Vietnam Journal of Mathematics **38**:1(2010) 55-62

Vietnam Journal
of
MATHEMATICS

On Arithmetical Rings and the Radical Formula

A. Nikseresht and A. Azizi

Department of Mathematics, College of Sciences,
Shiraz University, 71457-44776, Shiraz, Iran

Abstract. In this paper we investigate the radical formula in modules over arithmetical rings. Particularly we prove that every module over a finite dimensional arithmetical ring and every Prüfer module over an arbitrary arithmetical ring satisfies the radical formula. Also we give a necessary and sufficient condition for a serial module to satisfy the radical formula.

2000 Mathematics Subject Classification: 13C99, 13A15, 13F05, 13C13.

Key words: prime submodule, radical formula, arithmetical ring, invertible submodule.

1. Introduction

In this paper all rings are commutative and with identity, all modules are unitary, R denotes a ring and M denotes an R-module.

Prime ideals of a ring play an important role in the study of properties of the ring. So this concept was generalized to modules, see for example [2]. A proper submodule N of M is called *prime* when from $rm \in N$ for some $r \in R$ and $m \in M$, we can conclude either $m \in N$ or $rM \subseteq N$. Let $(N : M)$ be the set of all $r \in R$ such that $rM \subseteq N$. One can easily verify that N is a prime submodule of M if and only if $\frac{M}{N}$ is a torsion-free $\frac{R}{(N:M)}$-module. In this case $\mathfrak{p} = (N : M)$ is a prime ideal of R and we say that N is $\mathfrak{p}$-prime.

Recall that for an ideal I of R, the intersection of all prime ideals of R containing I is called the radical of I and is denoted by $\sqrt{I}$. It is well-known that

$$\sqrt{I} = \left\{ r \in R | r^k \in I \quad \text{for some k} \in \mathbb{N} \right\}.$$

To find a similar characterization for the intersection of prime submodules of M containing a specific submodule N— which is called the *radical of N in*

M and we denote it by $\mathrm{Rad}_M(N)$— the notion of envelope of a submodule was introduced in [7]. The *envelope* of a submodule N of M, $\mathrm{E}_M(N)$, is the set of all $x \in M$ for which there exist $r \in R$, $m \in M$ and $k \in \mathbb{N}$, such that $x = rm$ and $r^k m \in N$. If $\mathrm{Rad}_M(N) = R\mathrm{E}_M(N)$, where $R\mathrm{E}_M(N)$ is the submodule generated by $\mathrm{E}_M(N)$, it is said that N *satisfies the radical formula* (*s.t.r.f.*) in M. A module M s.t.r.f. when every submodule of M s.t.r.f. in M. Also we say that R s.t.r.f., if every R-module s.t.r.f.

In [4] it is proved that every Dedekind domain s.t.r.f. Also we know that if R is a Prüfer domain then $R \oplus R$ s.t.r.f. as an R-module (see [3, Theorem 2.4]).

Recall that an *arithmetical ring* is one in which for all ideals I, J and K, we have $I + (J \cap K) = (I + J) \cap (I + K)$ (see [5, 6]). Thus every Prüfer and Dedekind domain is an arithmetical ring.

In [1], the *nth envelope* of N in M was defined recursively by $\mathrm{E}_k(N) = \mathrm{E}(R\mathrm{E}_{k-1}(N))$ and $\mathrm{E}_0(N) = N$ (when there is no subtlety, we drop the M in $\mathrm{Rad}_M(N)$ and $\mathrm{E}_M(N)$). There it was said that a ring *s.t.r.f. of degree* k, if for every R-module M and every submodule N of M, we have $\mathrm{Rad}(N) = R\mathrm{E}_k(N)$. Moreover it was proved that every arithmetical ring with finite Krull dimension k, s.t.r.f. of degree k.

In [8] the concept of an invertible submodule was introduced. For an R-module M, let $\mathrm{Z}(M)$ be the set of zero divisors of M, that is $\mathrm{Z}(M) = \{r \in R | \exists\, 0 \neq m \in M \quad rm = 0\}$. Set $T = R \setminus (\mathrm{Z}(R) \cup \mathrm{Z}(M))$. For every submodule N of M let $N' = \left\{x \in T^{-1}R | xN \subseteq M\right\}$. Then N is called invertible if $N'N = M$. If every nonzero cyclic submodule (resp. finitely generated submodule, submodule) of M is *invertible*, M is called a *D1 module* (resp. *Prüfer module, Dedekind module*).

Here we continue the investigation of radical formula in arithmetical rings. Especially we show that every arithmetical ring with DCC on prime ideals and every D1 module over an arithmetical ring s.t.r.f. Moreover in our study we present a necessary and sufficient condition for a *serial module*, that is a module in which every two submodules are comparable, to s.t.r.f.

2. Results

The following lemma is of much use throughout this paper.

Lemma 2.1. *Let M be an R-module and N be a submodule of M.*

(i) $N \subseteq RE(N) \subseteq Rad(N)$.

(ii) *If $N_\mathfrak{p}$ s.t.r.f. in $M_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R, then N s.t.r.f. in M.*

(iii) *For all submodules K of M which contain N, K s.t.r.f. in M if and only if $\frac{K}{N}$ s.t.r.f. in $\frac{M}{N}$.*

Proof. Part (i) is trivial. Also (iii) follows the equalities $\frac{\mathrm{Rad}_M(K)}{N} = \mathrm{Rad}_{\frac{M}{N}}\left(\frac{K}{N}\right)$ and $\frac{R\mathrm{E}_M(K)}{N} = R\mathrm{E}_{\frac{M}{N}}\left(\frac{K}{N}\right)$. For (ii), see for example [9, Lemma 1.5 and Proposition 1.6]. ■

A ring in which every two ideals are comparable, is called a *chained ring*. For example every valuation domain is a chained ring.

Lemma 2.2. *A ring R is arithmetical if and only if $R_{\mathfrak{p}}$ is a chained ring for every prime ideal $\mathfrak{p}$ of R.*

Proof. See [5, Theorem 1]. ■

A proper submodule N of M is called *primary*, if from $rm \in N$ for some $r \in R$ and $m \in M$, we can deduce either $m \in N$ or $r^k M \subseteq N$ for some $k \in \mathbb{N}$. If N is a primary submodule of M, then it is easy to check that $\mathfrak{p} = \sqrt{(N : M)}$ is a prime ideal of R. In this case we say that N is $\mathfrak{p}$-primary.

Theorem 2.3. *If R is an arithmetical ring, then every primary submodule of M s.t.r.f. in M.*

Proof. Suppose that N is a $\mathfrak{p}$-primary submodule of M. Then for any prime ideal $\mathfrak{q}$ of R, $N_{\mathfrak{q}}$ is either the whole $M_{\mathfrak{q}}$ or a primary submodule of it. Hence by (2.1)(ii), we can assume that R is local and by (2.2), a chained ring. Also since N is $\mathfrak{p}$-primary, the zero submodule in $\frac{M}{N}$ is $\mathfrak{p}$-primary. So by (2.1)(iii), we can assume $N = 0$.

By [10, Lemma 1.3] $R\mathrm{E}(0) = \mathfrak{p}M$. We will show that $\mathfrak{p}M$ is a prime submodule of M and hence $\mathrm{Rad}(0) \subseteq \mathfrak{p}M = R\mathrm{E}(0)$. Then by (2.1)(i), the assertion is proved.

Suppose $rm \in \mathfrak{p}M$. So $rm = \sum p_i m_i$ where each $p_i \in \mathfrak{p}$ and $m_i \in M$. Since R is a chained ring, the ideals $\{Rp_i\}$ are pairwise comparable and thus have a maximum, say Rp_1. Therefore we can write $rm = p_1 m'$ for some $m' \in M$. If $Rr \subseteq Rp_1$, then $r \in \mathfrak{p}$ and we are finished. So suppose $r \notin \mathfrak{p}$ and $Rp_1 \subseteq Rr$, say $p_1 = r'r$ for some $r \in R$. Since $\mathfrak{p}$ is prime we must have $r' \in \mathfrak{p}$. Now $r(m - r'm') = 0$ hence, by the zero submodule being $\mathfrak{p}$-primary and $r \notin \mathfrak{p}$, we get $m - r'm' = 0$. Thus $m = r'm' \in \mathfrak{p}M$ which is the claimed result. ■

Next with help of a series of lemmas, we show that every arithmetical ring with DCC on prime ideals s.t.r.f.

Lemma 2.4. *Assume that N_1 and N_2 are submodules of the R-modules M_1 and M_2, respectively. Set $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$. Then $RE_M(N) = RE_{M_1}(N_1) \oplus RE_{M_2}(N_2)$. Also if N_1 and N_2 s.t.r.f. in M_1 and M_2, respectively, then N s.t.r.f. in M.*

Proof. The first statement is easy to prove and left to the reader. Also it is clear that if P is a prime submodule of M_1 (resp. M_2) then $P \oplus M_2$ (resp. $M_1 \oplus P$) is a prime submodule of M. Thus $\mathrm{Rad}_M(N) \subseteq \mathrm{Rad}_{M_1}(N_1) \oplus \mathrm{Rad}_{M_2}(N_2)$. So if N_1 and N_2 s.t.r.f., then $\mathrm{Rad}_M(N) \subseteq R\mathrm{E}_{M_1}(N_1) \oplus R\mathrm{E}_{M_2}(N_2) = R\mathrm{E}_M(N)$. ■

Lemma 2.5. *Assume that R is a chained ring and $M = \bigoplus_{i\in I} R_i$ where each $R_i \cong R$. If N is a finitely generated submodule of M, then there is an automorphism of M which maps N to $\bigoplus_{i\in I} N_i$, where N_i is a submodule of R_i.*

Proof. First note that every finitely generated submodule of M is in fact a submodule of $\bigoplus_{i\in F} R_i$, where F is a finite subset of I. So we assume that I is finite and prove the statement by induction on $n = |I|$.

The basic step is trivial, thus suppose $n > 1$ and N is the submodule generated by $\{(a_{ij})_{j=1}^{n} | 1 \leq i \leq k\}$. Since R is a chained ring we can assume that Ra_{kn} is the maximum element of $\{Ra_{ij}\}$. Therefore there are r_{ij}'s in R, such that $a_{ij} = r_{ij}a_{kn}$. Now the set

$$S = \{(1, 0, 0, \cdots, 0), \cdots, (0, 0, \cdots, 0, 1, 0), (r_{k1}, r_{k2}, \cdots, r_{k(n-1)}, 1)\}$$

forms a basis for M.

Define $\phi : M \to M$ by $\phi((r_{k1}, r_{k2}, \cdots, r_{k(n-1)}, 1)) = (0, 0, \cdots, 0, 1)$ and $\phi(x) = x$ for all other $x \in S$. Clearly ϕ is an automorphism of M. Set $N_n = Ra_{kn}$. Now $\phi(R(a_{k1}, \cdots, a_{kn})) = 0 \oplus 0 \oplus \cdots \oplus N_n$. Also if π_n denotes the natural projection from M to R_n, then obviously $\pi_n(\phi(N)) \subseteq N_n$, so in fact $\pi_n(\phi(N)) = N_n$. Thus $\phi(N) = A \oplus N_n$ for some finitely generated submodule A of $B = \bigoplus_{i=1}^{n-1} R_i$. Now by the induction hypothesis there is an automorphism ψ of B, which maps A to $\bigoplus_{i=1}^{n-1} N_i$ for some submodules N_i's of R_i's. Combining ϕ and ψ we get the desired automorphism. ∎

Proposition 2.6. *Let R be an arithmetical ring and M be a free R-module. Every finitely generated submodule of M s.t.r.f. in M.*

Proof. Let N be a finitely generated submodule of $M \cong \bigoplus_{i\in I} R_i$, where $R_i \cong R$. Then $N_{\mathfrak{p}}$ is also a finitely generated $R_{\mathfrak{p}}$-submodule of $M_{\mathfrak{p}}$, so by (2.1)(ii) and (2.2), we can assume R is a chained ring. Thus according to (2.5), by an automorphism of M, N is mapped to $\bigoplus_{i\in I} N_i$, where N_i is a submodule of R_i. Now since every ring obviously s.t.r.f. as a module over itself, by (2.4) we see that N s.t.r.f. in M. ∎

For any prime ideal $\mathfrak{p}$ of R and submodule N of M, define $M(\mathfrak{p}, N)$ to be the set $\{m \in M | sm \in \mathfrak{p}M + N \text{ for some } s \in R \setminus \mathfrak{p}\}$. It can easily be checked that $M(\mathfrak{p}, N)$ is either the whole M or a $\mathfrak{p}$-prime submodule of M. Also every $\mathfrak{p}$-prime submodule of M which contains N must contain $M(\mathfrak{p}, N)$, too. Thus always we have $\mathrm{Rad}_M(N) = \bigcap M(\mathfrak{p}, N)$, where $\mathfrak{p}$ runs through the prime ideals of R.

Lemma 2.7. *Suppose that R has finitely many prime ideals and N is a submodule of M. Then $Rad_M(N) = \bigcup Rad_M(N_f)$, where the union is taken over all finitely generated submodules N_f of N.*

Proof. It is obvious that $\bigcup \mathrm{Rad}_M(N_f) \subseteq \mathrm{Rad}_M(N)$. Let $x \in \mathrm{Rad}_M(N)$ and suppose that prime ideals of R are $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k$. So by the above remarks for each $1 \leq i \leq k$, there are $s_i \in R \setminus \mathfrak{p}_i$, $m_i \in \mathfrak{p}_i M$ and $n_i \in N$ such that $s_i x = m_i + n_i$. Set $N' =< n_1, n_2, \cdots, n_k >$. Thus $s_i x \in \mathfrak{p}_i M + N'$, which means that $x \in M(\mathfrak{p}_i, N')$ for all i. Therefore $x \in \mathrm{Rad}_M(N')$ and N' is a finitely generated submodule of N. This shows that $\mathrm{Rad}_M(N) \subseteq \bigcup \mathrm{Rad}_M(N_f)$ and completes the proof. ■

Now we are ready to state the following theorem which generalizes the fact that a Dedekind domain s.t.r.f.

Theorem 2.8. *Every arithmetical ring with DCC on prime ideals s.t.r.f. In particular every finite dimensional arithmetical ring s.t.r.f.*

Proof. Let R be an arithmetical ring with DCC on prime ideals. By (2.1)(iii), we need to prove that every free module over R s.t.r.f. and by (2.1)(ii), it suffices to show that for all prime ideals $\mathfrak{p}$ of R, every free $R_{\mathfrak{p}}$ module s.t.r.f. But $R_{\mathfrak{p}}$ satisfies DCC on prime ideals and since $R_{\mathfrak{p}}$ is a chained ring, this means that $R_{\mathfrak{p}}$ is finite dimensional and hence with finitely many prime ideals. Let M be a free $R_{\mathfrak{p}}$-module and N be a submodule of M. By (2.7), $\mathrm{Rad}(N) = \bigcup \mathrm{Rad}(N_f)$, where the union is taken over all finitely generated submodules N_f of N. Now by (2.6), for such N_f we have $\mathrm{Rad}(N_f) = R\mathrm{E}(N_f)$. Also obviously $R\mathrm{E}(N_f) \subseteq R\mathrm{E}(N)$. Thus $\mathrm{Rad}(N) = \bigcup R\mathrm{E}(N_f) \subseteq R\mathrm{E}(N)$ and by (2.1)(i) this concludes the proof. ■

Now we consider D1 modules over arithmetical rings. But first some lemmas.

Lemma 2.9. *Every torsion-free serial module (over an integral domain) s.t.r.f.*

Proof. Suppose that M is a torsion-free serial module. First we show that $\mathrm{E}(N)$ is a submodule of M, for all submodules N of M. Let $x,\ x' \in \mathrm{E}(N)$ then obviously $Rx,\ Rx' \subseteq \mathrm{E}(N)$ and they are comparable. Thus for example $Rx \subseteq Rx'$, and hence $Rx + Rx' \subseteq Rx' \subseteq \mathrm{E}(N)$.

Now to complete the proof, we show that $\mathrm{E}(N)$ is a prime submodule of M. Assume that $rm \in \mathrm{E}(N)$ for some $r \in R$ and $m \in M \setminus \mathrm{E}(N)$. We will show that $rM \subseteq \mathrm{E}(N)$. Since $rm \in \mathrm{E}(N)$, there are $r' \in R$, $m' \in M$ and $k \in \mathbb{N}$ such that $rm = r'm'$ and $r'^k m' \in N$. We will prove $rm' \in \mathrm{E}(N)$. Then if $m'' \in M$, either $Rm'' \subseteq Rm'$ or $Rm' \subseteq Rm''$. In the former case $rm'' \in Rrm' \subseteq \mathrm{E}(N)$. In the latter case there is an $r'' \in R$ such that $m' = r''m''$, so $rm = r'r''m''$ and $(r'r'')^k m'' = r''^{k-1} r'^k m' \in N$. Thus from the start, we could choose m' and r' to be m'' and $r'r''$ respectively, and get $rm'' \in N$.

If $Rm' \subseteq Rm$, then clearly $rm' \in \mathrm{E}(N)$. Therefore suppose $m = sm'$ for some $s \in R$. Since $m \notin \mathrm{E}(N)$, for every $n \in \mathbb{N}$ we have $s^n m' \notin N$, in particular $s^n m' \notin Rr'^k m'$. Hence $r'^k m' = a_n s^n m'$ for some nonunit $a_n \in R$. Especially for $n = 2k$, using the torsion-freeness of M, we get $r'^k = as^{2k}$ for some nonunit $a \in R$. If $rm' \notin \mathrm{E}(N)$, then by a similar argument as above, we see that $r'^k = br^{2k}$ for some nonunit $b \in R$.

Now $r'^{2k} = abs^{2k}r^{2k}$, but since $r'm' = rm = rsm'$ and M is torsion-free, we must have $r' = rs$. Thus $r'^{2k} = abr'^{2k}$ whence a, b are units, which is a contradiction. From this contradiction we conclude $rm' \in \mathrm{E}(N)$ and this completes the proof. ■

Lemma 2.10. *Suppose that R is a Prüfer domain and K is its quotient field. If M is isomorphic to an R-submodule of K, then M s.t.r.f.*

Proof. By (2.1)(ii), we can assume R is local and hence a valuation domain. Suppose that M is an R-submodule of K, $m, m' \in M$ and v is the valuation on K with ring R. Thus either $v(m) \leq v(m')$ or $v(m') \leq v(m)$. This means that either $Rm \subseteq Rm'$ or $Rm' \subseteq Rm$. Therefore M is a torsion-free serial module and by (2.9), s.t.r.f. ■

Proposition 2.1 *Let R be a Prüfer domain. If M has an invertible nonzero cyclic submodule, then M s.t.r.f.*

Proof. Let $0 \neq Rm$ be the invertible cyclic submodule of M and J be the set $\{x \in S^{-1}R | xm \in M\}$, where $S = R \backslash \mathrm{Z}(M)$. Thus J is an R-submodule of K, the quotient field of R. Define $\phi : J \to M$ by $\phi(x) = xm$. Clearly ϕ is well-defined and a homomorphism. Also since Rm is invertible, $Jm = JRm = M$ and hence ϕ is onto. Therefore $M \cong \frac{J}{\ker(\phi)}$. Now by (2.10), J s.t.r.f., and so by (2.1)(iii), $M \cong \frac{J}{\ker(\phi)}$ s.t.r.f. ■

Lemma 2.11. *If M is a D1 module over R and N is an invertible R-submodule of M, then $Ann(M)$ is a prime ideal of R and N is an invertible $\frac{R}{Ann(M)}$-submodule of M. Particularly M is a D1 module over $\frac{R}{Ann(M)}$.*

Proof. Set $I = \mathrm{Ann}(M)$. By [8, Corollary 2.2] I is a prime ideal of R and $\frac{R}{I}$ is a domain. Also it is clear that $\mathrm{Z}_{\frac{R}{I}}(M) = \mathrm{Z}_R(M) + I$. So if $S = R \backslash (\mathrm{Z}_R(R) \cup \mathrm{Z}_R(M))$ and $T = \frac{R}{I} \setminus \mathrm{Z}_{\frac{R}{I}}(M)$, then $r \in S$ implies $r + I \in T$. Let N be an invertible submodule of M, $N' = \{x \in S^{-1}R | xN \subseteq M\}$ and $N'' = \{x \in T^{-1}\frac{R}{I} | xN \subseteq M\}$. We must show $N''N = M$. Obviously $N''N \subseteq M$. Now let $m \in M$. Since $N'N = M$, we can write $m = \sum \frac{r_i}{s_i} n_i$ for some $r_i \in I$, $s_i \in S$ and $n_i \in N$, such that $\frac{r_i}{s_i} N \subseteq M$. Thus $m = \sum \frac{r_i + I}{s_i + I} n_i$, $s_i + I \in T$ and $\frac{r_i + I}{s_i + I} N \subseteq M$. Hence $m \in N''N$ and $N''N = M$. ■

Theorem 2.12. *Every D1 module (and hence every Prüfer and Dedekind module) over an arithmetical ring s.t.r.f.*

Proof. Suppose that R is an arithmetical ring, M is a D1 R-module and $I = \mathrm{Ann}(M)$. By (2.11), M is a D1 $\frac{R}{I}$-module and I is a prime ideal of R. But by [6, Chapter VI, Exercise 19.e] $\frac{R}{I}$ is a Prüfer domain. So by (2.1), M s.t.r.f. as an $\frac{R}{I}$-module and hence as an R-module. ■

Finally we state another consequence of (2.9). We call a module *almost factorable*, when the relation $0 \neq rm = r'm'$ for some $r, r' \in R$ and $m, m' \in M$, implies that there exists $m'' \in M$, such that $m, m' \in Rm''$. It is easy to check that every serial module and every torsion-free module over a chained ring is almost factorable.

Lemma 2.13. *If M is an almost factorable module and N is a submodule of M, then $E(N) = E(E(N))$.*

Proof. By (2.1)(i), $\mathrm{E}(N) \subseteq \mathrm{E}(\mathrm{E}(N))$. So let $x \in \mathrm{E}(\mathrm{E}(N))$. Hence there exist $r, r' \in R$, $m, m' \in M$ and $k, k' \in \mathbb{N}$ such that $x = rm,\ r^k m = r'm',\ r'^{k'} m' \in N$.

If $r^k m = 0$, then $x \in \mathrm{E}(0) \subseteq \mathrm{E}(N)$. Therefore suppose that $r^k m \neq 0$. Now by almost factorability, there are $m'' \in M$ and $a, a' \in R$ such that $m = am''$ and $m' = a'm''$. Now

$$(ra)^k m'' = a^{k-1}\left(r^k a m''\right) = a^{k-1}\left(r^k m\right) = a^{k-1}\left(r'm'\right) = \left(a^{k-1} r' a'\right) m''.$$

Thus if we set $c = (ra)^k$ and $d = a^{k-1} r' a'$, we have

$$c^{k'} m'' = c^{k'-1} (ra)^k m'' = c^{k'-1} d m'' = \cdots = d^{k'} m'' \in R r'^{k'} a' m'' = R r'^{k'} m' \subseteq N.$$

So $x = (ra)m''$ and $(ra)^{kk'} m'' \in N$, which means that $x \in \mathrm{E}(N)$. ■

By a *radical submodule* N of M, we mean a proper submodule N such that $\mathrm{Rad}(N) = N$.

Lemma 2.14. *If M is a serial module, then every radical submodule of M is prime in M.*

Proof. Let N be a radical submodule of M. Suppose that $rm \in N$ for some $r \in R$ and $m \in M \setminus N$. We prove that $rM \subseteq N$. Since $m \notin \mathrm{Rad}(N) = N$, there is a prime submodule P of M containing N but not m. Because M is serial, all prime submodules of M either contain P or are contained in P. Thus, if $P' \leq_p M$ means that P' is a prime submodule of M, then

$$N = \mathrm{Rad}(N) = \bigcap_{N \subseteq P' \leq_p M} P' = \bigcap_{N \subseteq P' \leq_p M,\, P' \subseteq P} P'.$$

Now for each prime submodule P' of M containing N and contained in P, we have $m \notin P'$ and $rm \in P'$. Therefore $rM \subseteq P'$ for all such P' and hence rM is a subset of their intersection which equals N. ■

Theorem 2.15. *Let M be a serial module. Then M s.t.r.f. if and only if for some $n \in \mathbb{N}$, the zero submodule s.t.r.f. of degree n in M.*

Proof. By definition, if M s.t.r.f., then the zero submodule s.t.r.f. (of degree 1). So assume that the zero submodule s.t.r.f. of degree n in M, for some $n \in \mathbb{N}$.

By the first paragraph of the proof of (2.9), $R\mathrm{E}(0) = \mathrm{E}(0)$ and hence by (2.13), $R\mathrm{E}_n(0) = R\mathrm{E}_1(0) = R\mathrm{E}(0)$. Thus $R\mathrm{E}(0) = \mathrm{Rad}(0)$ and by (2.14), is a prime submodule of M. Therefore $\frac{M}{R\mathrm{E}(0)}$ is a torsion-free $\frac{R}{(R\mathrm{E}(0):M)}$-module and by (2.9), s.t.r.f. Let N be an arbitrary submodule of M. If $R\mathrm{E}(0) \subseteq N$, then by (2.1)(iii), N s.t.r.f. in M. If $N \subseteq R\mathrm{E}(0)$, then $R\mathrm{E}(0) \subseteq R\mathrm{E}(N) \subseteq \mathrm{Rad}(N) \subseteq \mathrm{Rad}(0) = R\mathrm{E}(0)$ and again N s.t.r.f. in M. Hence M s.t.r.f. ■

Acknowledgement. The authors would like to thank the referee for his/her suggestions.

References

1. A. Azizi, Radical formula and prime submodules, *J. Algebra*, **307** (2007), 454–460.
2. A. Azizi and H. Sharif, On prime submodules, *Honam Math. J.* **21** (1) (1999), 1–12.
3. Dilek Buyruk and Dilek Pusat-Yilmaz, Modules over Prüfer domains which satisfy the radical formula, *Glasgow Math. J.* **49** (2007), 127–131.
4. James Jenkins and Patrick F. Smith, On the prime radical of a module over a commutative ring, *Comm. Algebra* **20** (12) (1992), 3593–3602.
5. C. Jensen, Arithmetical rings, *Acta. Math. Hungar.* **17** (1966), 115–123.
6. M. D. Larsen and P. J. McCarthy, *Multiplicative Theory of Ideals*, Academic Press, 1971.
7. R. McCasland and M. Moore, On radicals of submodules of finitely generated modules, *Canad. Math. Bull.* **29** (1) (1986), 37–39.
8. Adil G. Naoum and Firas H. Al-Alwan, Dedekind modules, *Comm. Algebra* **24** (2) (1996), 397–412.
9. H. Sharif, Y. Sharifi and S. Namazi, Rings satisfying the radical formula, *Acta Math. Hungar.* **71** (1996), 103–108.
10. Patrick F. Smith, Primary modules over commutative rings, *Glasgow Math. J.* **43** (2001), 103–111.