beta k ds}ds] \\
&= e^{\beta kt}\int_0^t \beta\varepsilon e^{\alpha s-\beta ks}ds \\
&= \frac{\beta\varepsilon}{\alpha-\beta t}(e^{\alpha t}-e^{\beta kt}).
\end{aligned}$$

So we have

$$\begin{aligned}
||X(t)-Y(t)|| &\leq \frac{\beta}{|\alpha-\beta k|}(1-e^{\beta kt-\alpha t})\varepsilon \\
&\leq \frac{\beta}{|\alpha-\beta k|}e^{|\beta k-\alpha|t}\varepsilon.
\end{aligned} \tag{19}$$

Now if we choose $T(\varepsilon) = \dfrac{1}{|\alpha-\beta k|}\ln(\dfrac{1}{\varepsilon})^\delta$ (where $0 < \delta < 1$) then $T(\varepsilon) \to \infty$ when $\varepsilon \to 0$. For all $t, 0 < t \leq T(\varepsilon)$ we shall get

$$||X(t)-Y(t)|| \leq \frac{\beta}{|\alpha-\beta k|}\varepsilon^{1-\delta} = c.\varepsilon^{1-\delta}.$$

where c is a constant independent of ε. This completes the proof of Lemma 2.3. ■

Proof of Theorem 2.1. We notice that the function $A(.)$ in (9) now has the form

$$A[f(t), f'(t), f''(t)] = \frac{f''(t)}{f(t)} - \frac{[f'(t)]^2}{f^2(t)} = 1,$$

where $f(t)$ is the characteristic function of the normal law, $f(t) \neq 0$ for all $t \in \mathbb{R}^1$. So, the condition $\dfrac{\partial A}{\partial f''} \neq 0$ is satisfied and we have

$$f''(t) = \frac{[f'(t)]^2}{f(t)} + f(t). \tag{20}$$

That means the function $g(x_1, x_2)$ has the form

$$g(x_1, x_2) = \frac{x_2^2}{x_1} + x_1, (x_1 \neq 0). \tag{21}$$

For every complex numbers x_1, x_2, y_1, y_2 we have the estimation

$$\begin{aligned}
|g(x_1, x_2) - g(y_1, y_2)| &= |(\frac{x_2^2}{x_1} + x_1) - (\frac{y_2^2}{y_1} + y_1)| \\
&\leq |\frac{x_2^2}{x_1} - \frac{y_2^2}{y_1}| + |x_1 - y_1| \\
&\leq \frac{|y_1 x_2^2 - x_1 y_2^2|}{|x_1|.|y_1|} + |x_1 - y_1| \\
&\leq \frac{|y_1 x_2^2 - y_1 y_2^2 + y_1 y_2^2 - x_1 y_2^2|}{|x_1|.|y_1|} + |x_1 - y_1| \\
&\leq \frac{|y_1||x_2^2 - y_2^2| + |y_2^2||x_1 - y_1|}{|x_1|.|y_1|} + |x_1 - y_1| \\
&\leq \frac{1}{|x_1|}|x_2 - y_2|.|x_2 + y_2| + \frac{|y_2^2|}{|x_1||y_1|}|x_1 - y_1| + |x_1 - y_1|.
\end{aligned} \tag{22}$$

But we notice that x_1, y_1 are characteristic functions and for every normal characteristic function $f(t)$, there always exists a constant q such that

$$q = \inf_{t \in \mathbb{R}^1} |f(t)| > 0.$$

That means there exist two constants q_1, q_2 such that

$$|x_1| \geq q_1 \ ; \ |y_1| \geq q_2.$$

Therefore, from (22) we can get

$$|g(x_1, x_2) - g(y_1, y_2)| \leq N_1|x_1 - y_1| + N_2|x_2 - y_2|,$$

hence the Lipschitz's condition (13) is satisfied. By using Lemma 2.1, we shall get the estimation $|f(t) - f_\varepsilon(t)| < c.\varepsilon^{1-\delta}$ for all $t, |t| \leq T(\varepsilon) = k.\ln(\frac{1}{\varepsilon})^\delta$ (where $T(\varepsilon) \to \infty$ as $\varepsilon \to 0$), where $f(t)$ is a solution of the differential equation (2) and $f_\varepsilon(t)$ is a solution of the differential equation (3). That means

$$\lambda(F, \phi) = \min_{T>0} \max \left\{ \sup_{|t| \leq T(\varepsilon)} |f_\varepsilon(t) - f(t)|; \frac{1}{T(\varepsilon)} \right\} \leq c_4 \max \{ \varepsilon^{1-\delta}; \frac{1}{\delta \ln \frac{1}{\varepsilon}} \}.$$

■

Proof of Theorem 2.2. By arguments similar to those of the proof of Theorem 2.1, and from (5) the function $g(x_1, x_2)$ has the form

$$g(x_1, x_2) = \frac{x_2^2}{x_1} - \sigma^2 x_1.$$

Therefore, for every complex numbers x_1, x_2, y_1, y_2, we have

$$|g(x_1, x_2) - g(y_1, y_2)| \leq \sigma^2 |x_1 - y_1| + \left| \frac{x_2^2}{x_1} - \frac{y_2^2}{y_1} \right|.$$

So, with the same arguments in the proof of Theorem 2.1 we shall get the conclusion (8). ■

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