Vietnam Journal of Mathematics **38**:1(2010) 79-87

Vietnam Journal
of
MATHEMATICS

The Solvability of a Higher-order Nonlinear Neutral Delay Difference Equation

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Received September 23, 2009
Revised September 06, 2009

Abstract. This paper studies the solvability of the following higher-order nonlinear neutral delay difference equation

$$\Delta\Big(a_{kn}\cdots\Delta\big(a_{2n}\Delta(a_{1n}\Delta(x_n+b_nx_{n-d}))\big)\Big)+\sum_{j=1}^{s}p_{jn}f_j(x_{n-r_{jn}})=q_n,\quad n\geq n_0,$$

where $n_0\geq 0, n\geq 0, d>0, k>0, j>0, s>0$ are integers, $\{a_{in}\}_{n\geq n_0}(i=1,2,\cdots,k)$, $\{b_n\}_{n\geq n_0}$, $\{p_{jn}\}_{n\geq n_0}(j=1,2,\cdots,s)$ and $\{q_n\}_{n\geq n_0}$ are real sequences, $\bigcup_{j=1}^{s}\{r_{jn}\}_{n\geq n_0}\subseteq\mathbb{Z}$, $f_j\in C(\mathbb{R},\mathbb{R})$ and $xf_j(x)\geq 0$ for any $x\neq 0\quad(j=1,2,\cdots,s)$. Some sufficient conditions for existence of nonoscillatory solutions of this equation are established and expatiated through five theorems according as the range of value of the sequence b_n.

2000 Mathematics Subject Classification: 34K15, 34C10.

Key words: nonoscillatory solution, neutral delay difference equation, contraction mapping.

1. Introduction and Preliminaries

Recently, the interest in the study of qualitative analysis of difference equations has been increasing (see [1–16] and references cited therein). Some authors have payed their attention to various difference equations. For example,

$$\Delta(a_n\Delta x_n)+p_nx_{g(n)}=0,\quad n\geq 0,([13])\tag{1}$$

$$\Delta(a_n\Delta x_n) = q_n x_{n+1}, \quad \Delta(a_n\Delta x_n) = q_n f(x_{n+1}), \quad n \geq 0, ([10]) \tag{2}$$

$$\Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \geq n_0, ([5]) \tag{3}$$

$$\Delta^2(x_n + px_{n-k}) + f(n, x_n) = 0, \quad n \geq 1, \quad ([9]) \tag{4}$$

$$\Delta^2(x_n - px_{n-\tau}) = \sum_{i=1}^{m} q_i f_i(x_{n-\sigma_i}), \quad n \geq n_0, \quad ([8]) \tag{5}$$

$$\Delta(a_n\Delta(x_n + bx_{n-\tau})) + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \cdots, x_{n-d_{kn}}) = c_n, \quad n \geq n_0, \qquad ([7]) \tag{6}$$

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} = 0, \ n \geq n_0, \ ([14]) \tag{7}$$

$$\Delta^m(x_n + c_n x_{n-k}) + p_n f(x_{n-r}) = 0, \quad n \geq n_0, \quad ([3, 4, 11, 12]) \tag{8}$$

$$\Delta^m(x_n + cx_{n-k}) + \sum_{s=1}^{u} p_n^s f_s(x_{n-r_s}) = q_n, \quad n \geq n_0, \quad ([15]) \tag{9}$$

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} - q_n x_{n-l} = 0, \quad n \geq n_0. \quad ([16]) \tag{10}$$

The purpose of this paper is to investigate the following higher-order nonlinear neutral delay difference equation

$$\Delta\Big(a_{kn} \cdots \Delta\big(a_{2n}\Delta(a_{1n}\Delta(x_n + b_n x_{n-d}))\big)\Big) + \sum_{j=1}^{s} p_{jn} f_j(x_{n-r_{jn}}) = q_n, \quad n \geq n_0, \tag{11}$$

where $n_0 \geq 0, n \geq 0, d > 0, k > 0, j > 0, s > 0$ are integers, $\{a_{in}\}_{n\geq n_0}(i = 1, 2, \cdots, k)$, $\{b_n\}_{n\geq n_0}$, $\{p_{jn}\}_{n\geq n_0}(j = 1, 2, \cdots, s)$ and $\{q_n\}_{n\geq n_0}$ are real sequences, $\bigcup_{j=1}^{s}\{r_{jn}\}_{n\geq n_0} \subseteq \mathbb{Z}$, $f_j \in C(\mathbb{R}, \mathbb{R})$ and $xf_j(x) \geq 0$ for any $x \neq 0 \quad (j = 1, 2, \cdots, s)$. Clearly, difference equations (1)–(10) are special cases of Eq. (11). By using Banach contraction principle, the existence of nonoscillatory solutions of Eq. (11) is established.

The forward difference Δ is defined as usual, i.e., $\Delta x_n = x_{n+1} - x_n$. The higher-order difference for a positive integer m is defined as $\Delta^m x_n = \Delta(\Delta^{m-1}x_n), \Delta^0 x_n = x_n$. Throughout this paper, assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ and $\mathbb{Z}$ stand for the sets of all positive integers and integers, respectively, $\alpha = \inf\{n - r_{jn} : 1 \leq j \leq s, n \geq n_0\}$, $\beta = \min\{n_0 - d, \alpha\}$, $\lim_{n\to\infty}(n - r_{jn}) = +\infty, 1 \leq j \leq s$, l_β^∞ denotes the set of real sequences defined on the set of positive integers larger than β where any individual sequence is bounded with respect to the usual supremum norm $||x|| = \sup_{n\geq\beta}|x_n|$ for $x = \{x_n\}_{n\geq\beta} \in l_\beta^\infty$. It is well known that l_β^∞ is a Banach space under the supremum norm. Let

$$A(M, N) = \big\{x = \{x_n\}_{n\geq\beta} \in l_\beta^\infty : M \leq x_n \leq N, n \geq \beta\big\} \quad \text{for} \quad N > M > 0.$$

Obviously, $A(M, N)$ is a bounded closed and convex subset of l_β^∞. Put

$$\overline{b} = \limsup_{n\to\infty} b_n \quad \text{and} \quad \underline{b} = \liminf_{n\to\infty} b_n.$$

By a solution of Eq. (11), we mean a sequence $\{x_n\}_{n\ge\beta}$ with a positive integer $N_0 \ge n_0 + d + |\alpha|$ such that Eq. (11) is satisfied for all $n \ge N_0$. As is customary, a solution of Eq. (11) is said to be oscillatory about zero, or simply oscillatory if the terms x_n of the sequence $\{x_n\}_{n\ge\beta}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

2. Existence of nonoscillatory solutions

In this section, a few sufficient conditions of the existence of nonoscillatory solutions of Eq. (11) are given.

Theorem 2.1. *Assume that there exist constants M and N with $N > M > 0$ and that*

$$|b_n| \le b < \frac{N-M}{2N}, \text{ eventually,} \tag{12}$$

$$\sum_{t=n_0}^{\infty} \max\Big\{\frac{1}{|a_{it}|}, |p_{jt}|, |q_t| : 1 \le i \le k, 1 \le j \le s\Big\} < +\infty, \tag{13}$$

and

$$f_j \ (j = 1, 2, \cdots, s) \text{ is Lipschitz continuous on } \mathbb{R}. \tag{14}$$

Then Eq. (11) *has a nonoscillatory solution in* $A(M, N)$.

Proof. Choose $L \in (M + bN, N - bN)$. By (12)–(14), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$|b_n| \le b < \frac{N-M}{2N}, \ \forall n \ge N_0, \tag{15}$$

$$\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{F\big|\sum_{j=1}^{s} p_{jt}\big| + |q_t|}{\big|\prod_{i=1}^{k} a_{it_i}\big|} \le \min\{L - bN - M, N - bN - L\}, \tag{16}$$

where $F = \max_{M\le x\le N}\{f_j(x) : 1 \le j \le s\}$, and

$$\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\big|\sum_{j=1}^{s} p_{jt}\big|}{\big|\prod_{i=1}^{k} a_{it_i}\big|} < \frac{1-b}{K}, \tag{17}$$

where K is the biggest Lipschitz coefficient of f_j.

Define a mapping $T : A(M, N) \to X$ by

$$(Tx)_n = \begin{cases} L - b_n x_{n-d} + \\ \quad (-1)^k \sum\limits_{t_1=n}^{\infty} \sum\limits_{t_2=t_1}^{\infty} \cdots \sum\limits_{t_k=t_{k-1}}^{\infty} \sum\limits_{t=t_k}^{\infty} \dfrac{\sum\limits_{j=1}^{s} p_{jt} f_j(x_{t-r_{jt}}) - q_t}{\prod_{i=1}^{k} a_{it_i}}, & n \geq N_0 \\ (Tx)_{N_0}, & \beta \leq n < N_0 \end{cases} \tag{18}$$

for all $x \in A(M, N)$.

For every $x \in A(M, N)$ and $n \geq N_0$, it follows from (14), (15) and (16) that

$$\begin{aligned} (Tx)_n \geq & L - bN - \sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left| \sum_{j=1}^{s} p_{jt} f_j(x_{t-r_{jt}}) - q_t \right|}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\ \geq & L - bN - \sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{F\left| \sum_{j=1}^{s} p_{jt} \right| + |q_t|}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\ \geq & M \end{aligned}$$

and

$$\begin{aligned} (Tx)_n \leq & L + bN + \sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{F\left| \sum_{j=1}^{s} p_{jt} \right| + |q_t|}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\ \leq & N. \end{aligned}$$

That is, $T(A(M, N)) \subseteq A(M, N)$. It is claimed that T is a contraction mapping on $A(M, N)$. In fact, (13), (15) and (17) guarantee that for any $x, y \in A(M, N)$ and $n \geq N_0$

$$\begin{aligned} & \left| (Tx)_n - (Ty)_n \right| \\ \leq & |b_n| |x_{n-d} - y_{n-d}| \\ & + \sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left| \sum_{j=1}^{s} p_{jt} \right| |f_j(x_{t-r_{jt}}) - f_j(y_{t-r_{jt}})|}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\ \leq & b\|x - y\| + \\ & \sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{K\left| \sum_{j=1}^{s} p_{jt} \right| \max\{|x_{t-r_{jt}} - y_{t-r_{jt}}| : 1 \leq j \leq s\}}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\ \leq & b\|x - y\| + \|x - y\| \sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{K\left| \sum_{j=1}^{s} p_{jt} \right|}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\ = & \theta\|x - y\|, \end{aligned}$$

where $\theta = b + \sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{K\left| \sum_{j=1}^{s} p_{jt} \right|}{\left| \prod_{i=1}^{k} a_{it_i} \right|} < 1$. This implies that

$$\|Tx - Ty\| \leq \theta\|x - y\|, \quad \forall x, y \in A(M, N),$$

that is, T is a contraction mapping on $A(M, N)$. Consequently T has a unique fixed point $x \in A(M, N)$, which is a bounded nonoscillatory solution of Eq. (11). This completes the proof. ■

Theorem 2.2. *Assume that*

$$b_n \geq 0, \ \textit{eventually, and } 0 \leq \underline{b} \leq \overline{b} < 1, \tag{19}$$

there exist constants M and N with $N > \frac{2-\underline{b}}{1-\overline{b}} M > 0$, and that (13)–(14) *hold. Then Eq.* (11) *has a nonoscillatory solution in $A(M, N)$.*

Proof. Choose $L \in (M + \frac{1+\overline{b}}{2} N, N + \frac{\underline{b}}{2} M)$. By (19), (13) and (14), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\frac{\underline{b}}{2} \leq b_n \leq \frac{1+\overline{b}}{2}, \ \forall n \geq N_0, \tag{20}$$

$$\begin{aligned} &\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{F\left|\sum_{j=1}^{s} p_{jt}\right| + |q_t|}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\ &\leq \min\Big\{L - M - \frac{1+\overline{b}}{2} N, N - L + \frac{\underline{b}}{2} M\Big\}, \end{aligned} \tag{21}$$

where $F = \max_{M \leq x \leq N}\{f_j(x) : 1 \leq j \leq s\}$, and

$$\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left|\sum_{j=1}^{s} p_{jt}\right|}{\left|\prod_{i=1}^{k} a_{it_i}\right|} < \frac{1-\overline{b}}{2K}, \tag{22}$$

where K is the biggest Lipschitz coefficient of f_j.

Define a mapping $T : A(M, N) \to X$ as (18). The rest of proof is similar to that in Theorem 2.1. This completes the proof. ■

Theorem 2.3. *Assume that*

$$b_n \leq 0, \ \textit{eventually, and } -1 < \underline{b} \leq \overline{b} \leq 0, \tag{23}$$

there exist constants M and N with $N > \frac{2+\overline{b}}{1+\underline{b}} M > 0$, and that (13)–(14) *hold. Then Eq.* (11) *has a nonoscillatory solution in $A(M, N)$.*

Proof. Choose $L \in (\frac{2+\overline{b}}{2} M, \frac{1+\underline{b}}{2} N)$. By (23), (13) and (14), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\frac{\underline{b} - 1}{2} \leq b_n \leq \frac{\overline{b}}{2}, \ \forall n \geq N_0, \tag{24}$$

$$\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{F\left|\sum_{j=1}^{s}p_{jt}\right|+|q_t|}{\left|\prod_{i=1}^{k}a_{it_i}\right|}\le\min\left\{L-\frac{2+\overline{b}}{2}M,\frac{1+\underline{b}}{2}N-L\right\},\tag{25}$$

where $F=\max_{M\le x\le N}\{f_j(x):1\le j\le s\}$, and

$$\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{\left|\sum_{j=1}^{s}p_{jt}\right|}{\left|\prod_{i=1}^{k}a_{it_i}\right|}<\frac{1+\underline{b}}{2K},\tag{26}$$

where K is the biggest Lipschitz coefficient of f_j.

Define a mapping $T:A(M,N)\to X$ as (18). The rest of proof is similar to that in Theorem 2.1. This completes the proof. ■

Theorem 2.4. *Assume that*

$$b_n>1,\ \textit{eventually},\ 1<\underline{b}\ \textit{and}\ \overline{b}<\underline{b}^2<+\infty,\tag{27}$$

there exist constants M and N with $N>\frac{\underline{b}(\overline{b}^2-\underline{b})}{\overline{b}(\underline{b}^2-\overline{b})}M>0$, and that (13)–(14) *hold. Then Eq.* (11) *has a nonoscillatory solution in $A(M,N)$.*

Proof. Take $\varepsilon\in(0,\underline{b}-1)$ sufficiently small satisfying

$$1<\underline{b}-\varepsilon<\overline{b}+\varepsilon<(\underline{b}-\varepsilon)^2\tag{28}$$

and

$$\left((\overline{b}+\varepsilon)(\underline{b}-\varepsilon)^2-(\overline{b}+\varepsilon)^2\right)N>\left((\overline{b}+\varepsilon)^2(\underline{b}-\varepsilon)-(\underline{b}-\varepsilon)^2\right)M.\tag{29}$$

Choose $L\in\left((\overline{b}+\varepsilon)M+\frac{\overline{b}+\varepsilon}{\underline{b}-\varepsilon}N,(\underline{b}-\varepsilon)N+\frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}M\right)$. By (28), (13) and (14), an integer $N_0>n_0+d+|\alpha|$ can be chosen such that

$$\underline{b}-\varepsilon<b_n<\overline{b}+\varepsilon,\quad\forall b\ge N_0,\tag{30}$$

$$\begin{aligned}&\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{F\left|\sum_{j=1}^{s}p_{jt}\right|+|q_t|}{\left|\prod_{i=1}^{k}a_{it_i}\right|}\\&\quad\le\min\left\{\frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}L-(\underline{b}-\varepsilon)M-N,\frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}M+(\underline{b}-\varepsilon)N-L\right\},\end{aligned}\tag{31}$$

where $F=\max_{M\le x\le N}\{f_j(x):1\le j\le s\}$, and

$$\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{\left|\sum_{j=1}^{s}p_{jt}\right|}{\left|\prod_{i=1}^{k}a_{it_i}\right|}<\frac{\underline{b}-\varepsilon-1}{K},\tag{32}$$

where K is the biggest Lipschitz coefficient of f_j.

Define a mapping $T : A(M,N) \to X$ by

$$(Tx)_n = \begin{cases} \frac{L}{b_{n+d}} - \frac{x_{n+d}}{b_{n+d}} \\ \quad + \frac{(-1)^k}{b_{n+d}} \sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\sum_{j=1}^{s} p_{jt} f_j(x_{t-r_{jt}}) - q_t}{\prod_{i=1}^{k} a_{it_i}}, & n \geq N_0 \\ (Tx)_{N_0}, & \beta \leq n < N_0 \end{cases} \tag{33}$$

for all $x \in A(M,N)$. The rest of proof is similar to that in Theorem 2.1. This completes the proof. ■

Theorem 2.5. *Assume that*

$$b_n < -1, \text{ eventually, } -\infty < \underline{b} \text{ and } \overline{b} < -1, \tag{34}$$

there exist constants M and N with $N > \frac{1+\underline{b}}{1+\overline{b}} M > 0$, and that (13)–(14) hold. Then Eq. (11) has a nonoscillatory solution in $A(M,N)$.

Proof. Take $\epsilon \in \big(0, -(1+\overline{b})\big)$ sufficiently small satisfying

$$\underline{b} - \epsilon < \overline{b} + \epsilon < -1 \tag{35}$$

and

$$(1+\overline{b}+\epsilon)N < (1+\underline{b}-\epsilon)M. \tag{36}$$

Choose $L \in \big((1+\overline{b}+\epsilon)N, (1+\underline{b}-\epsilon)M\big)$. By (35), (13) and (14), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\underline{b} - \epsilon < b_n < \overline{b} + \epsilon, \quad \forall n \geq N_0, \tag{37}$$

$$\begin{aligned} &\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{F\left|\sum_{j=1}^{s} p_{jt}\right| + |q_t|}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\ &\leq \min\left\{\left(\overline{b} + \epsilon + \frac{\overline{b}+\epsilon}{\underline{b}-\epsilon}\right)M - \frac{\overline{b}+\epsilon}{\underline{b}-\epsilon}L, L - (1+\overline{b}+\epsilon)N\right\}, \end{aligned} \tag{38}$$

where $F = \max_{M \leq x \leq N}\{f_j(x) : 1 \leq j \leq s\}$, and

$$\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left|\sum_{j=1}^{s} p_{jt}\right|}{\left|\prod_{i=1}^{k} a_{it_i}\right|} < \frac{-\overline{b} - \epsilon - 1}{K}, \tag{39}$$

where K is the biggest Lipschitz coefficient of f_j.

Define a mapping $T : A(M,N) \to X$ as (33). The rest of proof is similar to that in Theorem 2.1. This completes the proof. ■

Remark 2.6. Theorems 2.1–2.5 extend and improve Theorem 1 of Cheng [5], Theorems 2.3–2.7 of Liu, Xu and Kang [7], Theorems 1–5 of Zhou and Huang [15] and corresponding theorems in [3,4,8–14,16].

3. Examples

In this section, some examples are presented to illustrate the advantage of the above results.

Example 3.1. Consider the following third-order nonlinear neutral delay difference equation:

$$\Delta\Big(3^n\Delta\big(2^n\Delta(x_n+2^{-n}x_{n-1})\big)\Big)=0,\quad n\geq 1. \tag{40}$$

Choose $M=1$ and $N=2$. It is easy to verify that the conditions of Theorem 2.1 are satisfied. Therefore Theorem 2.1 ensures that Eq. (40) has a nonoscillatory solution in $A(1,2)$. However, the results in [5,7,15] are not applicable for Eq. (40).

Example 3.2. Consider the following fourth-order nonlinear neutral delay difference equation:

$$\begin{aligned}&\Delta\Big((2^n-n)\Delta\Big(n(n+1)(n+2)\Delta\big((n^2-n+1)\Delta(x_n+\frac{2^n-1}{3^n}x_{n-2})\big)\Big)\Big)\\&+\frac{\sin nx}{n}\big(\cos(x_{n-3})+3\big)+\frac{\cos nx}{n}\big(\sin(x_{n-4})+4\big)\\=&\frac{2n-1}{2^n},\quad n\geq 5,\end{aligned} \tag{41}$$

where

$$\begin{aligned}&a_{1n}=n^2-n+1,\quad a_{2n}=n(n+1)(n+2),\quad a_{3n}=2^n-n,\\&b_n=\frac{2^n-1}{3^n},\quad p_{1n}=\frac{\sin nx}{n},\quad p_{2n}=\frac{\cos nx}{n},\\&f_1(x)=\cos x+3,\quad f_2(x)=\sin x+4,\quad q_n=\frac{2n-1}{2^n}.\end{aligned}$$

Choose $M=2$ and $N=5$. It can be verified that the assumptions of Theorem 2.2 are fulfilled. It follows from Theorem 2.2 that Eq. (41) has a nonoscillatory solution in $A(2,5)$. However, the results in [5,7,15] are unapplicable for Eq. (41).

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