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Hankel Operators with Vector Valued Symbols on the Hardy Space

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Abstract. In this paper we have shown that the sequence $\left\{J^m_{qI_{n\times n}} T_\Phi T^m_{(zq)I_{n\times n}}\right\}$ converges strongly to the Hankel operator $S_\Phi \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ for $\Phi \in L^\infty_{M_n}(\mathbb{T})$ and for all inner functions $q \in H^\infty(\mathbb{T})$. Here T_Φ is the Toeplitz operator on the Hardy space $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}), I_{n\times n}$ is the identity matrix of order $n, J^m_{qI_{n\times n}} = \text{diag}[J^m_q, J^m_q, ...J^m_q]$ where for $i \geq 0, m \geq 0, J^{m^\star}_q z^i = q^m z^{m-i}, 0 \leq i \leq m$ and 0, otherwise.

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1. Introduction

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $L^2(\mathbb{T})$ be the space of square integrable, measurable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure on $\mathbb{T}$. The sequence $\{e_n(z)\}^\infty_{n=-\infty} = \{z^n\}^\infty_{n=-\infty}$ forms an orthonormal basis for $L^2(\mathbb{T})$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of f are $C_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta, n \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. The Hardy space $H^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ consisting of functions f with $C_n(f) = 0$ for all negative integers n. Since $C_n = C_n(\cdot)$ is a bounded linear functional on $L^2(\mathbb{T})$ for any fixed n and $H^2(\mathbb{T}) = \bigcap_{n<0} \ker C_n$, it follows that $H^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$ and therefore a Hilbert space. Let $\widetilde{P}$ denote the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Let $L^\infty(\mathbb{T})$ be the space of all essentially bounded measurable functions on $\mathbb{T}$. For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_φ from $H^2(\mathbb{T})$ into itself is defined by $T_\varphi f = \widetilde{P}(\varphi f)$ and the Hankel operator S_φ from $H^2(\mathbb{T})$ into itself is defined by $S_\varphi f = \widetilde{P}(\widetilde{J}(\varphi f))$. Here $\widetilde{J}$ is the mapping from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$ defined by $\widetilde{J}f(z) = f(\overline{z})$. Let $\mathcal{L}(H)$ denote the algebra of bounded, linear operators from a Hilbert space H into itself and $\mathcal{LC}(H)$ be the set of all compact linear operators from H into itself. Let $\mathcal{LF}(H)$ be the set of all finite rank linear operators from H into itself. Let $H^\infty(\mathbb{T}) = \{f \in L^\infty(\mathbb{T}) : C_n(f) = 0 \quad \text{for } n < 0\}$. A function $q \in H^\infty(\mathbb{T})$ is said to be an inner function if $|q(e^{it})| = 1$ almost everywhere on $\mathbb{T}$. Let $I_{n\times n}$ be the identity matrix of order n.

Let $L^2_{\mathbb{C}^n}(\mathbb{T})$ denote the Hilbert space of $\mathbb{C}^n$-valued, norm-square integrable, measurable functions on $\mathbb{T}$ and $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ the corresponding Hardy space of functions in $L^2_{\mathbb{C}^n}(\mathbb{T})$ with vanishing negative Fourier coefficients. We note that $L^2_{\mathbb{C}^n}(\mathbb{T}) = L^2(\mathbb{T}) \otimes \mathbb{C}^n$ and $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = H^2(\mathbb{T}) \otimes \mathbb{C}^n$ where the Hilbert space tensor product is used. If Φ is a bounded measurable M_n-valued function (where M_n denotes the algebra of matrices of order n with complex entries) in $L^\infty_{M_n}(\mathbb{T}) = L^\infty(\mathbb{T}) \otimes M_n$, then T_Φ denotes the Toeplitz operator defined on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ by $T_\Phi f = P(\Phi f)$ for $f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ where P is the orthogonal projection of $L^2_{\mathbb{C}^n}(\mathbb{T})$ onto $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. The Hankel operator S_Φ is a mapping from $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ into itself defined by $S_\Phi f = P(J(\Phi f))$ where $J : L^2_{\mathbb{C}^n}(\mathbb{T}) \to L^2_{\mathbb{C}^n}(\mathbb{T})$ is defined by $Jf(z) = f(\overline{z})$. In this paper we have shown that if $\Phi \in L^\infty_{M_n}(\mathbb{T})$ then for all inner functions $q \in H^\infty(\mathbb{T})$, the sequence $\{J^m_{qI_{n\times n}} T_\Phi T^m_{(zq)I_{n\times n}}\}$ converges strongly to the Hankel operator $S_\Phi \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$. This is an extension of a result given in [2] and [3]. It relates Toeplitz and Hankel operators in some asymptotic sense.

2. Hankel Operators and Inner Functions

In this section we obtain a characterization of Hankel operators on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ in terms of inner functions in $H^\infty(\mathbb{T})$. In fact we show that if S is a bounded linear operator on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ then S is a Hankel operator if and only if $ST_{qI_{n\times n}} = T^*_{q^+ I_{n\times n}} S$ for all inner functions $q \in H^\infty(\mathbb{T})$ where $q^+(z) = \overline{q(\overline{z})}$. We first prove a result for Hankel operators on $H^2(\mathbb{T})$.

Lemma 2.1. *Let $S \in \mathcal{L}(H^2(\mathbb{T}))$. Then $T^*_{q^+} S = ST_q$ for all inner functions $q \in H^\infty(\mathbb{T})$ if and only if there exists $\varphi \in L^\infty(\mathbb{T})$ such that $S = S_\varphi$, a Hankel operator. Here $q^+(z) = \overline{q(\overline{z})}$.*

Proof. Let $\mathcal{A} = \{\overline{\eta} h : \eta \text{ is inner and } h \in H^2(\mathbb{T})\}$. Then $\mathcal{A}$ is a dense linear [4] subspace of $L^2(\mathbb{T})$. Define $\widetilde{J} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ as $\widetilde{J}f(z) = f(\overline{z})$ for all $z \in \mathbb{T}$ and $\widetilde{S}f = Sf$ if $f \in H^2(\mathbb{T})$ and $\widetilde{S}$ is bounded linear on $L^2(\mathbb{T})$. Thus $\widetilde{S}|_{H^2(\mathbb{T})} = S$. Define a map $\Omega : \mathcal{A} \to \mathbb{C}$ as $\Omega(\overline{\eta} h) = \langle \widetilde{S}(\overline{\eta} h), 1\rangle$. Then Ω is well-defined and linear. In fact, if $\overline{\eta_1} h_1 = \overline{\eta_2} h_2$, then we have $\Omega(\overline{\eta}_1 h_1) = \Omega(\overline{\eta}_2 h_2)$. Further

$|\Omega(\overline{\eta}h)| \leq \|\widetilde{S}\|\|\overline{\eta}h\|$. So Ω is a bounded, linear functional on $\mathcal{A}$. Since $\mathcal{A}$ is dense in $L^2(\mathbb{T})$, there exists a unique $\varphi \in L^2(\mathbb{T})$ such that $\Omega(\overline{\eta}h) = \langle \overline{\eta}h, \overline{\varphi}\rangle$ and $|\langle \overline{\eta}h, \overline{\varphi}\rangle| \leq \|\widetilde{S}\|\|\overline{\eta}h\|$. Thus $\varphi \in L^\infty(\mathbb{T})$. Define $\Gamma_\varphi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ as $\Gamma_\varphi f = \widetilde{J}(\varphi f)$. Notice that Γ_φ is bounded on $L^2(\mathbb{T})$ and $\|\Gamma_\varphi\| \leq \|\varphi\|_\infty$. Further, $\langle \widetilde{S}(\overline{\eta}h), 1\rangle = \Omega(\overline{\eta}h) = \langle \overline{\eta}h, \overline{\varphi}\rangle = \langle \overline{\eta^+}h^+, \widetilde{J}\varphi\rangle = \langle \eta^+\widetilde{J}h, \varphi^+\rangle = \langle (\widetilde{J}\varphi)\eta^+\widetilde{J}h, 1\rangle = \langle \widetilde{J}(\varphi\overline{\eta}h), 1\rangle = \langle \Gamma_\varphi(\overline{\eta}h), 1\rangle$. Since $\mathcal{A}$ is dense in $L^2(\mathbb{T})$, we have $\langle \widetilde{S}f, 1\rangle = \langle \Gamma_\varphi f, 1\rangle$ for all $f \in L^2(\mathbb{T})$. In particular, for $g \in H^2(\mathbb{T}), \langle Sg, 1\rangle = \langle \widetilde{S}g, 1\rangle = \langle \Gamma_\varphi g, 1\rangle = \langle \widetilde{P}\Gamma_\varphi g, 1\rangle = \langle S_\varphi g, 1\rangle$. Now since $T^*_{q^+}S = ST_q$ for all inner functions $q \in H^\infty(\mathbb{T})$, we obtain in particular, $T^*_{z^+}S = ST_z$. That is, $T^*_{z}S = T_{\overline{z}}S = ST_z$. Thus for polynomials p, u in z,we have $\langle S(pu^+), 1\rangle = \langle ST_{pu^+}1, 1\rangle = \langle T^*_{(pu^+)^+}S1, 1\rangle = \langle T_{\widetilde{J}(pu^+)}S1, 1\rangle = \langle S1, T_{(pu^+)^+}1\rangle = \langle S1, (p^+u)1\rangle = \langle S1, T_{p^+}u\rangle = \langle T^*_{p^+}S1, u\rangle = \langle ST_p1, u\rangle = \langle Sp, u\rangle$. Thus $\langle Sp, u\rangle = \langle S(pu^+), 1\rangle = \langle S_\varphi(pu^+), 1\rangle = \langle S_\varphi p, u\rangle$. Since polynomials are dense in $H^2(\mathbb{T})$, we have for $f, g \in H^2(\mathbb{T}), \langle Sf, g\rangle = \langle S_\varphi f, g\rangle$. Hence $S = S_\varphi$.

Conversely, if S_φ is a Hankel operator in $\mathcal{L}(H^2(\mathbb{T}))$, then for any inner function $\eta \in H^\infty(\mathbb{T})$ and for all $n, m \geq 0, \langle T^*_{\eta^+}S_\varphi z^n,\ z^m\rangle = \langle S_\varphi z^n,\ T_{\eta^+}z^m\rangle = \langle S_\varphi z^n,\ \eta^+z^m\rangle = \langle \widetilde{P}\widetilde{J}(\varphi z^n),\ \eta^+z^m\rangle = \langle \widetilde{J}(\varphi z^n),\ \eta^+z^m\rangle = \langle \varphi z^n,\ \overline{\eta}\ \overline{z}^m\rangle = \langle \varphi\eta z^n,\ \overline{z}^m\rangle = \langle \widetilde{J}(\varphi\eta z^n),\ z^m\rangle = \langle S_\varphi(\eta z^n),\ z^m\rangle = \langle S_\varphi T_\eta z^n,\ z^m\rangle$. Thus $T^*_{\eta^+}S_\varphi = S_\varphi T_\eta$ for all inner functions $\eta \in H^\infty(\mathbb{T})$. ∎

Theorem 2.2. *Let $S \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$. The operator S is a Hankel operator on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ if and only if $ST_{qI_{n\times n}} = T^*_{q^+I_{n\times n}}S$ for all inner functions $q \in H^\infty(\mathbb{T})$.*

Proof. Let $S \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$. Since $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = H^2(\mathbb{T}) \oplus H^2(\mathbb{T}) \oplus ... \oplus H^2(\mathbb{T})$, hence

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$

where $S_{ij} \in \mathcal{L}(H^2(\mathbb{T})), 1 \leq i, j \leq n$. Thus $T^*_{q^+I_{n\times n}}S = ST_{qI_{n\times n}}$ for all inner functions $q \in H^\infty(\mathbb{T})$ if and only if $T^*_{q^+}S_{ij} = S_{ij}T_q$ for all inner functions $q \in H^\infty(\mathbb{T}), 1 \leq i, j \leq n$. But from Lemma 2.1, it follows that $S_{ij} = S_{\varphi_{ij}} \in \mathcal{L}(H^2(\mathbb{T}))$, a Hankel operator on $H^2(\mathbb{T})$ with symbol $\varphi_{ij} \in L^\infty(\mathbb{T}), 1 \leq i, j \leq n$. Hence

$$S = \begin{bmatrix} S_{\varphi_{11}} & S_{\varphi_{12}} & \cdots & S_{\varphi_{1n}} \\ S_{\varphi_{21}} & S_{\varphi_{22}} & \cdots & S_{\varphi_{2n}} \\ \vdots & \vdots & & \vdots \\ S_{\varphi_{n1}} & S_{\varphi_{n2}} & \cdots & S_{\varphi_{nn}} \end{bmatrix} = S_\Phi \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})),$$

a Hankel operator with symbol $\Phi \in L^\infty_{M_n}(\mathbb{T})$ and

$$\Phi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{bmatrix}.$$

Now suppose $\Phi \in L^{\infty}_{M_n}(\mathbb{T})$ and S_Φ is a Hankel operator on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Let

$$\Phi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{bmatrix}.$$

Then since $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = H^2(\mathbb{T}) \oplus .... \oplus H^2(\mathbb{T})$, hence

$$S_\Phi = \begin{bmatrix} S_{\varphi_{11}} & S_{\varphi_{12}} & \cdots & S_{\varphi_{1n}} \\ S_{\varphi_{21}} & S_{\varphi_{22}} & \cdots & S_{\varphi_{2n}} \\ \vdots & \vdots & & \vdots \\ S_{\varphi_{n1}} & S_{\varphi_{n2}} & \cdots & S_{\varphi_{nn}} \end{bmatrix}$$

where $S_{\varphi_{ij}} \in \mathcal{L}(H^2(\mathbb{T}))$ is a Hankel operator on $H^2(\mathbb{T})$ with symbol $\varphi_{ij} \in L^\infty(\mathbb{T})$. From Lemma 2.1, it follows that $T^*_{q^+} S_{\varphi_{ij}} = S_{\varphi_{ij}} T_q, 1 \le i, j \le n$. Hence

$$T^*_{q^+ I_{n\times n}} S_\Phi = S_\Phi T_{qI_{n\times n}}. \qquad \blacksquare$$

3. The Hankel Sequence Associated with Toeplitz Operators

In this section we construct a sequence $\{J^m_{qI_{n\times n}} T_\Phi T^m_{(zq)I_{n\times n}}\}$ associated with the Toeplitz operator T_Φ on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ that converges strongly to the Hankel operator S_Φ on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ for all inner functions $q \in H^\infty(\mathbb{T})$. This sequence is referred to as the Hankel sequence associated with T_Φ. We further prove that if $\Phi \in L^\infty_{M_n}(\mathbb{T})$, then $\{J^m_{qI_{n\times n}} S_\Phi T^m_{(zq)I_{n\times n}}\}$ converges strongly to 0 for all inner functions $q \in H^\infty(\mathbb{T})$.

Theorem 3.1. *If $\varphi \in L^\infty(\mathbb{T})$, then for all inner functions $q \in H^\infty(\mathbb{T})$, the sequence $\{J^m_q T_\varphi T^m_{(zq)}\}$ converges strongly to $S_\varphi \in \mathcal{L}(H^2(\mathbb{T}))$, the Hankel operator with symbol φ and where for $i \ge 0, m \ge 0$,*

$$J_q^{m^*} z^i = \begin{cases} q^m z^{m-i}, & 0 \le i \le m \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Notice that for $i, j \ge 0, i, j \in \mathbb{Z}, \langle S_\varphi z^j, z^i\rangle = \langle \widetilde{P}\widetilde{J}(\varphi z^j),\, z^i\rangle = \langle \widetilde{J}(\varphi z^j), z^i\rangle = \langle \varphi z^j,\, z^{-i}\rangle = \langle \varphi,\, z^{-(i+j)}\rangle = \hat{\varphi}(-(i+j))$, where $\hat{\varphi}(n)$ is the n-th Fourier coefficient of φ. Further, for $i, j \ge 0, i, j \in \mathbb{Z}$,

$$\langle J^m_q T_\varphi T^m_{(zq)} z^j, z^i\rangle = \langle T_\varphi(z^m q^m z^j), J_q^{m^*} z^i\rangle$$

$$=\begin{cases} \langle \widetilde{P}(\varphi z^m q^m z^j), q^m z^{m-i}\rangle, & 0 \le i \le m \\ 0, & \text{otherwise} \end{cases}$$

$$=\begin{cases} \langle \varphi z^j, z^{-i}\rangle, & 0 \le i \le m \\ 0, & \text{otherwise} \end{cases}$$

$$=\langle P_m S_\varphi z^j, z^i\rangle$$

where P_m is the orthogonal projection from $H^2(\mathbb{T})$ onto span $\{1, z, ..., z^m\}$. Thus $J_q^m T_\varphi T_{(zq)}^m = P_m S_\varphi$. Since $P_m \longrightarrow I$ strongly, hence the sequence $\{J_q^m T_\varphi T_{(zq)}^m\}$ converges strongly to the Hankel operator S_φ. ■

Theorem 3.2. *Let $\Phi \in L^\infty_{M_n}(\mathbb{T})$. Then for all inner functions $q \in H^\infty(\mathbb{T})$, the sequence $\{J^m_{qI_{n\times n}} T_\Phi T^m_{(zq)I_{n\times n}}\}$ converges strongly to the Hankel operator $S_\Phi \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ where $J^m_{qI_{n\times n}} = diag[J_q^m, J_q^m, ..., J_q^m]$ and $I_{n\times n}$ is the identity matrix of order n.*

Proof. Let

$$\Phi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & \vdots & & \vdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{bmatrix}.$$

Then

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & T_{\varphi_{12}} & \cdots & T_{\varphi_{1n}} \\ T_{\varphi_{21}} & T_{\varphi_{22}} & \cdots & T_{\varphi_{2n}} \\ \vdots & \vdots & & \vdots \\ T_{\varphi_{n1}} & T_{\varphi_{n2}} & \cdots & T_{\varphi_{nn}} \end{bmatrix}.$$

Now for $f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}), f = (f_1, f_2, ..., f_n)$,

$$\left\| J^m_{qI_{n\times n}} T_\Phi T^m_{(zq)I_{n\times n}} f - S_\Phi f \right\|^2 \le \sum_{i=1}^n \sum_{j=1}^n \left\| J_q^m T_{\varphi_{ij}} T^m_{(zq)} f_j - S_{\varphi_{ij}} f_j \right\|^2.$$

By Theorem 3.1, for all inner functions $q \in H^\infty(\mathbb{T})$, $\left\| J_q^m T_{\varphi_{ij}} T^m_{(zq)} f_j - S_{\varphi_{ij}} f_j \right\| \longrightarrow 0$ for $1 \le i, j \le n$ as $m \longrightarrow \infty$. That is, $J_q^m T_{\varphi_{ij}} T^m_{(zq)} \to S_{\varphi_{ij}}$ strongly for $1 \le i, j \le n$ as $m \longrightarrow \infty$. Thus $\left\| J^m_{qI_{n\times n}} T_\Phi T^m_{(zq)I_{n\times n}} f - S_\Phi f \right\| \longrightarrow 0$ as $m \to \infty$ and for all $f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. ■

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $H^2(\mathbb{D})$ be the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2_{H^2(\mathbb{D})} = \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

If $f \in H^2(\mathbb{D})$, then Fatou's theorem implies that the limit

$$\widetilde{f}(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$$

exists for almost every θ and it is well known [5] that $\widetilde{f} \in H^2(\mathbb{T})$. If $g \in H^2(\mathbb{T})$ then it is also true [5] that

$$\|g\|^2_{H^2(\mathbb{T})} = \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |\hat{g}(re^{i\theta})|^2 d\theta$$

where $\hat{g}$ is the harmonic extension of g to $\mathbb{D}$. Since there is an isometrical isomorphism between $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$, we shall not distinguish between $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$.

Theorem 3.3. *Let $K \in \mathcal{LC}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$. Then $T^{*m}_{q^+ I_{n\times n}} \longrightarrow 0$ in the strong operator topology for all inner functions $q \in H^\infty(\mathbb{T})$ and $T^{*m}_{q^+ I_{n\times n}} K \to 0$ in the norm topology as $m \to \infty$.*

Proof. For $\lambda \in \mathbb{D}$, let $K_\lambda(t) = \frac{1}{1-\overline{\lambda}e^{it}}$ be the reproducing kernel of the Hardy space $H^2(\mathbb{T})$ and k_λ be the normalized reproducing kernel of the Hardy space $H^2(\mathbb{T})$, that is, $k_\lambda(t) = \frac{\sqrt{1-|\lambda|^2}}{1-\overline{\lambda}e^{it}}$. Let $f = \sum_{i=1}^{p} c_i k_{\lambda_i}$. Then

$$T^{*m}_{q^+}\left(\sum_{i=1}^{p} c_i k_{\lambda_i}\right) = \sum_{i=1}^{p} c_i [q(\overline{\lambda}_i)]^m k_{\lambda_i}.$$

Hence

$$\left\| T^{*m}_{q^+}\left(\sum_{i=1}^{p} c_i k_{\lambda_i}\right)\right\| \le \sum_{i=1}^{p} |c_i||q(\overline{\lambda}_i)|^m \|k_{\lambda_i}\| = \sum_{i=1}^{p} |c_i||q(\overline{\lambda}_i)|^m \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

This is so as q is inner and therefore $|q(z)| < 1$ for $z \in \mathbb{D}$. Since the reproducing kernels $K_\lambda, \lambda \in \mathbb{D}$, span $H^2(\mathbb{D})$, we obtain $T^{*m}_{q^+} \longrightarrow 0$ in the strong operator topology for all inner functions $q \in H^\infty(\mathbb{T})$.It is not difficult now to verify that $T^{*m}_{q^+ I_{n\times n}} \longrightarrow 0$ in the strong operator topology for all inner functions $q \in H^\infty(\mathbb{T})$. For a rank one operator, $f \otimes g$ (here $(f \otimes g)(h) = \langle h, g\rangle f$), we have $T^{*m}_{q^+}(f \otimes g) = (T^{*m}_{q^+} f) \otimes g$. Since $\overline{\mathcal{LF}(H^2(\mathbb{T}))} = \mathcal{LC}(H^2(\mathbb{T}))$, it is proved that $T^{*m}_{q^+} L \to 0$ in norm as $m \to \infty$ for all compact operators $L \in \mathcal{LC}(H^2(\mathbb{T}))$. Now let $K \in \mathcal{LC}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$. Then

$$K = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix}$$

where $K_{ij} \in \mathcal{LC}(H^2(\mathbb{T})), 1 \le i, j \le n$. Hence $T^{*m}_{q^+ I_{n\times n}} K = [T^{*m}_{q^+} K_{ij}]_{1 \le i,j \le n}$. Therefore, for $F \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}), F = (f_1, f_2, ..., f_n), f_i \in H^2(\mathbb{T})$,

$$\left\|T^{*m}_{q^+I_{n\times n}}KF\right\|^2 = \int_0^{2\pi}\left(\sum_{i=1}^{n}\left|\sum_{j=1}^{n}T^{*m}_{q^+}K_{ij}f_j(e^{i\theta})\right|^2\right)\frac{d\theta}{2\pi}$$

$$= \sum_{i=1}^{n}\int_0^{2\pi}\left|\sum_{j=1}^{n}T^{*m}_{q^+}K_{ij}f_j(e^{i\theta})\right|^2\frac{d\theta}{2\pi} \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{2\pi}|T^{*m}_{q^+}K_{ij}f_j(e^{i\theta})|^2\frac{d\theta}{2\pi}$$

$$= \sum_{i,j=1}^{n}\|T^{*m}_{q^+}K_{ij}f_j\|^2 \longrightarrow 0 \text{ as } m\to\infty.$$

Hence $T^{*^m}_{q^+I_{n\times n}}K \to 0$ strongly. Further, it follows that

$$\|T^{*m}_{q^+I_{n\times n}}KF\|^2 \leq \|T^{*m}_{q^+}K_{11}\|^2\sum_{j=1}^{n}\|f_j\|^2 = \left\|T^{*^m}_{q^+}K_{11}\right\|^2\|F\|^2$$

if $\left\|T^{*^m}_{q^+}K_{11}\right\| = \max_{1\leq i,j\leq n}\|T^{*^m}_{q^+}K_{ij}\|$. Thus $\|T^{*m}_{q^+I_{n\times n}}K\|^2 \leq \|T^{*m}_{q^+}K_{11}\|^2 \to 0$ as $m\to\infty$. Thus $T^{*m}_{q^+I_{n\times n}}K \longrightarrow 0$ in norm as $m\to\infty$. ■

Lemma 3.4. *Let $\Phi \in L^\infty_{M_n}(\mathbb{T})$ and S_Φ be the Hankel operator with symbol Φ on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Then $S_\Phi T^m_{qI_{n\times n}} \longrightarrow 0$ strongly as $m\to\infty$.*

Proof. Let $\Phi = [\varphi_{ij}]_{1\leq i,j\leq n}$. Then $S_\Phi = [S_{\varphi_{ij}}]_{1\leq i,j\leq n}$ and hence

$$S_\Phi T^m_{qI_{n\times n}} = T^{*m}{}_{q^+I_{n\times n}}S_\Phi = [T^{*^m}_{q^+}S_{\varphi_{ij}}]_{1\leq i,j\leq n}.$$

Now for $F\in\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}), F=(f_1,f_2,...,f_n), f_i\in H^2(\mathbb{T}), 1\leq i\leq n$, we have

$$\begin{aligned}\left\|S_\Phi T^m_{qI_{n\times n}}F\right\|^2 &= \int_0^{2\pi}\left(\sum_{i=1}^{n}\left|\sum_{j=1}^{n}T^{*m}_{q^+}S_{\varphi_{ij}}f_j(e^{i\theta})\right|^2\right)\frac{d\theta}{2\pi}\\ &= \sum_{i=1}^{n}\int_0^{2\pi}\left|\sum_{j=1}^{n}T^{*m}_{q^+}S_{\varphi_{ij}}f_j(e^{i\theta})\right|^2\frac{d\theta}{2\pi}\\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{2\pi}|T^{*m}_{q^+}S_{\varphi_{ij}}f_j(e^{i\theta})|^2\frac{d\theta}{2\pi}\\ &= \sum_{i,j=1}^{n}\|T^{*m}_{q^+}S_{\varphi_{ij}}f_j\|^2.\end{aligned}$$

Since $T_{q^+}^{*m} \longrightarrow 0$ strongly, hence $\sum_{i,j=1}^{n} \left\| T_{q^+}^{*m} S_{\varphi_{ij}} f_j \right\|^2 \to 0$ as $m \to \infty$ and therefore $\left\| S_\Phi T_{qI_{n\times n}}^m F \right\|^2 \longrightarrow 0$ as $m \to \infty$. Thus $S_\Phi T_{qI_{n\times n}}^m \longrightarrow 0$ strongly as $m \to \infty$. ∎

Theorem 3.5. *If $\Phi \in L_{M_n}^\infty(\mathbb{T})$ then $\{J_{qI_{n\times n}}^m S_\Phi T_{(zq)I_{n\times n}}^m\}$ converges strongly to 0 for all inner functions $q \in H^\infty(\mathbb{T})$.*

Proof. Notice that since $J_q^{m*} z^i = \begin{cases} q^m z^{m-i}, & 0 \le i \le m \\ 0, & \text{otherwise} \end{cases}$
hence $J_q^{m^*} z^i = M_{(zq)^m} J P_m z^i$ where $J : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined as $Jf(z) = f(\overline{z})$ and $M_\varphi : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ is the multiplication operator with symbol $\varphi \in L^\infty(\mathbb{T})$ and P_m is the projection of $H^2(\mathbb{T})$ onto span $\{1, z, ..., z^m\}$. Thus

$$J_q^m = P_m J M_{(zq)^m}^* = P_m J M_{\overline{z}^m \overline{q}^m}$$

and hence

$$J_{qI_{n\times n}}^m S_\Phi T_{(zq)I_{n\times n}}^m = [J_q^m S_{\varphi_{ij}} T_{(zq)}^m]_{1\le i,j\le n}$$

if $\Phi = [\varphi_{ij}]_{1\le i,j\le n}, \varphi_{ij} \in L^\infty(\mathbb{T})$. Now

$$\begin{aligned} J_q^m S_{\varphi_{ij}} T_{(zq)}^m &= P_m J M_{\overline{z}^m \overline{q}^m} S_{\varphi_{ij}} T_{z^m q^m} = P_m J M_{\overline{z}^m \overline{q}^m} T^*_{(z^m q^m)^+} S_{\varphi_{ij}} \\ &= P_m J M_{\overline{z}^m \overline{q}^m} T_{\overline{z}^m (Jq)^m} S_{\varphi_{ij}} = P_m J M_{\overline{z}^m \overline{q}^m} T^*_{z^m q^{+m}} S_{\varphi_{ij}} \\ &= P_m J (T_{z^m q^{+m}} M_{z^m q^m})^* S_{\varphi_{ij}} = P_m J T^*_{z^{2m} q^m q^{+m}} S_{\varphi_{ij}}. \end{aligned}$$

Thus for $F \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$, $F = (f_1, f_2, ..., f_n), f_i \in H^2(\mathbb{T}), 1 \le i \le n$, we have

$$\begin{aligned} \left\| J_{qI_{n\times n}}^m S_\Phi T_{(zq)I_{n\times n}}^m F \right\|^2 &= \int_0^{2\pi} \left(\sum_{i=1}^n \left| \sum_{j=1}^n J_q^m S_{\varphi_{ij}} T_{(zq)}^m f_j(e^{i\theta}) \right|^2 \right) \frac{d\theta}{2\pi} \\ &\le \sum_{i=1}^n \sum_{j=1}^n \int_0^{2\pi} \left| J_q^m S_{\varphi_{ij}} T_{(zq)}^m f_j(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} \\ &= \sum_{i,j=1}^n \left\| J_q^m S_{\varphi_{ij}} T_{(zq)}^m f_j \right\|^2 \\ &= \sum_{i,j=1}^n \left\| P_m J T^*_{z^{2m} q^m q^{+m}} S_{\varphi_{ij}} f_j \right\|^2 \\ &\le \sum_{i,j=1}^n \left\| T^*_{z^{2m} q^m q^{+m}} S_{\varphi_{ij}} f_j \right\|^2 \end{aligned}$$

since $\|J\| \le 1$ and$\|P_m\| \le 1$. Thus by Lemma 3.4,

$$\left\| J^m_{qI_{n\times n}} S_\Phi T^m_{(zq)I_{n\times n}} F \right\|^2 \leq \sum_{i,j=1}^{n} \left\| S_{\varphi_{ij}} T_{z^{2m}q^{+m}q^m} f_j \right\|^2 \longrightarrow 0 \text{ as } m \to \infty.$$

Hence $J^m_{qI_{n\times n}} S_\Phi T^m_{(zq)I_{n\times n}} \longrightarrow 0$ strongly as $m \to \infty$. ■

4. The Hankel Sequence Associated with Multiplication Operators

In this section we consider the multiplication operators M_Φ defined on $L^2_{\mathbb{C}^n}(\mathbb{T})$ with symbol $\Phi \in L^\infty_{M_n}(\mathbb{T})$. We construct a sequence using the multiplication operator M_Φ which converges strongly to a bounded linear operator B_Φ on $L^2_{\mathbb{C}^n}(\mathbb{T})$ and $PB_\Phi|_{\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})} = S_\Phi$, a Hankel operator.

For $\Phi \in L^\infty_{M_n}(\mathbb{T})$, define the multiplication operator $M_\Phi : L^2_{\mathbb{C}^n}(\mathbb{T}) \longrightarrow L^2_{\mathbb{C}^n}(\mathbb{T})$ with symbol Φ as $M_\Phi f = \Phi f$. Let

$$Q_m = P_m I_{n\times n} = \begin{bmatrix} P_m & 0 & \cdots & 0 \\ 0 & P_m & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & P_m \end{bmatrix}$$

and $U^m_{qI_{n\times n}} = Q_m M^m_{q^+ I_{n\times n}} J$ for all inner functions $q \in H^\infty(\mathbb{T}), m \in \mathbb{Z}_+$. Consider the sequence $\{\sigma_m(M_\phi)\} = \{U^m_{qI_{n\times n}} M_\Phi M^m_{qI_{n\times n}}\}$. This sequence is referred to as the Hankel sequence associated with multiplication operator M_Φ on $L^2_{\mathbb{C}^n}(\mathbb{T})$.

Theorem 4.1. *For $\Phi \in L^\infty_{M_n}(\mathbb{T})$, the sequence $\{\sigma_m(M_\Phi)\}$ converges strongly to a bounded linear operator $B_\Phi \in \mathcal{L}(L^2_{\mathbb{C}^n}(\mathbb{T}))$ for all inner functions $q \in H^\infty(\mathbb{T})$ and $PB_\Phi|_{\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})} = S_\Phi$, the Hankel operator on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ with symbol Φ.*

Proof. We shall first verify that $JM_\Phi M_{qI_{n\times n}} = M^*_{q^+ I_{n\times n}} JM_\Phi$ for all inner functions $q \in H^\infty(\mathbb{T})$ where $q^+(z) = \overline{q(\overline{z})}$.
Notice that if $\Phi = (\varphi_{ij})_{1\leq i,j\leq n}$ then $JM_\Phi M_{qI_{n\times n}} = M^*_{q^+ I_{n\times n}} JM_\Phi$ for all inner functions $q \in H^\infty(\mathbb{T})$ if and only if $\widetilde{J} M_{\varphi_{ij}} M_q = M^*_{q^+} \widetilde{J} M_{\varphi_{ij}}$ for all inner functions $q \in H^\infty(\mathbb{T})$ and for all $i, j \in \{1, 2, .., n\}$. Again

$$\langle \widetilde{J} M_{\varphi_{ij}} M_q z^j, z^i \rangle = \langle M_{\varphi_{ij} q} z^j, z^{-i} \rangle = \langle \varphi_{ij} q z^j, z^{-i} \rangle$$

and

$$\begin{aligned} \langle M^*_{q^+} \widetilde{J} M_{\varphi_{ij}} z^j, z^i \rangle =& \langle \widetilde{J}(\varphi_{ij} z^j), M_{q^+} z^i \rangle = \langle \varphi_{ij} z^j, \widetilde{J}(q^+ z^i) \rangle = \langle \varphi_{ij} z^j, \overline{q} z^{-i} \rangle \\ =& \langle \varphi_{ij} q z^j, z^{-i} \rangle \end{aligned}$$

for all $i, j \in \mathbb{Z}$. Hence $\widetilde{J} M_{\varphi_{ij}} M_q = M^*_{q^+} \widetilde{J} M_{\varphi_{ij}}, 1 \leq i, j \leq n$ and for all inner functions $q \in H^\infty(\mathbb{T})$. Thus

$$U^m_{qI_{n\times n}} M_\Phi M^m_{qI_{n\times n}} = Q_m M^m_{q^+ I_{n\times n}} JM_\Phi M^m_{qI_{n\times n}}$$

$$= \left(P_m M^m_{q^+} \widetilde{J} M_{\varphi_{ij}} M^m_q\right)_{1\le i,j\le n}$$
$$= \left(P_m M^m_{q^+} M^{*m}_{q^+} \widetilde{J} M_{\varphi_{ij}}\right)_{1\le i,j\le n}$$
$$= \left(P_m M^m_{q^+} M^m_{\widetilde{J}q} \widetilde{J} M_{\varphi_{ij}}\right)_{1\le i,j\le n}$$
$$= \left(P_m \widetilde{J} M_{\varphi_{ij}}\right)_{1\le i,j\le n}$$

for all inner functions $q \in H^\infty(\mathbb{T})$. Now for $f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$; $f = (f_1, f_2, ..., f_n)$,

$$\left\| U^m_{qI_{n\times n}} M_\Phi M^m_{qI_{n\times n}} f - JM_\Phi f \right\|^2 \le \sum_{i=1}^n \sum_{j=1}^n \left\| P_m \widetilde{J} M_{\varphi_{ij}} f_j - \widetilde{J} M_{\varphi_{ij}} f_j \right\|^2 \longrightarrow 0$$

as $m \longrightarrow \infty$. Hence $U^m_{qI_{n\times n}} M_\Phi M^m_{qI_{n\times n}} \longrightarrow JM_\Phi$ strongly as $m \longrightarrow \infty$ for all inner functions $q \in H^\infty(\mathbb{T})$ and $PJM_\Phi|_{\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})} = S_\Phi$, the Hankel operator on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ with symbol Φ. ∎

Corollary 4.2. *Let $\Phi \in L^\infty_{M_n}(\mathbb{T})$ be such that $U^m_{qI_{n\times n}} M_\Phi M^m_{qI_{n\times n}} \longrightarrow 0$ strongly as $m \to \infty$ for all inner functions $q \in H^\infty(\mathbb{T})$. Then $\Phi \in M_{zI_{n\times n}} H^\infty_{M_n}(\mathbb{T})$ where $H^\infty_{M_n}(\mathbb{T}) = H^\infty(\mathbb{T}) \otimes M_n$.*

Proof. Suppose $\Phi \in L^\infty_{M_n}(\mathbb{T})$ and $\Phi = (\varphi_{ij})_{1\le i,j\le n}$ and $U^m_{qI_{n\times n}} M_\Phi M^m_{qI_{n\times n}} \longrightarrow 0$ strongly as $m \to \infty$ for all inner functions $q \in H^\infty(\mathbb{T})$. Then by Theorem 4.1, $JM_\Phi \equiv 0$ and therefore $S_\Phi \equiv 0$. Hence $S_{\varphi_{ij}} \equiv 0$ on $H^2(\mathbb{T})$ for all $i, j \in \{1, 2, ..., n\}$. That is, $\varphi_{ij} \in zH^\infty(\mathbb{T})$ and $\Phi \in M_{zI_{n\times n}} H^\infty_{M_n}(\mathbb{T})$. ∎

These asymptotic results are useful in obtaining distance formulas for operators on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ relating to Toeplitz and Hankel operators. Such distance formulas for Toeplitz and Hankel operators on $H^2(\mathbb{T})$ were studied in [1],[2] and [3]. This will be taken up in a future work for operators on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$.

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