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Some Results on Random Equations*

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Abstract. In this paper, some sufficient conditions on the existence of solutions of some random operator equations are obtained. Some applications to the random integral equations are also presented.

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1. Introduction and Preliminaries

The theory of random equations including random fixed point theorems is an important topic of the stochastic analysis and has been investigated by various authors (see e.g [9], [10], [13], [16]). The purpose of the paper is to prove some theorems on the existence of solutions of some random equations and to present some applications of these results to random integral equations. We begin by explaining why only continuous random mappings are concerned. Section 3 deals with the random equation $T(\omega, x) = \eta(\omega)$. Theorem 3.1 is an extension of Propo-

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sition 3.1 in [13] and Theorem 3.4 gives a sufficient condition for this random equation to have a unique solution. Section 4 treats the random equation with perturbations of the form $T(\omega, x) + k(\omega)x = \eta(\omega)$ and looks at some examples.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X, Y be separable metric spaces. A mapping $\xi : \Omega \to X$ is called a X-valued random variable if ξ is $(\mathcal{F}, \mathcal{B})$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra of X. The set of all (equivalent classes) X-valued random variables is denoted by $L_0^X(\Omega)$.

We recall now the concept of random mappings and the concept of random equations which can be found in e.g. [5], [10].

Definition 1.1. A mapping $T : \Omega \times X \to Y$ is said to be a random mapping (or random operator) from X into Y if for each fixed $x \in X$, the mapping $\omega \mapsto T(\omega, x)$ is measurable. A random mapping from X into X is called a random mapping on X and random mapping from X into the real line $\mathbb{R}$ is called a random function on X.

From a viewpoint of the probability, a random mapping from X into Y is a rule that assigns to each element $x \in X$ a Y-valued random variable. In other words, a random mapping from X into Y is nothing but a mapping from X into $L_0^Y(\Omega)$.

Definition 1.2. A random mapping T is said to be continuous if for each $\omega \in \Omega$ the mapping $x \mapsto T(\omega, x)$ is continuous.

Definition 1.3. Let T be a random mapping from X into Y and $\eta(\omega)$ is a Y-valued random variable.

1. A random variable $\xi(\omega) \in L_0^X(\Omega)$ is said to be a solution of the random equation $T(\omega, x) = \eta(\omega)$ if $T(\omega, \xi(\omega)) = \eta(\omega)$ a.s.
2. A random variable $\xi(\omega) \in L_0^X(\Omega)$ is said to be a random fixed point of T if $T(\omega, \xi(\omega)) = \xi(\omega)$ a.s.

In the sequel we need the following random contraction mapping theorem.

Theorem 1.4 (see [5], Th.7). *Let T be a random contraction mapping on a separable Banach space X in the sense that there exists a nonnegative real-valued random variable such that $k(\omega) < 1$ a.s. and $\|T(\omega, x_1) - T(\omega, x_2)\| \le k(\omega)\|x_1 - x_2\|$ a.s. for every pair of elements $x_1, x_2 \in X$.*

Then T has a unique random fixed point. Moreover, for each $x_0(\omega) \in L_0^X(\Omega)$ the sequence of random variables $\{x_n(\omega)\}$ defined by $x_{n+1}(\omega) = T(\omega, x_n(\omega))$ $(n = 0, 1, ...)$ converges a.s. to this unique random fixed point.

2. Why only Continuous Random Mappings are Considered?

Definition 2.1. Let T_1 and T_2 be two random mappings from X into Y. The random mapping T_2 is said to be a modification of T_1 if for all $x \in X$ we have $T_1(\omega, x) = T_2(\omega, x)$ a.s. Note that the exceptional set can depend on x.

From a probability viewpoint, a random mapping is not distinguishable from its modification because they define the same mapping from X into $L_0^Y(\Omega)$. Hence, if T_2 is a modification of T_1 then it is expected that the set of solutions of the random equation $T_1(\omega, x) = \eta(\omega)$ and the set of solutions of the random equation $T_2(\omega, x) = \eta(\omega)$ should be the same. However, the following simple example shows that this is not true.

Example 2.2. Let $(\Omega, \mathcal{A}, \mathbb{P}) = ([0;1], \mathcal{B}, \mu)$ and $X = Y = \mathbb{R}$. Let T_1, T_2 be two random mappings on $\mathbb{R}$ given by

$$T_1(\omega, x) = \begin{cases} x.\omega & \text{if } x \neq \omega \\ 1 & \text{if } x = \omega \end{cases}, \quad T_2(\omega, x) = x.\omega \ \ \forall \omega \in \Omega \ \forall x \in X.$$

It is clear that T_1 is a modification of T_2. Consider the random variable $\xi(\omega)$ given by $\xi(\omega) = \omega$. It is easy to check that it is a solution of the equation $T_1(\omega, x) = 1$ but it is not a solution of the equation $T_2(\omega, x) = 1$. Moreover, it is a solution of the equation $T_2(\omega, x) = \omega^2$ but the equation $T_1(\omega, x) = \omega^2$ has no solution.

The following proposition shows that for continuous random mappings the above-mentioned desire becomes reality.

Proposition 2.3. *Let X, Y be two separable metric spaces, T_1, T_2 be two continuous random mappings from X into Y and T_1 be a modification of T_2. If $\xi(\omega)$ is a solution of the random equation $T_1(\omega, x) = \eta(\omega)$ then $\xi(\omega)$ is also a solution of the random equation $T_2(\omega, x) = \eta(\omega)$.*

Proof. Because of separability of X, there exists a sequence $\{x_n\}$ dense in X. For each x_n, there exists a set Ω_n of probability one such that $T_1(\omega, x_n) = T_2(\omega, x_n)$ $\forall \omega \in \Omega_n$. Let $\Omega_1' = \cap_{n=1}^{\infty} \Omega_n$. Clearly, Ω_1' has probability one and we have

$$T_1(\omega, x_n) = T_2(\omega, x_n) \ \forall \omega \in \Omega_1' \ \forall n. \tag{1}$$

Let Ω_2' be the set of ω such that $T_1(\omega, \xi(\omega)) = \eta(\omega)$. Then the set $\Omega_0 = \Omega_1' \cap \Omega_2'$ has probability one.

For each fixed $\omega \in \Omega_0$, we have $T_1(\omega, \xi(\omega)) = \eta(\omega)$. By the density of $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ converging to $\xi(\omega)$. By the continuity of T_1 and T_2, we have for each $i = 1, 2$

$$\lim_{k \to \infty} T_i(\omega, x_{n_k}) = T_i(\omega, \xi(\omega)). \tag{2}$$

By (1) and (2) we conclude that $T_1(\omega, \xi(\omega)) = T_2(\omega, \xi(\omega))$ $\forall \omega \in \Omega_0$. Thus, $T_2(\omega, \xi(\omega)) = \eta(\omega)$ $\forall \omega \in \Omega_0$ and we are done. ∎

3. Random Equations on Hilbert Spaces

Theorem 3.1. *Let $r = r(\omega, t)$ be a function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$ such that $r(\omega, .)$ is continuous and $\lim_{t\to+\infty} r(\omega, t) = +\infty$ a.s. Assume that X is a finite dimensional Hilbert space and T is a continuous random mapping on X such that for each $x \in X$*

$$\langle T(\omega, x), x\rangle \geq r(\omega, ||x||)||x|| \text{ a.s.} \tag{3}$$

(Note that the exceptional set can depend on x). Then, for any $\eta \in L_0^X(\Omega)$, the random equation $T(\omega, x) = \eta(\omega)$ has a solution.

Remark 3.2. The above theorem is an extension of Proposition 3.1 in [13], where it is assumed that η is a bounded random variable and $r = r(t)$ is a function not depending on ω.

Proof. By the separability of X, there exist a countable subspace X_0 dense in X and a set Ω_0 which has probability one such that (3) is satisfied for all x in X_0 and all ω in Ω_0. By the density of X_0 in X, the continuity of $T(\omega, .)$ and $r(\omega, .)$, inequality (3) holds for all x in X and ω in Ω_0. Hence, without lost of generality, we can assume that the inequality (3) holds for all $\omega \in \Omega$ and for all x in X.

At first, we show that the theorem holds for the case $\eta(\omega) = 0\ \forall\omega \in \Omega$. For each fixed $\omega \in \Omega$, by (3) and Lemma 4 in [7], the mapping $x \mapsto T(\omega, x)$ is onto. Hence, there exists $x(\omega)$ such that $T(\omega, x(\omega)) = 0$. Let $F : \Omega \to 2^X$ be a mapping defined by $F(\omega) = \{x|T(\omega, x) = 0\}$. By the above claim and the continuity of the mapping $x \mapsto T(\omega, x)$, F is a multifunction which has nonempty closed values. Now, we prove the measurability of F. Let $B = \{0\}$ then B is a closed subset of X and $F(\omega) = \{x|T(\omega, x) \in B\}$. By Theorem 6.4 in [12], F is measurable hence F has measurable graph by Theorem 3.5 in [12]. Finally, by Theorem 5.7 in [12], there exists a $X-$valued random variable ξ such that $\xi(\omega) \in F(\omega)$ a.s., i.e. $T(\omega, \xi(\omega)) = 0$ a.s.

Now for an arbitrary random variable $\eta(\omega)$, we define a random mapping S by $S(\omega, x) = T(\omega, x) - \eta(\omega)$. Then S has the same properties as T. Indeed,

$$\begin{aligned}\langle S(\omega, x), x\rangle &= \langle T(\omega, x), x\rangle - \langle \eta(\omega), x\rangle \\ &\geq \big[r(\omega, ||x||) - ||\eta(\omega)||\big].||x|| = s(\omega, ||x||).||x||,\end{aligned}$$

where $\lim_{t\to+\infty} s(\omega, t) = \lim_{t\to+\infty} r(\omega, t) - ||\eta(\omega)|| = +\infty$ a.s. As shown before, there exists a $X-$valued random variable ξ such that $S(\omega, \xi(\omega)) = 0$ a.s., i.e. $T(\omega, \xi(\omega)) = \eta(\omega)$. ■

Taking $r(\omega, t) = k(\omega)t^{p-1}$ where $p > 1$ and $k(\omega)$ is a positive random variable we obtain

Corollary 3.3. *Let X be a finite dimensional Hilbert space and T be a continuous random mapping on X such that $\langle T(\omega, x), x\rangle \geq k(\omega)||x||^p$ a.s. for all $x \in X$, where $k(\omega)$ is a positive random variable and $p > 1$. Then, for any $\eta \in L_0^X(\Omega)$, the random equation $T(\omega, x) = \eta(\omega)$ has a solution.*

In the case of an infinite dimensional Hilbert space we have the following

Theorem 3.4. *Let X be a separable Hilbert space and T be a continuous random mapping on X such that it is strongly monotone, i.e. there exists a positive real-valued random variable $m(\omega)$ such that, for all x_1, x_2 in X,*

$$\langle T(\omega, x_1) - T(\omega, x_2), x_1 - x_2 \rangle \geq m(\omega)||x_1 - x_2||^2 \text{ a.s.} \tag{4}$$

(Note that the exceptional set can depend on x_1, x_2). Then, for any $\eta \in L_0^X(\Omega)$, the random equation $T(\omega, x) = \eta(\omega)$ has a unique solution.

Besides, if T is a lipschitzian random mapping, i.e. there exists a positive real-valued random variable $M(\omega)$ such that, for all $x_1, x_2 \in X$,

$$||T(\omega, x_1) - T(\omega, x_2)|| \leqslant M(\omega)||x_1 - x_2|| \text{ a.s.},$$

then the sequence of random variables $\{x_n\}$ defined by

$$x_{n+1} = x_n - \alpha[T(\omega, x_n) - \eta]$$

converges a.s. to this unique solution, where the initial approximation $x_0(\omega)$ is an arbitrary $X-$valued random variable and α is a real-valued random variable such that $0 < \alpha(\omega) < 2m(\omega)/M^2(\omega)$ a.s.

Proof. By a similar argument as in the proof of Theorem 3.1, we can assume that inequality (4) holds for all x in X and all ω in Ω.

For each fixed $\omega \in \Omega$, by inequality (4), the mapping $x \mapsto T(\omega, x)$ is a strongly monotone mapping on X. Therefore, by Theorem 1 in [6], it has continuous inverse, denoted by $x \mapsto T^{-1}(\omega, x)$. Let $\xi(\omega) = T^{-1}(\omega, \eta(\omega))$ then $T(\omega, \xi(\omega)) = \eta(\omega)$ for each $\omega \in \Omega$. It remains to prove the measurability of $\xi(\omega)$. By the continuity of the mapping $x \mapsto T^{-1}(\omega, x)$, for each $\omega \in \Omega$, from the Corollary 1.6 and Theorem 1.9 in [16] (see also Theorem 2.15 in [4]) the mapping $(\omega, x) \mapsto T^{-1}(\omega, x)$ is a continuous random mapping denoted by T^{-1}. By Theorem 2.14 in [4] (see also Theorem 3.3 in [17]), $\xi(\omega) = T^{-1}(\omega, \eta(\omega))$ is a X-valued random variable for any $\eta \in L_0^X(\Omega)$. The uniqueness follows clearly from the existence of T^{-1}.

Next suppose that T enjoys additionally the lipschitzian property. Define a random mapping T_α by $T_\alpha(\omega, x) = x - \alpha[T(\omega, x) - \eta(\omega)]$. It follows easily that $T_\alpha(\omega, x_1) - T_\alpha(\omega, x_2) = x_1 - x_2 - \alpha[T(\omega, x_1) - T(\omega, x_2)]$ and

$$\begin{aligned} ||T_\alpha(\omega, x_1) - T_\alpha(\omega, x_2)||^2 &= ||x_1 - x_2||^2 - 2\alpha\langle T(\omega, x_1) - T(\omega, x_2), x_1 - x_2\rangle \\ &\quad + \alpha^2||T(\omega, x_1) - T(\omega, x_2)||^2 \\ &\leqslant [1 - 2\alpha.m(\omega) + \alpha^2.M^2(\omega)].||x_1 - x_2||^2 \\ &= k^2(\omega)||x_1 - x_2||^2, \end{aligned}$$

where $k^2(\omega) = 1 - 2\alpha.m(\omega) + \alpha^2.M^2(\omega)$. Since $0 < \alpha(\omega) < 2m(\omega)/M^2(\omega)$, we have $k(\omega) < 1$. Therefore, T_α is a random contraction mapping. By Theorem 1.4, T_α has a unique random fixed point, denoted by ξ, and the sequence of random variables $\{x_n(\omega)\}$ defined by $x_{n+1}(\omega) = T_\alpha(\omega, x_n(\omega)) = x_n(\omega) - \alpha(\omega).[T(\omega, x_n(\omega)) - \eta(\omega)]$ converges to ξ, where the initial approximation $x_0(\omega)$ is an arbitrary $X-$valued random variable. From the equality $T_\alpha(\omega, \xi(\omega)) = \xi(\omega)$ we obtain $T(\omega, \xi(\omega)) = \eta(\omega)$ showing that $\xi(\omega)$ is a solution of the random equation $T(\omega, x) = \eta(\omega)$. ■

4. Random Equations with Perturbations

In this section, we consider random equations with perturbations of the form $T(\omega, x) + k(\omega)x = \eta(\omega)$, where $k(\omega)$ is a positive real-valued random variable.

Recall that a mapping f on a Hilbert space X is said to be compact if f is continuous and it maps each bounded subset of X into a strongly precompact subset of X. A random mapping T is said to be compact if for each $\omega \in \Omega$ the mapping $x \mapsto T(\omega, x)$ is compact.

Theorem 4.1. *Let X be a separable Hilbert space, T be a compact random mapping on X, $k(\omega)$ be a positive real-valued random variable and $r = r(\omega, t)$ be a function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$ such that $r(\omega, .)$ is continuous and $\lim_{t \to +\infty} r(\omega, t) = +\infty$ a.s. Assume that for each $x \in X$*

$$\big\langle T(\omega, x), x \big\rangle \geq r(\omega, ||x||).||x|| - k(\omega).||x||^2 \ a.s. \tag{5}$$

Then, for any $\eta \in L_0^X(\Omega)$, the random equation $T(\omega, x) + k(\omega)x = \eta(\omega)$ has a solution.

Proof. By a similar argument as in the proof of Theorem 3.1, we can suppose that inequality (5) holds for all $x \in X$ and for all $\omega \in \Omega$.

At first, we prove for the case $\eta = 0$. Let G be a random mapping on X defined by $G(\omega, x) = T(\omega, x) + k(\omega)x$. We have to show that the random equation $G(\omega, x) = 0$ has a solution. Let S be a random mapping from $X \times X$ into X defined by $S(\omega, x, y) = k(\omega)x$ for all $x, y \in X$ and for all $\omega \in \Omega$. Clearly, we have $G(\omega, x) = T(\omega, x) + S(\omega, x, x)$ for all $x \in X$ and for all $\omega \in \Omega$. Fix $\omega \in \Omega$. We shall show that the mapping $x \mapsto G(\omega, x)$ is a surjection on X by checking that all assumptions stated in Theorem 2 in [8] are satisfied. Indeed, by inequality (5), we have $\big\langle G(\omega, x), x \big\rangle / ||x|| \geq r(\omega, ||x||) \to +\infty, ||x|| \to +\infty$. Since $k(\omega) > 0$, for each fixed $y \in X$, $S(\omega, x, y) = k(\omega)x$ is continuous and monotone with respect to x. For each fixed $x \in X$, $S(\omega, x, y) = k(\omega)x$ is a constant mapping so it is completely continuous with respect to y. By the compactness of T, $x \mapsto T(\omega, x)$ is a compact mapping. Finally, if $\big\langle S(\omega, x_n, x_n) - S(\omega, x, x_n), x_n - x \big\rangle = k(\omega)||x_n - x||^2 \to 0$ then x_n converges strongly to x which implies that $T(\omega, x_n)$ converges to $T(\omega, x)$. Hence, by Theorem 2 in [8], the mapping $x \mapsto G(\omega, x)$ is a surjection on X. Thus, there exists an element of X, denoted by $x(\omega)$, such that $G(\omega, x(\omega)) = 0$. By the same arguments as in the proof of Theorem 3.1, it can be shown that there exists a $X-$valued random variable ξ such that $G(\omega, \xi(\omega)) = 0$ a.s., i.e. the random equation $G(\omega, x) = 0$ has a solution as desired.

The general case is reduced to the special case $\eta = 0$ as follows. Put $T'(\omega, x) = T(\omega, x) - \eta(\omega)$. Then T' has the same properties as T. Indeed, for all x in X

$$\begin{aligned}\big\langle T'(\omega, x), x\big\rangle &= \big\langle T(\omega, x), x\big\rangle - \big\langle \eta(\omega), x\big\rangle \\ &\geq \big[r(\omega, ||x||) - ||\eta(\omega)||\big].||x|| - k(\omega).||x||^2 \\ &= r'(\omega, ||x||).||x|| - k(\omega).||x||^2 \text{ a.s.,}\end{aligned}$$

where and $r'(\omega, t) = r(\omega, t) - ||\eta(\omega)||$. Consequently, there exists a $X-$valued random variable ξ such that $T'(\omega, \xi(\omega)) + k(\omega)\xi(\omega) = 0$ a.s., i.e. $T(\omega, \xi(\omega)) + k(\omega)\xi(\omega) = \eta(\omega)$. ■

Corollary 4.2. *Let X be a separable Hilbert space, $L(\omega)$ be a positive real-valued random variable and T be a compact random mapping on X such that for each $x \in X$*

$$||T(\omega, x)|| \leqslant L(\omega)||x|| \ a.s.$$

If $L(\omega) < k(\omega)$ a.s. then the random equation $T(\omega, x) + k(\omega)x = \eta(\omega)$ has a solution for any $\eta \in L_0^X(\Omega)$.

Proof. Put $\epsilon(\omega) = k(\omega) - L(\omega) > 0$ a.s. We have for each x in X

$$\big\langle T(\omega, x), x\big\rangle \geq -||T(\omega, x)||.||x|| \geq \epsilon(\omega).||x||^2 - k(\omega)||x||^2 \text{ a.s.}$$

Let $r(\omega, t) = \epsilon(\omega)t$ then (5) is satisfied. Thus, the proof is complete. ■

Theorem 4.3. *Let X be a separable Banach space and T be a random mapping on X such that for each $\omega \in \Omega$ the mapping $x \mapsto T(\omega, x)$ is a linear continuous operator. Denote by $\|T\|(\omega)$ the norm of the linear continuous operator $x \mapsto T(\omega, x)$. If $\|T\|(\omega) < k(\omega)$ a.s. then, for any $\eta \in L_0^X(\Omega)$, the random equation $T(\omega, x) + k(\omega)x = \eta(\omega)$ has a unique solution.*

Proof. Fix $\omega \in \Omega$. Since $\|T\|(\omega) < k(\omega)$, the linear continuous operator $x \mapsto A(\omega, x) = T(\omega, x) + k(\omega)x$ has continuous inverse, denoted by $x \mapsto A^{-1}(\omega, x)$. Let $\xi(\omega) = A^{-1}(\omega, \eta(\omega))$ then we have $T(\omega, \xi(\omega)) + k(\omega)\xi(\omega) = \eta(\omega)$ for each fixed $\omega \in \Omega$. The measurability of $\xi(\omega)$ can be shown by using the same arguments as in the proof of Theorem 3.4. ■

Theorem 4.4. *Let T be a continuous random mapping on a separable Hilbert space X and satisfy the monotone property, i.e. for all $x_1, x_2 \in X$,*

$$\big\langle T(\omega, x_1) - T(\omega, x_2), x_1 - x_2\big\rangle \geq 0 \ a.s.$$

Then, for any $\eta \in L_0^X(\Omega)$, the random equation $T(\omega, x) + k(\omega)x = \eta(\omega)$ has a unique solution.

Proof. The random equation under consideration is of the form $S(\omega, x) = \eta(\omega)$, where S is the random mapping given by $S(\omega, x) = T(\omega, x) + k(\omega)x$. Then S is continuous and for all $x_1, x_2 \in X$, we have

$$\begin{aligned}\big\langle S(\omega, x_1) - S(\omega, x_2), x_1 - x_2\big\rangle \\ &= \big\langle T(\omega, x_1) - T(\omega, x_2), x_1 - x_2\big\rangle + k(\omega)||x_1 - x_2||^2 \\ &\geq k(\omega)||x_1 - x_2||^2 \text{ a.s.}\end{aligned}$$

By Theorem 3.4, for any $\eta \in L_0^X(\Omega)$, the random equation $S(\omega, x) = \eta(\omega)$ has a unique solution. Thus, the theorem is proved. ■

Theorem 4.5. *Let T be a continuous random mapping on a separable Hilbert space X and satisfy the lipschitzian property, i.e. there exists a nonnegative real-valued random variable $L(\omega)$ such that for all $x_1, x_2 \in X$*

$$||T(\omega, x_1) - T(\omega, x_2)|| \leqslant L(\omega)||x_1 - x_2|| \ a.s.$$

If $L(\omega) < k(\omega)$ a.s. then, for any $\eta \in L_0^X(\Omega)$, the random equation $T(\omega, x) + k(\omega)x = \eta(\omega)$ has a unique solution.

Proof. The random equation under consideration is of the form $S(\omega, x) = \eta(\omega)$, where S is the random mapping given by $S(\omega, x) = T(\omega, x) + k(\omega)x$. By the lipschitzian property of T, we have for all $x_1, x_2 \in X$

$$\begin{aligned}\big\langle S(\omega, x_1) - S(\omega, x_2), x_1 - x_2\big\rangle \\ &= \big\langle T(\omega, x_1) - T(\omega, x_2), x_1 - x_2\big\rangle + k(\omega).||x_1 - x_2||^2 \\ &\geq k(\omega).||x_1 - x_2||^2 - ||T(\omega, x_1) - T(\omega, x_2)||.||x_1 - x_2|| \\ &\geq [k(\omega) - L(\omega)].||x_1 - x_2||^2 = m(\omega).||x_1 - x_2||^2 \text{ a.s.,}\end{aligned}$$

where $m(\omega) = k(\omega) - L(\omega) > 0$. By Theorem 3.4, for any $\eta \in L_0^X(\Omega)$, the random equation $S(\omega, x) = \eta(\omega)$ has a unique solution. ■

Finally, we will present some applications to random integral equations.

Example 4.6. Consider the random integral equation

$$K(\omega, t)\int_0^t K(\omega, s)x(s)ds + h(\omega, t)x(t) = \eta(\omega, t) \tag{6}$$

where $K(\omega, t), h(\omega, t)$ are continuous random functions and $\eta(\omega, t)$ is a random function such that $\int_0^1 \eta^2(\omega, t)dt < +\infty$ a.s.

Put $L(\omega) = \min_{t\in[0;1]} h(\omega, t); U(\omega) = \max_{t\in[0;1]} K(\omega, t)$. We claim that if $L(\omega) > U^2(\omega)/\sqrt{2}$ a.s. then the integral equation (6) has a unique solution $x(\omega, t)$ such that $\int\limits_0^1 x^2(\omega, t)dt < +\infty$ a.s.

Indeed, we define a random mapping T on $X = L_2[0;1]$ as follows: For $x = x(t) \in X$, $T(\omega,x)(t) = K(\omega,t)\int\limits_0^t K(\omega,s)x(s)ds + h(\omega,t)x(t)$. It is easy to check that T is a continuous random operator on X and $T(\omega,x_1) - T(\omega,x_2) = T(\omega,x_1 - x_2)$. We have

$$\big\langle T(\omega,x),x\big\rangle = \int_0^1 K(\omega,t)x(t)\big(\int_0^t K(\omega,s)x(s)ds\big)dt + \int_0^1 h(\omega,t)x^2(t)dt$$

and

$$\begin{aligned}
&\Big(\int_0^1 K(\omega,t)x(t)\big(\int_0^t K(\omega,s)x(s)ds\big)dt\Big)^2\\
&\leqslant \int_0^1 K^2(\omega,t)x^2(t)dt. \int_0^1 \big(\int_0^t K^2(\omega,s)x^2(s)ds. \int_0^t ds\big)dt\\
&\leqslant \int_0^1 K^2(\omega,t)x^2(t)dt. \int_0^1 K^2(\omega,s)x^2(s)ds. \int_0^1 tdt\\
&= \frac{1}{2}\Big(\int_0^1 K^2(\omega,t)x^2(t)dt\Big)^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\big\langle T(\omega,x),x\big\rangle &\geq \int_0^1 h(\omega,t)x^2(t)dt - \int_0^1 1/\sqrt{2}.K^2(\omega,t)x^2(t)dt\\
&= \int_0^1 \big(h(\omega,t) - 1/\sqrt{2}.K^2(\omega,t)\big)x^2(t)dt \geq m(\omega)\|x\|^2 \text{ a.s.}
\end{aligned}$$

where $m(\omega) = L(\omega) - U^2(\omega)/\sqrt{2}$ is an a.s. positive real-valued random variable. Consequently,

$$\big\langle T(\omega,x_1) - T(\omega,x_2), x_1 - x_2\big\rangle = \big\langle T(\omega,x_1 - x_2), x_1 - x_2\big\rangle \geq m(\omega)||x_1 - x_2||^2 \text{ a.s.}$$

The random integral equation (6) can be written in the form $T(\omega,x) = \eta(\omega)$. Consequently, the claim follows from Theorem 3.4.

Example 4.7. Consider the random integral equation

$$\int_0^1 K(\omega,t,s)x(s)ds + h(\omega)x(t) = \eta(\omega,t), \tag{7}$$

where $K(\omega,t,s)$ and $\eta(\omega,t)$ are continuous random functions defined on $[0;1]\times[0;1]$ and $[0;1]$, respectively; $h(\omega)$ is a positive real-valued random variable.

Let $M(\omega) = \max\limits_{[0;1]\times[0;1]} |K(\omega,t,s)|$. We claim that if $M(\omega) < h(\omega)$ a.s. then the integral equation (7) has a unique solution $x(\omega,t)$ which is a continuous random function on $[0;1]$.

Indeed, we define a random mapping T on $X = C[0;1]$ as follows: For $x = x(t) \in X$, $T(\omega, x)(t) = \int\limits_0^1 K(\omega, t, s)x(s)ds$. It is easy to verify that for each ω, $x \mapsto T(\omega, x)$ is a continuous linear operator on X. We have

$$||T(\omega, x)(t)|| = \max_{[0;1]} |\int_0^1 K(\omega, t, s)x(s)ds| \leqslant \int_0^1 M(\omega). \max_{[0;1]} |x(s)|ds = M(\omega).||x||$$

for all $x \in C[0;1]$ which implies that $\|T\|(\omega) \leqslant M(\omega) < h(\omega)$ a.s. The random integral equation (7) can be written in the form $T(\omega, x) + h(\omega)x = \eta(\omega)$. Consequently, the claim follows from Theorem 4.3.

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