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Unicity Theory of Entire Functions that Share One Value IM

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Abstract. In the paper we study the uniqueness of entire functions concerning differential polynomials sharing one value IM. Our results will supplement two recent results of Zhang and Lin [13] and at the same time include a recent result of Banerjee [1].

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1. Introduction, Definitions and Results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \{\infty\} \cup \mathbb{C}$ we say that f and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities and we say that f and g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities.
We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. For any $a \in \{\infty\} \cup \mathbb{C}$, we define

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

Corresponding to one famous question of Hayman [6], Fang and Hua [3], Yang and Hua [11] obtained the following theorem.

Theorem A. *Let f and g be two non-constant entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) =$*

c_2e^{-cz}, *where* c_1, c_2 *and* c *are three constants satisfying* $(c_1c_2)^{n+1}c^2 = -1$ *or* $f \equiv tg$ *for a constant* t *such that* $t^{n+1} = 1$.

Considering kth derivative instead of 1st derivative Fang [4] proved the following theorems.

Theorem B ([4]). *Let* f *and* g *be two non-constant entire functions, and let* n, k *be two positive integers with* $n > 2k + 4$. *If* $[f^n]^{(k)}$ *and* $[g^n]^{(k)}$ *share* 1 *CM, then either* $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, *where* c_1, c_2 *and* c *are three constants satisfying* $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ *or* $f \equiv tg$ *for a constant* t *such that* $t^n = 1$.

Theorem C ([4]). *Let* f *and* g *be two non-constant entire functions, and let* n, k *be two positive integers with* $n \geq 2k + 8$. *If* $[f^n(f-1)]^{(k)}$ *and* $[g^n(g-1)]^{(k)}$ *share* 1 *CM, then* $f \equiv g$.

Now a natural question arises:
Is Theorem B and Theorem C hold for some general differential polynomials like $[f^n(f^m - 1)]^{(k)}$ or $[f^n(f-1)^m]^{(k)}$?
Recently X.Y. Zhang and W.C. Lin [13] answered the above question and proved the following theorems.

Theorem D ([13]). *Let* f *and* g *be two non-constant entire functions, and let* n, m *and* k *be three positive integers with* $n \geq 2k + m^* + 4$, *and* λ, μ *be constants such that* $|\lambda| + |\mu| \neq 0$. *If* $[f^n(\mu f^m + \lambda)]^{(k)}$ *and* $[g^n(\mu g^m + \lambda)]^{(k)}$ *share* 1 *CM, then one of the following holds:*
(i) *If* $\lambda\mu \neq 0$, *then* $f \equiv g$.
(ii) *If* $\lambda\mu = 0$, *then either* $f \equiv tg$, *where* t *is a constant satisfying* $t^{n+m^*} = 1$ *or* $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, *where* c_1, c_2 *and* c *are three constants satisfying*

$$(-1)^k\lambda^2(c_1c_2)^{n+m^*}[(n+m^*)c]^{2k} = 1$$

or

$$(-1)^k\mu^2(c_1c_2)^{n+m^*}[(n+m^*)c]^{2k} = 1$$

and m^* *is defined by* $m^* = \chi_\mu m$, *where*

$$\chi_\mu = \begin{cases} 0 \text{ if } \mu = 0 \\ 1 \text{ if } \mu \neq 0. \end{cases}$$

Theorem E ([13]). *Let* f *and* g *be two non-constant entire functions, and let* n, m *and* k *be three positive integers with* $n > 2k + m + 4$. *If* $[f^n(f-1)^m]^{(k)}$ *and* $[g^n(g-1)^m]^{(k)}$ *share* 1 *CM, then either* $f \equiv g$ *or* f *and* g *satisfy the algebraic equation* $R(f,g) = 0$, *where* $R(w_1, w_2) = w_1^n(w_1-1)^m - w_2^n(w_2-1)^m$.

Now one may ask the following question which is the motivation of the paper. Can one obtain the same result as in Theorem D and Theorem E when CM is

replaced by IM?
It is worth mentioning that in the above area some investigations has already been carried out by A. Banerjee [2]. Banerjee proved the following result.

Theorem F ([2])**.** *Let f and g be two non-constant entire functions and n and k be two positive integers with $n > 5k + 7$. Let $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share b IM for a non-zero constant b. Then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = b^2$ or $f \equiv tg$ for some nth root of unity 1.*

In the paper we will concentrate our attention to find a possible solution of the above question.
We now state the main results of the paper which also include the above result of A. Banerjee.

Theorem 1.1. *Let f and g be two non-constant entire functions, and let n, m and k be three positive integers. Let $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share the value 1 IM where λ, μ are constants such that $|\lambda| + |\mu| \neq 0$. Then one of the following holds:*
(i) *If $\lambda\mu \neq 0$ and $n > 5k + 6m + 7$, then $f \equiv g$.*
(ii) *If $\lambda\mu = 0$ and $n > 5k - m^* + 7$, then either $f \equiv tg$, where t is a constant satisfying $t^{n+m^*} = 1$ or $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying*

$$(-1)^k\lambda^2(c_1c_2)^{n+m^*}[(n + m^*)c]^{2k} = 1$$

or

$$(-1)^k\mu^2(c_1c_2)^{n+m^*}[(n + m^*)c]^{2k} = 1,$$

where m^* is the same as in Theorem D.

Theorem 1.2. *Let f and g be two non-constant entire functions, and let $n(\geq 1)$, $m(\geq 0)$ and $k(\geq 1)$ be three integers. Let $[f^n(f - 1)^m]^{(k)}$ and $[g^n(g - 1)^m]^{(k)}$ share the value 1 IM. Then one of the following holds:*
(i) *when $m = 0$ and $n > 5k + 7$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$;*
(ii) *when $m \geq 1$ and $n > 5k + 4m + 9$, then either $f \equiv g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(x, y) = x^n(x - 1)^m - y^n(y - 1)^m$.*

Though the standard definitions and notations of the value distribution theory available in [7], we explain some definitions and notations which are used in the paper.

Definition 1.3 ([8]). For $a \in \{\infty\} \cup \mathbb{C}$ we denote by $N(r, a; f \mid= 1)$ the counting functions of simple a-points of f.
For a positive integer p we denote by $N(r, a; f \mid\leq p)$ $(N(r, a; f \mid\geq p))$ the counting function of those a-points of f whose multiplicities are not greater (less) than

p, where each a-point is counted according to its multiplicity.
$\overline{N}(r,a;f\mid\leq p)$ and $\overline{N}(r,a;f\mid\geq p)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.
Also $N(r,a;f\mid< p)$ and $N(r,a;f\mid> p)$ are defined analogously.

Definition 1.4 ([9]). Let p be a positive integer or infinity. We denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\leq p$ and p times if $m>p$. That is
$N_p(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+...+\overline{N}(r,a;f\mid\geq p)$.

Definition 1.5. For $a\in C\cup\{\infty\}$ we put

$$\delta_k(a,f)=1-\limsup_{r\to\infty}\frac{N_k(r,a;f)}{T(r,f)}.$$

Definition 1.6. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. We define by $N_{11}(r,1;f)$ the counting function for common simple 1-points of f and g where multiplicity is not counted.

Definition 1.7. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1 -point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p>q$, with multiplicity being not counted. $\overline{N}_L(r,1;g)$ is defined analogously.

2. Lemmas and Propositions

In this section we present some lemmas and propositions which will be needed in the sequel.

Proposition 2.1 ([13]). *Let f be a transcendental entire function, and n, m, k be three positive integers with $n\geq k+2$, and λ, μ are complex numbers such that $|\lambda|+|\mu|\neq 0$. Then $[f^n(\mu f^m+\lambda)]^{(k)}=1$ has infinitely many solutions.*

Proposition 2.2 ([13]). *Let f be a transcendental entire function, and n, m, k be three non-negative integers such that $n\geq k+2\geq 3$. Then $[f^n(f-1)^m]^{(k)}=1$ has infinitely many solutions.*

Lemma 2.3 ([10]). *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r,a_i(z))=S(r,f)$ for $i=0,1,2,...,n$. Then*

$$T(r,a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2.4 ([1]). *Let f and g be two non-constant meromorphic functions, and let p and k be two positive integers. Then*

$$N_p(r,0;f^{(k)}) \le N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)-\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;\frac{f^{(k)}}{f}\;|\ge m\right)+S(r,f).$$

Lemma 2.5 ([7, 12]). *Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r,a_i(z)) = S(r,f)$, i=1,2. Then*

$$T(r,f) \le \overline{N}(r,\infty;f)+\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f).$$

Lemma 2.6 ([7, 12]). *Let f be a transcendental entire function, and let k be a positive integer. Then for any non-zero finite complex number c*

$$\begin{aligned} T(r,f) &\le N(r,0;f)+N(r,c;f^{(k)})-N(r,0;f^{(k+1)})+S(r,f)\\ &\le N_{k+1}(r,0;f)+\overline{N}(r,c;f^{(k)})-N_0(r,0;f^{(k+1)})+S(r,f), \end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\ne 0$.

Lemma 2.7 ([5]). *Let $f(z)$ be a non-constant entire function, and let $k\ge 2$ be a positive integer. If $f(z)f^{(k)}(z)\ne 0$, then $f(z)=e^{az+b}$, where $a\ne 0$, b are constants.*

Lemma 2.8. *Let f and g be two non-constant entire functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and*

$$\Delta=\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+3\delta_{k+1}(0,g)>6, \tag{1}$$

then either $f^{(k)}g^{(k)}\equiv 1$ or $f\equiv g$.

Proof. Let

$$H(z)=\frac{F''}{F'}-\frac{2F'}{F-1}-\frac{G''}{G'}+\frac{2G'}{G-1}, \tag{2}$$

where $F\equiv f^{(k)}$ and $G\equiv g^{(k)}$. From (2) we see that if z_0 is a common simple 1-point of F and G, then it is a zero of H. Thus we have

$$\begin{aligned} N_{11}(r,1;F)=N_{11}(r,1;G) &\le \overline{N}(r,0;H)\le T(r,H)+O(1)\\ &\le N(r,\infty;H)+S(r,f)+S(r,g). \end{aligned} \tag{3}$$

It is easy to see that $H(z)$ has poles only at zeros of F' and G' and 1-points of F whose multiplicities are not equal to the multiplicities of the corresponding 1-points of G. So from (2) we have

$$N(r,\infty;H) \le \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) \\ + N_0(r,0;F') + N_0(r,0;G'), \qquad (4)$$

where $N_0(r,0;F')$ and $N_0(r,0;G')$ have the same meaning as in Lemma 2.6. By Lemma 2.6 we have

$$T(r,f) \le N_{k+1}(r,0;f) + \overline{N}(r,1;F) - N_0(r,0;F') + S(r,f), \qquad (5)$$

and

$$T(r,g) \le N_{k+1}(r,0;g) + \overline{N}(r,1;G) - N_0(r,0;G') + S(r,g). \qquad (6)$$

Since F and G share 1 IM, we obtain

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1;G) &\le N_{11}(r,1;F) + \overline{N}_L(r,1;G) + N(r,1;F) \\ &\le N_{11}(r,1;F) + \overline{N}_L(r,1;G) + T(r,F) + O(1) \\ &\le N_{11}(r,1;F) + \overline{N}_L(r,1;G) + m(r,F) + O(1) \\ &\le N_{11}(r,1;F) + \overline{N}_L(r,1;G) + m(r,f) + m\left(r,\frac{F}{f}\right) + O(1) \\ &\le N_{11}(r,1;F) + \overline{N}_L(r,1;G) + T(r,f) + S(r,f). \qquad (7) \end{aligned}$$

Now since f is an entire function and by Lemma 2.4 we have

$$\begin{aligned} \overline{N}_L(r,1;F) &\le N(r,1;F) - \overline{N}(r,1;F) \\ &\le N\left(r,\infty;\frac{F}{F'}\right) \\ &\le N\left(r,\infty;\frac{F'}{F}\right) + S(r,f) \\ &\le \overline{N}(r,0;F) + S(r,f) \\ &\le N_{k+1}(r,0;f) + S(r,f). \qquad (8) \end{aligned}$$

Similarly we have

$$\overline{N}_L(r,1;G) \le N_{k+1}(r,0;g) + S(r,g). \qquad (9)$$

From (3)-(9) we obtain

$$T(r,f) + T(r,g) \le \overline{N}(r,0;f) + \overline{N}(r,0;g) + 2N_{k+1}(r,0;f) + 3N_{k+1}(r,0;g) \\ + T(r,f) + S(r,f) + S(r,g),$$

i.e.

$$T(r,g) \le \overline{N}(r,0;f) + \overline{N}(r,0;g) + 2N_{k+1}(r,0;f) + 3N_{k+1}(r,0;g) \\ + S(r,f) + S(r,g).$$

We suppose that there exists a set I of infinite measure such that $T(r,f) \le T(r,g)$ for $r \in I$. Then for $r \in I$,

$$T(r,g) \le \{[7 - \Theta(0,f) - \Theta(0,g) - 2\delta_{k+1}(0,f) \\ -3\delta_{k+1}(0,g)] + \varepsilon\}T(r,g) + S(r,g),$$

for $r \in I$ and $0 < \varepsilon < \Delta - 6$. From this we get

$$(\Delta - 6)T(r,g) \le S(r,g)$$

for $r \in I$, which is a contradiction. This contradiction arises due to the assumption that $H(z) \not\equiv 0$. Hence $H(z) \equiv 0$.
This implies

$$\frac{F''}{F'} - \frac{2F'}{F-1} \equiv \frac{G''}{G'} - \frac{2G'}{G-1}.$$

Solving it we get

$$\frac{1}{F-1} \equiv \frac{BG + A - B}{G-1}, \tag{10}$$

where $A(\neq 0)$ and B are constants.
Now we consider the following cases.

Case I. Let $B \neq 0$ and $A = B$.
If $B = -1$, we obtain by (10) $FG \equiv 1$.
If $B \neq -1$, from (10) we get

$$\frac{1}{F} \equiv \frac{BG}{(1+B)G - 1}.$$

So by Lemma 2.4 we have

$$\overline{N}\left(r, \frac{1}{1+B}; G\right) \le \overline{N}(r,0;F) \le N_{k+1}(r,0;f) + S(r,f).$$

By Lemma 2.6 we obtain

$$\begin{aligned} T(r,g) &\le N_{k+1}(r,0;g) + \overline{N}\left(r, \frac{1}{1+B}; G\right) - N_0(r,0;G') + S(r,g) \\ &\le N_{k+1}(r,0;g) + N_{k+1}(r,0;f) + S(r,f) + S(r,g) \\ &\le \overline{N}(r,0;f) + \overline{N}(r,0;g) + 2N_{k+1}(r,0;f) + 3N_{k+1}(r,0;g) \\ &\quad + S(r,f) + S(r,g) \\ &\le (7 - \Delta)T(r,g) + S(r,g). \end{aligned}$$

Thus we obtain

$$(\Delta - 6)T(r,g) \le S(r,g),$$

$r \in I$, which is a contradiction.

Case II. Let $B \neq 0$ and $A \neq B$.
If $B = -1$, from (10) we obtain

$$F \equiv \frac{A}{-[G-(a+1)]}.$$

If $B \neq -1$, then we get from (10) that

$$F - \left(1 + \frac{1}{B}\right) \equiv \frac{-A}{B^2\left(G + \frac{A-B}{B}\right)}.$$

Since f is an entire function, by Lemma 2.4, Lemma 2.6 and by using the same argument as in Case I, we get a contradiction in both cases.

Case III. Let $B = 0$. Then we obtain from (10) that

$$f = \frac{1}{A} g + Q(z), \tag{11}$$

where $Q(z)$ is a polynomial. If $Q(z) \not\equiv 0$, then by Lemma 2.5, we have

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,Q;f) + S(r,f) \\ &\leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + S(r,f). \end{aligned} \tag{12}$$

Clearly by (11), we obtain

$$T(r,f) = T(r,g) + S(r,f). \tag{13}$$

So from (12), we obtain

$$T(r,f) \leq \{2 - [\Theta(0,f) + \Theta(0,g)] + \varepsilon\} T(r,f) + S(r,f),$$

where

$$0 < \varepsilon < 2 - 2\delta_{k+1}(0,f) + 3 - 3\delta_{k+1}(0,g).$$

Therefore we obtain

$$(\Delta - 6) T(r,f) \leq S(r,f)$$

for $r \in I$, which is a contradiction. Hence $Q(z) \equiv 0$ and so from (11) we get

$$f = \frac{1}{A} g. \tag{14}$$

Since $f^{(k)}$ and $g^{(k)}$ share 1 IM, we obtain from (14) that $A = 1$ and so $f \equiv g$. This proves the lemma. ■

3. Proofs of the Theorems

Proof of Theorem 1.1. We consider $F(z) = f^n(\mu f^m + \lambda)$ and $G(z) = g^n(\mu g^m + \lambda)$. Then $[F(z)]^{(k)}$ and $[G(z)]^{(k)}$ share 1 IM. Now

$$\varDelta = \varTheta(0, F) + \varTheta(0, G) + 2\delta_{k+1}(0, F) + 3\delta_{k+1}(0, G).$$

We consider the following three cases.

Case 1. Let $\lambda\mu \neq 0$. By using Lemma 2.3 we have

$$\begin{aligned} \varTheta(0, F) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; F)}{T(r, F)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; f^n(\mu f^m + \lambda))}{(n+m)T(r, f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{(m+1)T(r, f)}{(n+m)T(r, f)} \\ &\geq \frac{n-1}{n+m}. \end{aligned} \tag{15}$$

Similarly,

$$\varTheta(0, G) \geq \frac{n-1}{n+m}. \tag{16}$$

$$\begin{aligned} \delta_{k+1}(0, F) &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, 0; F)}{T(r, F)} \\ &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, 0; f^n(\mu f^m + \lambda))}{(n+m)T(r, f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{(k+m+1)T(r, f)}{(n+m)T(r, f)} \\ &\geq \frac{n-k-1}{n+m}. \end{aligned} \tag{17}$$

Similarly,

$$\delta_{k+1}(0, G) \geq \frac{n-k-1}{n+m}. \tag{18}$$

From (15)-(18) we obtain

$$\begin{aligned} \varDelta &= 2\frac{n-1}{n+m} + 5\frac{n-k-1}{n+m} \\ &= \frac{7n-5k-7}{n+m}. \end{aligned}$$

Since $n > 5k+6m+7$, we get $\Delta > 6$. So by Lemma 2.8 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$. Now we consider the following two subcases.

Subcase (i) Let
$$F^{(k)}G^{(k)} \equiv 1,$$
i.e.
$$[f^n(\mu f^m+\lambda)]^{(k)}[g^n(\mu g^m+\lambda)]^{(k)} \equiv 1. \tag{19}$$
Since f and g are entire functions and $n > 5k+6m+7$, it is clear that
$$f \neq 0 \qquad \text{and} \qquad g \neq 0. \tag{20}$$
Let $f(z) = e^{\alpha(z)}$, where $\alpha(z)$ is an entire function. Then we obtain
$$[\mu f^{n+m}]^{(k)} = t_1(\alpha', \alpha'', ..., \alpha^{(k)})e^{(n+m)\alpha(z)} \tag{21}$$
and
$$[\lambda f^n]^{(k)} = t_2(\alpha', \alpha'', ..., \alpha^{(k)})e^{n\alpha(z)}, \tag{22}$$
where $t_i(\alpha', \alpha'', ..., \alpha^{(k)}) \not\equiv 0$ $(i=1,2)$ are differential polynomials. Since g is an entire function, we have from (19) that
$$[f^n(\mu f^m+\lambda)]^{(k)} \neq 0.$$
So from (21) and (22) we get
$$t_1(\alpha', \alpha'', ..., \alpha^{(k)})e^{m\alpha(z)} + t_2(\alpha', \alpha'', ..., \alpha^{(k)}) \neq 0. \tag{23}$$
Since α is an entire function, we have
$$T(r, \alpha') = S(r, f)$$
and
$$T(r, \alpha^{(j)}) \leq T(r, \alpha') + S(r, f) = S(r, f)$$
for $j = 1, 2, ..., k$. Hence we have
$$T(r, t_i) = S(r, f) \tag{24}$$
for $i = 1, 2$. So by (23), (24), Lemma 2.3 and Lemma 2.5 we get
$$\begin{aligned} mT(r, f) &\leq T(r, t_1 e^{m\alpha}) + S(r, f) \\ &\leq \overline{N}(r, 0; t_1 e^{m\alpha}) + \overline{N}(r, 0; t_1 e^{m\alpha} + t_2) + S(r, f) \\ &\leq T\left(r, \frac{1}{t_1}\right) + S(r, f) \\ &= S(r, f), \end{aligned}$$

which is a contradiction.

Subcase (ii) Let

$$F \equiv G.$$

That is

$$f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda). \tag{25}$$

Let $h = \frac{f}{g}$. If $h \not\equiv 1$, from (25) we obtain

$$g^m = -\frac{\lambda}{\mu}\frac{1-h^n}{1-h^{n+m}}.$$

Since g is an entire function, every zero of $h^{n+m}-1$ is a zero of h^n-1 and hence of h^m-1. Since $n > 5k+6m+7$, we obtain that h is a constant, which is a contradiction as f and g are non-constant. Therefore $h \equiv 1$, that is $f \equiv g$.

Case 2. Let $\lambda = 0$ and $\mu \neq 0$. In this case $F = \mu f^{n+m}$ and $G = \mu g^{n+m}$. Proceeding in the same way as Case 1, we obtain

$$\Theta(0,F) \geq \frac{n+m-1}{n+m}, \tag{26}$$

$$\Theta(0,G) \geq \frac{n+m-1}{n+m}, \tag{27}$$

$$\delta_{k+1}(0,F) \geq \frac{n+m-k-1}{n+m}, \tag{28}$$

$$\delta_{k+1}(0,G) \geq \frac{n+m-k-1}{n+m}. \tag{29}$$

Since $n > 5k-m+7$, by (1), (26)-(29) we obtain $\Delta > 6$. So by Lemma 2.8 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$. Now we consider the following two subcases.

Subcase (i) Let

$$F^{(k)}G^{(k)} \equiv 1,$$

i.e.

$$[\mu f^{n+m}]^{(k)}[\mu g^{n+m}]^{(k)} \equiv 1. \tag{30}$$

By the nature of f and g it is clear that

$$[\mu f^{n+m}]^{(k)} \neq 0 \qquad \text{and} \qquad [\mu g^{n+m}]^{(k)} \neq 0. \tag{31}$$

If $k \geq 2$, then by (20), (30), (31) and Lemma 2.7 we obtain that $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k \mu^2 (c_1 c_2)^{n+m} [(n+m)c]^{2k} = 1.$$

Let $k = 1$. Let $f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$ where $\alpha(z)$ and $\beta(z)$ are two entire functions. So from (30) we have

$$\mu^2 (n+m)^2 \alpha' \beta' e^{(n+m)(\alpha+\beta)} \equiv 1. \tag{32}$$

Thus α' and β' have no zeros and we may take $\alpha' = e^{\gamma(z)}$ and $\beta' = e^{\delta(z)}$, where γ and δ are two entire functions. So from (32) we get

$$\mu^2 (n+m)^2 e^{(n+m)(\alpha+\beta)+\gamma+\delta} \equiv 1.$$

Differentiating it we get

$$(n+m)e^{\gamma} + \gamma' \equiv -((n+m)e^{\delta} + \delta'). \tag{33}$$

Since γ and δ are entire, we have $T(r, \gamma') = S(r, e^{\gamma})$ and $T(r, \delta') = S(r, e^{\delta})$. From this we have

$$T(r, e^{\gamma}) = T(r, e^{\delta}) + S(r, e^{\gamma}) + S(r, e^{\delta}).$$

This implies that $S(r, e^{\gamma}) = S(r, e^{\delta}) = S(r)$, say.
Let $\rho = -(\gamma' + \delta')$. Then $T(r, \rho) = S(r)$. If $\rho \not\equiv 0$, (33) can be written as

$$\frac{e^{\gamma}}{\rho} + \frac{e^{\delta}}{\rho} \equiv \frac{1}{n+m}.$$

From this and the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} T(r, e^{\gamma}) &\leq T\left(r, \frac{e^{\delta}}{\rho}\right) + S(r) \\ &\leq \overline{N}\left(r, \infty; \frac{e^{\delta}}{\rho}\right) + \overline{N}\left(r, 0; \frac{e^{\delta}}{\rho}\right) + \overline{N}\left(r, \frac{1}{n+m}; \frac{e^{\delta}}{\rho}\right) + S(r) \\ &\leq S(r), \end{aligned}$$

a contradiction. So by (33) we have

$$\alpha' + \beta' = e^{\gamma} + e^{\delta} = -\left(\frac{\gamma'}{n+m} + \frac{\delta'}{n+m}\right) \equiv 0,$$

i.e. $\gamma = \delta + (2s+1)\pi i$ for some integer s. Again $\gamma' + \delta' \equiv 0$ implies $\gamma + \delta = d$, where d is a constant. Taking $\gamma = d_1$ we get $\delta = d - d_1 = d_2$, where d_1, d_2 are constants. Again $\alpha' + \beta' \equiv 0$ implies $\alpha = cz + \log c_1$ and $\beta = -cz + \log c_2$. Since $f = e^{\alpha}$ and $g = e^{\beta}$, by (32) we obtain that $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k \mu^2 (c_1 c_2)^{n+m} [(n+m)c]^{2k} = 1.$$

Subcase (ii) Let

$$F \equiv G,$$

i.e.

$$\mu f^{n+m} \equiv \mu g^{n+m},$$

hence

$$f \equiv tg,$$

where t is a constant satisfying $t^{n+m} = 1$.

Case 3. Let $\lambda \neq 0$ and $\mu = 0$. Then $F = \lambda f^n$ and $G = \lambda g^n$. This case can be proved by using the same argument as in Case 2. This completes the proof of Theorem 1.1. ■

Proof of Theorem 1.2. Consider $F(z) = f^n(f-1)^m$ and $G(z) = g^n(g-1)^m$.

$$\begin{aligned} \Theta(0,F) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,0;F)}{T(r,F)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,0;f^n(f-1)^m)}{(n+m)T(r,f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{(1+m^{**})T(r,f)}{(n+m)T(r,f)} \\ &\geq \frac{n+m-1-m^{**}}{n+m}, \end{aligned} \tag{34}$$

where

$$m^{**} = \begin{cases} 0 \text{ if } m = 0 \\ 1 \text{ if } m \geq 1. \end{cases}$$

Similarly,

$$\Theta(0,G) \geq \frac{n+m-1-m^{**}}{n+m}. \tag{35}$$

$$\begin{aligned} \delta_{k+1}(0,F) &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r,0;F)}{T(r,F)} \\ &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r,0;f^n(f-1)^m)}{(n+m)T(r,f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{(k+m+1)T(r,f)}{(n+m)T(r,f)} \\ &\geq \frac{n-k-1}{n+m}. \end{aligned} \tag{36}$$

Similarly,

$$\delta_{k+1}(0,G) \geq \frac{n-k-1}{n+m}. \tag{37}$$

From (34)-(37) it is easily checked that $\Delta > 6$ provided $n > 5k+4m+2m^{**}+7$. Since

$$5k+4m+2m^{**}+7 = \begin{cases} 5k+7 & \text{if } m=0 \\ 5k+4m+9 & \text{if } m \geq 1, \end{cases}$$

by Lemma 2.8 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$. Now we consider the following two cases.

Case 1. Let

$$F^{(k)}G^{(k)} \equiv 1,$$

i.e.

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1. \tag{38}$$

Then we consider the following two subcases.

Subcase (i) Let $m=0$. Then by (38) and proceeding in the same way as in case 2 [subcase (i)] of Theorem 1.1 we obtain $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.

Subcase (ii) Let $m \geq 1$. Since f and g are entire functions, we have$f \neq 0$ and $g \neq 0$. Let $f(z) = e^{\alpha(z)}$, where $\alpha(z)$ is a non-constant entire function. Clearly

$$[f^{n+m}(z)]^{(k)} = s_m(\alpha', \alpha'', ..., \alpha^{(k)}) e^{(n+m)\alpha(z)}, \tag{39}$$

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$$(-1)^{m-i}[{}^mC_i f^{n+i}(z)]^{(k)} = s_i(\alpha', \alpha'', ..., \alpha^{(k)}) e^{(n+i)\alpha(z)}, \tag{40}$$

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$$(-1)^m [f^n(z)]^{(k)} = s_0(\alpha', \alpha'', ..., \alpha^{(k)}) e^{n\alpha(z)}, \tag{41}$$

where $s_i(\alpha', \alpha'', ..., \alpha^{(k)})$ $(i = 0,1,2,...,m)$ are differential polynomials. Obviously

$$s_i(\alpha', \alpha'', ..., \alpha^{(k)}) \not\equiv 0$$

for $i = 0,1,2,...,m$, and

$$[f^n(f-1)^m]^{(k)} \not\equiv 0.$$

From (39) – (41) we have

$$s_m(\alpha', \alpha'', ..., \alpha^{(k)})e^{m\alpha(z)} + ... + s_0(\alpha', \alpha'', ..., \alpha^{(k)}) \neq 0. \tag{42}$$

Since $\alpha(z)$ is an entire function, we obtain

$$T(r, \alpha') = S(r, f)$$

and

$$T(r, \alpha^{(j)}) = S(r, f)$$

for $j = 1, 2, ..., k$. Hence $T(r, s_i) = S(r, f)$ for $j = 1, 2, ..., k$. So from (42), Lemmas 2.3 and 2.5 we obtain

$$\begin{aligned} mT(r, f) &= T(r, s_m e^{m\alpha} + ...s_1 e^{\alpha}) + S(r, f) \\ &\leq \overline{N}(r, 0; s_m e^{m\alpha} + ... + s_1 e^{\alpha}) + \overline{N}(r, 0; s_m e^{m\alpha} + ... + s_1 e^{\alpha} + s_0) + S(r, f) \\ &\leq \overline{N}(r, 0; s_m e^{(m-1)\alpha} + ... + s_1) + S(r, f) \\ &\leq (m-1)T(r, f) + S(r, f), \end{aligned}$$

which is a contradiction.

Case 2. Let

$$F \equiv G,$$

i.e.

$$f^n(f-1)^m \equiv g^n(g-1)^m. \tag{43}$$

We now consider the following two subcases.

Subcase (i) Let $m = 0$. Then by (43) we have $f \equiv tg$ for a constant t such that $t^n = 1$.

Subcase (ii) Let $m \geq 1$. Then from (43) we get

$$\begin{aligned} f^n[f^m + ... + (-1)^i \ {}^mC_{m-i}\, f^{m-i} + ... + (-1)^m] = g^n[g^m + ... \\ +(-1)^i \ {}^mC_{m-i}\, g^{m-i} + ... + (-1)^m]. \end{aligned} \tag{44}$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = gh$ in (44) we get

$$g^{n+m}(h^{n+m}-1)+...+(-1)^i \ {}^mC_{m-i}\, g^{n+m-i}(h^{n+m-i}-1)+...+(-1)^m g^n(h^n-1) = 0,$$

which implies $h = 1$. Thus $f \equiv g$.
If h is not a constant, then from (43) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(x, y) = x^n(x-1)^m - y^n(y-1)^m$. This completes the proof of Theorem 1.2. ■

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