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Equivalent Conditions for (Weak) Corestriction Principle for Non-Abelian Étale Cohomology of Group Schemes *

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Abstract. We introduce the notion of (Weak) Corestriction Principle and prove some equivalent relations between the validity of this principle for various connecting maps in non-abelian étale cohomology.

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1. Introduction

Let G be a commutative algebraic group scheme defined over a ring R (always associative, commutative and with unity). For simplicity we consider only flat affine group schemes of finite type over affine schemes $\mathrm{Spec}(R)$, which will be called R-group schemes. Let $\mathrm{H}^i_*(R, G)$, where $* = et$ or $flat$ (or $fppf$), denote the usual étale or flat cohomology $\mathrm{H}^i_{et}(\mathrm{Spec}(R), G))$ in degree i, whenever it makes sense. As it is well-known, there exists corestriction homomorphism $\mathrm{Cores} := \mathrm{Cores}_{R'/R,G} : \mathrm{H}^i_{et}(R', G) \to \mathrm{H}^i_{et}(R, G)$ for any $i \geq 0$ and any finite étale extension R' of R (see e.g., next section), which gives rise to a map of

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functors $(G \mapsto \mathrm{H}^i_{et}(R', G)) \to (G \mapsto \mathrm{H}^i_{et}(R, G))$. In particular, if

$$1 \to A \xrightarrow{j} B \xrightarrow{p} C \to 1 \qquad (*)$$

is an exact sequence of commutative algebraic R-group schemes, and if $\{\alpha_1, \alpha_2, ...\}$ (resp. $\{\alpha'_1, \alpha'_2, ...\}$) denotes the sequence of homomorphisms appearing in the long exact sequence of cohomology over $\mathrm{Spec}(R)$ (resp. over $\mathrm{Spec}(R')$) deduced from $(*)$, then we have

$$\mathrm{Cores} \circ \alpha'_m = \alpha_m \circ \mathrm{Cores}$$

for all $m \geq 1$. However, if in $(*)$ one of the groups is not commutative, then it turns out that there is no corestriction map between these two long exact sequences in general. (In [20] Riehm has found some sufficient conditions for the existence of corestriction map in the case of fields). It leads us to the following definition. Let A, B be flat affine group schemes defined over R. Assume that we are given a map of functors

$$f : (S \mapsto \mathrm{H}^i_{et}(S, A)) \to (S \mapsto \mathrm{H}^j_{et}(S, B)),$$

where S denotes a finite étale extension of R, i.e. a collection of maps of cohomology sets

$$f_S : \mathrm{H}^i_{et}(S, A) \to \mathrm{H}^j_{et}(S, B),$$

where f_S is functorial in S. Assume further that among A, B, only B (resp. A) is a commutative algebraic R-group scheme and $0 \leq i \leq 1$ (resp. $0 \leq j \leq 1$).

Definition. We say that the *Corestriction Principle over R holds for the image* (*resp. the kernel*) *of* f_R if we have

$$\mathrm{Cores}_{S/R,B}(\mathrm{Im}\ (f_S)) \subset \mathrm{Im}\ (f_R)$$

(resp.

$$\mathrm{Cores}_{S/R,A}(\mathrm{Ker}\ (f_S)) \subset \mathrm{Ker}\ (f_R))$$

for any faithfully flat finite étale extension S of R. We call a map $\mathrm{H}^i_{et}(R, A) \to \mathrm{H}^j_{et}(R, B)$ *connecting* if it is appearing in the long exact sequence of cohomology deduced from a short exact sequence of R-groups involving A and B.

It is natural and important to investigate whether or not the Corestriction Principle always holds for connecting maps. In the case $i = 1, j = 2$, Rosset and Tate [22] constructed an example showing that, in general, Corestriction Principle for the image (or kernel) does not hold. Namely let k be a field and let

$$1 \to \mu_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1$$

be the exact sequence of algebraic k-groups, where μ_n denotes the center (=n-th roots of 1) of the special linear group SL_n, and PGL_n denotes the corresponding projective linear group. Then they showed that if $\mu_n(\bar{k}) \subset k$ then for any finite separable extension k'/k, any element of the image of the connecting (boundary)

map $\Delta' : \mathrm{H}^1(k', \mathrm{PGL}_n) \to \mathrm{H}^2(k', \mu_n)$ (the elements of which are called (cohomological) symbols over k') via the corestriction map $\mathrm{Cores}_{k'/k,\mu_n}$ is a *sum* (in the corresponding group) of elements of the image of Δ and may not be in the image. Thus the norm (corestriction) of a (cohomological) symbol may not be a symbol, though it is a *sum* of symbols. In other words, a weaker statement holds true : *the corestriction of a symbol over k' lies in the group generated by the symbols over k .* (In view of Merkurjev and Suslin's Theorem [16], if k contains a primitive n-th root of 1, then we have the following stronger statement: the image of the connecting map $\Delta : \mathrm{H}^1_{et}(k, \mathrm{PGL}_n) \to \mathrm{H}^2(k, \mu_n) = {}_nBr(k)$ *generates* the whole n-torsion subgroup ${}_nBr(k)$.)
So for a connecting map $\alpha_R : \mathrm{H}^p_{et}(R, G) \to \mathrm{H}^q_{et}(R, T)$, where G, T are connected reductive R-groups, T is a torus, $0 \le p \le 1, 0 \le q \le 2$ and a collection of connecting maps $\alpha_S : \mathrm{H}^p_{et}(S, G) \to \mathrm{H}^q_{et}(S, T)$ for any faithfully flat finite étale extension S of a ring R, it is natural to ask if the above generation phenomenon always holds.
Definition. We say that *Weak Corestriction Principle holds for the image of α_R,* if

$$\mathrm{Cores}_{S/R,T}(\mathrm{Im}\ (\alpha_S)) \subset \langle \mathrm{Im}\ (\alpha_R) \rangle,$$

where $\langle A \rangle$ denotes the *subgroup* generated by A in the corresponding group.

Similarly one may consider ($Weak$) Corestriction Principle for the *kernel* of a connecting map. Notice that some particular cases of the Corestriction Principle (in other terminology, Norm Principle), was proved to hold in [2, 3, 9, 15, 28–33], under certain restrictions either on the group G, or on the arithmetic nature of the field or ring R. The reader is referred to these papers for some more details of application of (Weak) Corestriction Principle to arithmetic. For example, we have proved recently the existence of a norm map for the class groups of reductive group schemes over Dedekind rings of global fields, satisfying certain condition of approximation (non-compactness) (see [31, 33]).

The purpose of this paper is to prove some equivalent relations (or rather, interrelations) between various statements about the validity of (Weak) Corestriction Principle for various types of connecting maps related with reductive group schemes. This work forms a basis for the proof of some important (Weak) Corestriction Principle for reductive group schemes over arithmetical (Dedekind) rings of local and global fields as is done in [31, 33]. After some preliminary results in Section 1, we prove some equivalent conditions (in the form of reduction theorems) in Section 2, which show how different statements about (Weak) Corestriction Principle are related and how useful they are in reducing the problem to simpler one. In Section 3 we reduce the problem to quasi-split case.

By convention, all reductive group schemes are *connected*.

2. Preliminary results

2.1. In this section we present some necessary facts related with the comparison of Čech, étale and flat cohomologies for smooth group schemes and well-known crossed-diagram construction by Ono (which also relates to the notion of z-extensions used by Langlands). We fix a ring R and consider in this section the category $\mathcal{G}Sch_R$ of flat group schemes over Spec(R) of finite type. We will need the following preliminary results. The first one is a slight extension of a result of [26], Exp. XXIV, Proposition 8.4, and its proof.[1]

Proposition 2.1. *Let X be a scheme and let G be a sheaf of groups in étale topology on X.*
a) *For any finite étale morphism of schemes $f : Y \to X$, we have isomorphisms*

$$\varphi_i : \mathrm{H}^i_{et}(X, f_* f^* G) \simeq \mathrm{H}^i_{et}(Y, f^* G)$$

for all $i \geq 0$, where $i = 0, 1$, if G is a non-abelian sheaf.
b) *If Y is as above, and if G is a smooth commutative X-group scheme, then there exists a functorial corestriction homomorphism*

$$\mathrm{Cores}_{Y/X} : \mathrm{H}^i_{et}(Y, f^* G) \to \mathrm{H}^i_{et}(X, G), for\ \ all\ \ i \geq 0.$$

Proof. a) We need the following slight extensions of Propositions 8.2, 8.3 and 8.4 in [26], Exp. XXIV.

Lemma 2.1.1 *Let $\mathcal{C}$ be a category with fiber products and equipped with a topology, which is weaker than the canonical one. Let $S' \to S$ be a morphism of $\mathcal{C}, G'$ a S'-sheaf of groups, $G := \mathrm{Res}_{S'/S}(G')$ the restriction of scalars of G'. Let $\mathrm{H}^i_S(S', G')$ be the set of all elements of $\mathrm{H}^i(S', G')$, being trivial while restricted to a sieve of S', which is obtained from a sieve of S by base change from S to S'. Then the canonical map $\mathrm{H}^i(S, G) \to \mathrm{H}^i(S', G')$ (which is the composition $\mathrm{H}^i(S, G) \to \mathrm{H}^i(S', G) \to \mathrm{H}^i(S', G')$) induces a bijection*

$$\mathrm{H}^i(S, G) \simeq \mathrm{H}^i_S(S', G').$$

Here $i \leq 1$ if G is a non-abelian sheaf.

Proof of Lemma 2.1.1. The same proof of (loc.cit) works through: we just use the isomorphisms (resp. bijections)

$$\mathrm{H}^i(T/S, G) \simeq \mathrm{H}^i(T'/S', G'),\ \text{for all}\ \ i \geq 0,$$

where $T \to S$ is any S-object of $\mathcal{C}$, $T' = T \times_S S'$, due to the canonical isomorphism $Hom_S(X, G) \simeq Hom_{S'}(X \times_S S', G')$. Then, replacing $T \to S$ by a covering $(S_i \to S)$, and take inductive limit over them, we are done. ∎

Lemma 2.1.2 *With notation as in Lemma 2.1.1, assume that there is a covering $(S_t \to S)_t$ such that for all t, with $S'_t := S' \times_S S_t$, we have $\mathrm{H}^i_{S_t}(S'_t, G') = \mathrm{H}^i(S'_t, G')$. Then we have $\mathrm{H}^i_S(S', G') = \mathrm{H}^i(S', G')$.*

[1] I thank the referee for her/his sugesstion, that one needs such a generalisation for the proof of Lemma 2.2.1 below.

Proof of Lemma 2.1.2. For $x \in \mathrm{H}^i(S', G')$, let x_t be the image of x via the canonical map $\mathrm{H}^i(S', G') \to \mathrm{H}^i(S'_t, G')$. By assumption, there is a covering $(S_{ts} \to S_t)$ such that $x_{ts} = 0$, where x_{ts} is the image of x_t via $\mathrm{H}^i(S'_t, G') \to \mathrm{H}^i(S'_{ts}, G')$, and where $S'_{ts} := S' \times_S S_{ts}$. But then $(S_{ts} \to S)_{t,s}$ is also a covering of S, so x is trivialized by passing to (S_{ts}), i.e., $x \in \mathrm{H}^i_S(S', G')$ as required. ■

Return to the proof of Proposition 2.1. Let X be a scheme, $f : Y \to X$, a finite étale morphism of schemes, and let G' be a sheaf of groups in étale topology on Y, and let $G = \mathrm{Res}_{Y/X}(G')$, which represents f_*G'. According to Lemma 2.1.2, we need just to show that there exists a covering $(X_t \to X)_t$ such that for all t, with $Y_t := Y \times_X X_t$, we have $\mathrm{H}^i_{X_t}(Y_t, G') = \mathrm{H}^i(Y_t, G')$. As in the proof of [26], Exp. XXIV, Proposition 8.4, we are reduced to considering the problem locally, thus we may assume that we have a decomposition $Y = X_1 \sqcup \cdots \sqcup X_m$, where each X_i is just a copy of X. Then, G' is given by its restrictions G_t to X_t, and we have

$$\mathrm{H}^i(Y, G') = \prod_t \mathrm{H}^i(X_t, G'_t) = \prod_t \mathrm{H}^i(X, G'_t). \qquad (*)$$

Since G' is given by (G'_t) on Y, and since $Y = X_1 \sqcup \cdots \sqcup X_m = X \sqcup \cdots \sqcup X$, each G'_t is a X-group scheme. It follows from the definition of $G = \mathrm{Res}_{Y/X}(G')$, that in this case $G \simeq \prod_{1 \le t \le m} G'_t$, so by [9], Chap. III, Sec.2, Remark 2.4.4, we have

$$\mathrm{H}^i(X, G) = \prod_t \mathrm{H}^i(X, G'_t). \qquad (**)$$

Now the proof of a) follows from Lemma 2.1.1 and Lemma 2.1.2 and $(*), (**)$.
b) It is known that if f is finite and étale, then it is also locally free (cf. [26], Proposition 18.2.3 and Remark 18.2.7). In this case, since the sheaf G is representable, one can define, according to Grothendieck - Deligne trace theory, a trace morphism

$$Tr_f : f_*f^*G \to G$$

(cf. [27], Exp. 17, 6.3.13), thus also trace map of between the cohomology sets

$$Tr^i_f : \mathrm{H}^i_{et}(X, f_*f^*G) \to \mathrm{H}^i_{et}(X, G).$$

According to a), we have the following isomorphisms $\varphi_i : \mathrm{H}^i_{et}(X, f_*f^*G) \simeq \mathrm{H}^i_{et}(Y, f^*G)$ for all $i \ge 0$, (and $i = 0, 1$, if G is a non-abelian sheaf). Therefore from a) above we have functorial homomorphisms for all i

$$\mathrm{Cores}_{Y/X} := Tr^i_f \circ \varphi_i^{-1} : \mathrm{H}^i_{et}(Y, f^*G) \to \mathrm{H}^i_{et}(X, G). \qquad ■$$

Remark 2.1.3 Another proof of Proposition 8.4 in [26], Exp. XXIV (for the case of H^1) makes use of an exact sequence

$$0 \to \mathrm{H}^1_{et}(X, f_*(F)) \to \mathrm{H}^1_{et}(Y, F) \to \mathrm{H}^0_{et}(X, R^1f_*(F))$$

(deduced from Leray spectral sequence $E_2^{p,q} = \mathrm{H}^p_{et}(X, R^q f_* F) \Rightarrow \mathrm{H}^{p+q}_{et}(Y, F)$, related to the morphism $f : Y \to X$, where F is an étale sheaf on Y), which is also true in the case of non-abelian sheaf. In the general case of $\mathrm{H}^i, i > 1, F$ is abelian, we may use this argument also to prove a) of Proposition 2.1.

Namely, in the case $Y \to X$ is finite, we have, by [27], Exp. VIII, Proposition 5.5, that $R^q f_*(F) = 0$ for $q > 0$. Thus from this and Leray spectral sequence it follows that $\mathrm{H}^p_{et}(X, f_*(F)) \simeq \mathrm{H}^p_{et}(Y, F)$ as desired.

2.2. Let G be a reductive R-group. Denote by $rad(G)$ the radical of G, $\tilde{G}$ the simply connected covering of the derived subgroup $G' := [G, G]$ of G,

$$\pi : \tilde{G} \times_{\mathrm{Spec}(R)} rad(G) \to G' \times_{\mathrm{Spec}(R)} rad(G) \to G$$

the composition of central isogenies (cf. [26], Exp. XXII, Prop. 6.2.4). Let $A =$ Ker (π). Recall that a reductive R-group H is a *z-extension* of a R-group G if H is an extension of G by an induced R-torus Z, such that the derived subgroup (called also the semisimple part) $[H, H]$ of H is simply connected. For a faithfully flat ring extension S/R and an element $x \in \mathrm{H}^1_{flat}(S, G)$, a z-extension of G over R is called *x-lifting* if $x \in \mathrm{Im}\ (\mathrm{H}^1_{flat}(S, H) \to \mathrm{H}^1_{flat}(S, G))$. The following lemmas are due to Borovoi and/or Kottwitz (see [5], Sec. 3, [13, 14]) in the case $S = R = k$ is a field. The method of proof is similar, but for the self-containedness and convenience of the reader, we give them here.

Lemma 2.2.1 a) *Let F be a finite flat R-group scheme of multiplicative type, which is split over a faithfully flat finite étale extension S/R. Then there exists a Galois extension S'/R, containing S, an induced R-torus Z, which is R-isomorphic to* $\mathrm{Res}_{S'/R}(\mathbf{G}_m)^n$ *for some n, and an embedding of R-group schemes $F \hookrightarrow Z$.*
b) *Let G be a reductive R-group, π, A as in 2.2 above. Let S be a faithfully flat finite étale extension of R, which splits A. Then there exists a z-extension $1 \to Z \to H \to G \to 1$ over R, such that $Z \simeq \mathrm{Res}_{S'/R}(\mathbf{G}_m)^n$ for some Galois extension S'/R, which contains S.*
c) *Let G be a reductive R-group, S a faithfully flat finite étale extension of R, x an element of* $\mathrm{H}^1_{et}(S, G)$. *Then there is a z-extension*

$$1 \to Z \to H \to G \to 1,$$

of G, where all groups and morphisms are defined over R, which is x-lifting.

Proof. a) We may choose any Galois extension S'/R, with Galois group Γ, which contains S. By [26], Exp. X, Corol. 1.2, there is an anti-equivalence between the category of R-multiplicative groups and the category of continuous Π-modules (i.e., the stabilizer in Π of any point of the module is open), where $\Pi = \pi_1(\mathrm{Spec}(R), \psi)$ the fundamental group of $\mathrm{Spec}(R)$ in the sense of Grothendieck (cf. [25], Exp. V) with respect to a geometric point $\psi : \mathrm{Spec}(k_s) \to \mathrm{Spec}(R)$. Here k_s denotes a separable closure of the quotient field k of R. In particular, Γ is a finite quotient group of Π. The corresponding functor is given by character group on the fiber at geometric point

$$H \mapsto M_H := Hom_{gr}(H_\psi, \mathbf{G}_{m,\psi}).$$

In our case, if F corrresponds to a Π-module M_F, then M_F is a finite $\mathbf{Z}[\Gamma]$-module, thus there is a surjective homomorphism of Γ-modules $M_B \to M_F$, where M_B is a free $\mathbf{Z}[\Gamma]$-module $\mathbf{Z}[\Gamma]^n$, where $n = Card(M_F)$, considered as a $\mathbf{Z}[\Pi]$-module, with trivial action of Ker $(\Pi \to \Gamma)$ on M_B. The R-torus B corresponding to M_B has the form $\mathrm{Res}_{S'/R}(\mathbf{G}_m)^n$. Due to the surjectivity of the homomorphism $M_B \to M_F$, the corresponding R-morphism $F \to B$ is injective.
b) By a), there exists an induced R-torus Z such that $A \hookrightarrow Z$. We set

$$H = (\tilde{G} \times_{\mathrm{Spec}(R)} rad(G) \times_{\mathrm{Spec}(R)} Z)/A,$$

where A is embedded into the product in an obvious way. Then $G = (\tilde{G} \times_{\mathrm{Spec}(R)} rad(G))/A$, and the obvious map $H \to G$ is clearly surjective. Its kernel is Z, and we have a z-extension as required.
c) We can find a faithfully flat étale , finite extension S'/R, which contains S, such that x comes from $\mathrm{H}^1_{et}(S'/S, G)$, i.e., x splits over S'. Thus, by enlarging S', we may assume that S'/R is Galois. By b), there exists a z-extension $1 \to Z \to H \to G \to 1$ of G, such that $Z \simeq \mathrm{Res}_{S'/R}(\mathbf{G}_m)^n$ for some n. Set $T := (\mathbf{G}_m)^n_{S'}$. Then we have by definition $Z(V) = T(V \otimes_R S')$ for any commutative R-algebra V. In particular, $Z(S) = T(S \otimes_R S')$. We consider the functor Z_1 which assigns to every S'-algebra U the group $Z_1(U) := T(S \otimes_R U)$, where U is considered as R -algebra via $R \hookrightarrow S'$. Then we have $Z(S) = T(S \otimes_R S') = Z_1(S')$, thus $Z = \mathrm{Res}_{S'/S}(Z_1)$. We have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^1_{et}(S,H) & \to & \mathrm{H}^1_{et}(S,G) & \xrightarrow{\Delta} & \mathrm{H}^2_{et}(S,Z) \\ \downarrow & & \downarrow & & \downarrow \eta \\ \mathrm{H}^1_{et}(S',H) & \to & \mathrm{H}^1_{et}(S',G) & \xrightarrow{\Delta'} & \mathrm{H}^2_{et}(S',Z) \end{array}$$

where all lines are exact, and the vertical arrows are restriction maps, and the maps Δ, Δ' are coboundary maps (see [9], Chap. IV, Sec. 3.5). Since $Z = \mathrm{Res}_{S'/S}(Z_1)$, by Lemma 2.1.1 (cf. also [26], Exp. XXIV, Remark 8.1.9, Proposition 8.4), we have

$$\begin{aligned} \mathrm{H}^2_{et}(S,Z) &= \mathrm{H}^2_{et}(S, \mathrm{Res}_{S'/S}(Z_1)) \\ &\simeq \mathrm{H}^2_{et}(S', Z_1) \\ &\hookrightarrow \mathrm{H}^2_{et}(S', \mathrm{Res}_{S'/S}(Z_1)_{S'}) \end{aligned}$$

due to the embedding $Z_{1,S'} \hookrightarrow \mathrm{Res}_{S'/S}(Z_1)_{S'}$, thus the induced map $\gamma' : \mathrm{H}^2_{et}(S,Z) \to \mathrm{H}^2_{et}(S',Z)$ is injective. In the following commutative diagram with exact lines,

$$\begin{array}{ccccc}
\mathrm{H}^1_{et}(R,H) & \to & \mathrm{H}^1_{et}(R,G) & \to & \mathrm{H}^2_{et}(R,Z) \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
\mathrm{H}^1_{et}(S,H) & \to & \mathrm{H}^1_{et}(S,G) & \stackrel{\Delta}{\to} & \mathrm{H}^2_{et}(S,Z) \\
\downarrow \alpha' & & \downarrow \beta' & & \downarrow \gamma' \\
\mathrm{H}^1_{et}(S',H) & \to & \mathrm{H}^1_{et}(S',G) & \stackrel{\Delta'}{\to} & \mathrm{H}^2_{et}(S',Z)
\end{array}$$

we have $x \in \mathrm{Ker}\ \beta'$, hence $0 = \Delta'(\beta'(x)) = \gamma'(\Delta(x))$. Since γ' is injective, it follows that $\Delta(x) = 0$, i.e., $x \in \mathrm{Im}\ (\mathrm{H}^1_{et}(S,H) \to \mathrm{H}^1_{et}(S,G))$. ■

Lemma 2.2.2 *Let $\alpha : G_1 \to G_2$ be a R-morphism of reductive R-group schemes, S a faithfully flat finite étale extension of R, x an element of $\mathrm{H}^1_{et}(S, G_1)$. Then there exists a x-lifting z-extension $\alpha' : H_1 \to H_2$ of α, i.e., H_i is a z-extension of G_i $(i = 1, 2)$, and we have the following commutative diagram*

$$\begin{array}{ccc}
H_1 & \stackrel{\alpha'}{\to} & H_2 \\
\downarrow & & \downarrow \\
G_1 & \stackrel{\alpha}{\to} & G_2,
\end{array}$$

with all groups and morphisms defined over R.

Proof. It follows from Lemma 2.2.1. ■

3. (Weak) Corestriction Principle: some relations

3.1. Let R be a commutative domain, and let G be a reductive R-group scheme. Denote by G' the derived subgroup scheme of G, $\tilde{G}$ the simply connected covering of G', $\bar{G}$ the adjoint group scheme of G (see [26], Exp. XXII, 4.3.3), $\tilde{F} := \mathrm{Ker}\ (\tilde{G} \to \bar{G})$, $F := \mathrm{Ker}\ (\tilde{G} \to G')$ and let $\tilde{Z} = \mathrm{Cent}(\tilde{G})$, $Z = \mathrm{Cent}(G)$ be the corresponding centers.

In this section we will discuss some relation between the validity of (Weak) Corestriction Principles for connecting maps of various type. First we assume once for all that, for all smooth *commutative* R-group schemes involved, there is a notion of corestriction homomorphism, that is, for any smooth commutative R-group scheme T and each extension R'/R belonging to certain category $\mathcal{C}_R$ of faithfully flat, étale extensions of finite type over R there is a functorial homomorphism

$$\mathrm{Cores}_{R'/R,T} : \mathrm{H}^i_{et}(R', T_{R'}) \to \mathrm{H}^i_{et}(R, T).$$

Here we denote $T_{R'} = T \times_R R'$ the R'-group scheme obtained by base change from R to R'. For a crossed module $F \stackrel{\alpha}{\to} G$, consisting of smooth group R-schemes F and G, for the étale topology, we can consider the Deligne hypercohomology $\mathbf{H}^*_{et}(R, F \to G)$ (see [7], Sec. 2.4, [6], Sec. 2.16, Sec. 4.2) and the derived long exact sequence of pointed sets

$$1 \to \mathbf{H}^{-1}_{et}(R, F \to G) \to \mathrm{H}^0_{et}(R, F) \to \mathrm{H}^0_{et}(R, G) \stackrel{\delta}{\to}$$

$$\stackrel{\delta}{\to} \mathbf{H}^0_{et}(R, F \to G) \to \mathrm{H}^1_{et}(R, F) \to \mathrm{H}^1_{et}(R, G) \stackrel{\Delta}{\to}$$

$$\stackrel{\Delta}{\to} \mathbf{H}^1_{et}(R, F \to G).$$

This exact sequence, if appropriately interpreted, also holds in any topos (loc.cit). Applying this to the crossed module $\tilde{G} \to G$, where $G, \tilde{G}$ are as above, and define $\mathrm{H}^0_{ab,et}(R, G) := \mathbf{H}^0_{et}(R, \tilde{G} \to G), \mathrm{H}^1_{ab,et}(R, G) := \mathbf{H}^1_{et}(R, \tilde{G} \to G)$, we have the following exact sequence, which is functorial in G and in R

$$\to \mathrm{H}^0_{et}(R, \tilde{G}) \to \mathrm{H}^0_{et}(R, G) \stackrel{ab^0_{G,et}}{\to} \mathbf{H}^0_{et}(R, \tilde{G} \to G) \to$$

$$\to \mathrm{H}^1_{et}(R, \tilde{G}) \to \mathrm{H}^1_{et}(R, G) \stackrel{ab^1_{G,et}}{\to} \mathbf{H}^1_{et}(R, \tilde{G} \to G).$$

The map $ab^i_{G,et}$ are called *abelianization maps*. For $i = 0$, it is a group homomorphism. Since $\tilde{Z}$ and Z are commutative, so if they are smooth, then the resulting cohomology sets $\mathbf{H}^i_{et}(R, \tilde{Z} \to Z)$ have natural structure of abelian groups. In the case of spectrum of a field of characteristic 0, it is known that there exists functorial corestriction homomorphisms for $\mathrm{H}^i_{ab,et}(R, G)$ (which follows from [7], Sec. 2.4.3, or [19], cf. [29]). It can be also extended to the case of positive characteristic, if we assume that the center $\tilde{Z}$ of $\tilde{G}$ is smooth. However, in the general (étale or flat) case, it is not clear whether such functorial homomorphisms always exist. Thus we need to make the following hypothesis (Hyp_R) with respect to the ring R.

(Hyp_R) *For $R' \in \mathcal{C}_R$, for any G as above such that $\tilde{Z}$ is smooth, there exist functorial corestriction homomorphisms*

$$\mathrm{Cores}_{R'/R} : \mathrm{H}^i_{ab,et}(R', G_{R'}) \to \mathrm{H}^i_{ab,et}(R, G), i = 0, 1.$$

Remark. We show briefly that if we take $\mathcal{C}_R$ as the category $\mathcal{C}_{R,ef}$ of all étale finite extensions of a ring of integers of a local field R, then we have

Theorem 3.2. *With above notation, for any fixed $R' \in \mathcal{C}_{R,ef}$, then*
a) *For any smooth commutative R-group scheme of finite type T there exist functorial corestriction homomorphisms* $\mathrm{Cores}_{R'/R} : \mathrm{H}^i_{et}(R', T) \to \mathrm{H}^i_{et}(R, T)$, *for all $i \geq 0$;*
b) *For an R-reductive group scheme G as above such that $\tilde{Z}$ is smooth, there exist functorial corestriction homomorphisms*

$$\mathrm{Cores}_{R'/R} : \mathrm{H}^i_{ab,et}(G_{R'}) \to \mathrm{H}^i_{ab,et}(G), i = 0, 1,$$

i.e., (Hyp_R) *also holds.*

Proof. a) It was proved in Proposition 2.1
b) The case $i = 0$ is clear. Assume that $i = 1$. Set for $j = 1, 2$, $T'_j := T_{j,R'} := T_j \times_R R'$. Then if $f : T_1 \to T_2$ is a morphism of commutative smooth R-group schemes, then as in [7], Sec. 2.4, we obtain also an additive functor (trace)

$$Tr_{R'/R} : [T'_1 \to T'_2] \to [T_1 \to T_2],$$

where $[T_1 \to T_2]$ denotes the category of T_1-torsors equipped with a T_2-trivialization (cf. [7], Sec. 2.4). In fact, since $R' \in \mathcal{C}_{R,ef}$, it follows from [26], Proposition 18.2.3 and Remark 18.2.7, that $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is also finite and locally free, and since our sheaves of groups are representable, so the Grothendieck - Deligne trace theory [27], Exp. XVII, Sec. 6.3, applies. Now, with notation as in 3.1., we take any reductive R-group scheme G with smooth $\tilde{Z}$ (thus also smooth Z), then we have (see loc.cit) equivalences of categories

$$[\tilde{Z} \to Z] \overset{\phi}{\simeq} [\tilde{G} \to G], \;\; [\tilde{Z}' \to Z'] \overset{\phi'}{\simeq} [\tilde{G}_{R'} \to G_{R'}],$$

hence also the following commutative diagram, where all arrows are functorial

$$\begin{array}{ccc} [\tilde{Z}' \to Z'] & \overset{\phi'}{\simeq} & [\tilde{G}_{R'} \to G_{R'}] \\ \downarrow \alpha & & \downarrow \beta \\ [\tilde{Z} \to Z] & \overset{\phi}{\simeq} & [\tilde{G} \to G] \end{array}$$

where $\alpha = Tr_{R'/R}$. Now by using ϕ, ϕ' and by taking the cohomology functor $\mathbf{H}^1_{et}$, we have the following commutative diagram with functorial arrows

$$\begin{array}{ccc} \mathbf{H}^1_{et}(R', \tilde{Z}' \to Z') & \overset{\varphi'}{\simeq} & \mathbf{H}^1_{et}(R', \tilde{G}_{R'} \to G_{R'}) \\ \mathrm{Cores}_{R'/R} \downarrow \gamma & & \downarrow \delta \\ \mathbf{H}^1_{et}(R, \tilde{Z} \to Z) & \overset{\varphi}{\simeq} & \mathbf{H}^1_{et}(R, \tilde{G} \to G) \end{array}$$

which follows from results of [29] and [33]. The theorem is proved. ■

3.3. Let $\alpha : \mathrm{H}^p_{et}(R, G) \to \mathrm{H}^q_{et}(R, T)$ be a connecting map of cohomologies and assume that an extension R'/R, $R' \in \mathcal{C}_R$, is fixed. Under the assumption of (Hyp_R), we consider the following statements.

a) *The (Weak) Corestriction Principle holds for the image of connecting map* $\alpha_R : \mathrm{H}^p_{et}(R, G) \to \mathrm{H}^q_{et}(R, T)$ *for any given reductive R-groups G, T with T a R-torus and given p,q with* $0 \le p \le 1$, $0 \le p \le q \le p + 1$. b) *The (Weak) Corestriction Principle holds for the image of the functorial map*

ab^p_G : $\mathrm{H}^p_{et}(R,G) \to \mathrm{H}^p_{ab,et}(R,G)$ *for any reductive R-group G and given p,* $0 \le p \le 1$.

c) *The (Weak) Corestriction Principle holds for the image of the coboundary map* $\delta_R : \mathrm{H}^p_{et}(R,G_1) \to \mathrm{H}^{p+1}_{et}(R,T)$, *where* $1 \to T \to G_1 \to G \to 1$ *is any short exact sequence of reductive R-groups with G semisimple, T a central smooth subgroup, and p is given,* $0 \le p \le 1$.

d) *The same statement as in* c)*, but* G_1 *and G are supposed to be semisimple R-groups.*

e) *The (Weak) Corestriction Principle holds for the image of the coboundary map* $\delta_R : \mathrm{H}^p_{et}(R,\bar{G}) \to \mathrm{H}^{p+1}_{et}(R,F')$ *where* $1 \to F' \to G_1 \to \bar{G} \to 1$ *is any short exact sequence of reductive R-groups with* G_1 *semisimple,* $\bar{G}$ *adjoint, and F' a finite central smooth subgroup and p is given,* $0 \le p \le 1$.

f) *The (Weak) Corestriction Principle holds for the image of connecting boundary map* $\mathrm{H}^p_{et}(R,\bar{G}) \to \mathrm{H}^{p+1}_{et}(R,\tilde{F})$, *for any adjoint group* $\bar{G}$ *with smooth fundamental group* $\tilde{F} = Cent(\tilde{G})$ *and for given p,* $0 \le p \le 1$.

For the statements a) - f) considered above, let us denote by $x(p,q)$ (resp. $y(p)$) the statement x) (resp. y)) evaluated at (p,q), for $0 \le p \le q \le 2$. For example, $a(1,2)$ means the statement a) with $p = 1, q = 2$, or $f(1)$ means the statement f) with $p = 1$. We say that the statement x) holds if for any possible values of (p,q) the corresponding statement is true. Note that for any $p, 0 \le p \le 1$, we have obvious implications : $c(p) \Rightarrow d(p) \Rightarrow e(p) \Rightarrow f(p)$, i.e., c) $\Rightarrow$ d) $\Rightarrow$ e) $\Rightarrow$ f).

4. (Weak) Corestriction Principle: some equivalent relations

4.1. The relations between the statements in Section 2.2 are given in the following results. We will give the proof only in the case of Weak Corestriction Principle since all proofs hold true simultaneously for Corestriction and Weak Corestriction Principles, except possibly Proposition 4.10. (There, in the part b), we have to restrict ourselves to Corestriction Principle only.) Also, most of the proofs are similar (in fact, almost identical) to the case of field (see [30], Section 2), due to its functoriality, so most of them will be omitted, and we give only necessary arguments where it is needed.

4.2. *For a given reductive R-group G, and a given p with* $0 \le p \le 1$, *if the (Weak) Corestriction Principle holds for the image of* ab^p_G *then it also holds for any possible connecting map (if any)* $\mathrm{H}^p_{et}(R,G) \to \mathrm{H}^q_{et}(R,T)$ *where T is a R-torus and* $0 \le q \le 2$. *In particular, if* $b(p)$ *holds for any p then* $a(p,q)$ *holds for any pairs* (p,q) *which make sense.*

Proof. The proof follows immediately from the functoriality of the maps ab^p_G : $\mathrm{H}^p_{et}(R,G) \to \mathrm{H}^p_{ab,et}(R,G)$, $p = 0,1$, proved by Deligne in [7], Sec. 2.4 for $p = 0$, and by Breen in [6], Sec. 4.2, for $p = 1$ and by using the fact that for commutative smooth R-group schemes T, Deligne hypercohomology (or Borovoi abelianized cohomology) is isomorphic to the usual étale cohomology

$$\mathrm{H}^i_{ab,et}(R,T) \simeq \mathrm{H}^i_{et}(R,T), i = 0,1$$

(see, e.g. [5] in the case of fields, and [6], Proposition 6.2, in general case). ∎

4.3. *For a given reductive R-group scheme G, let $\mathcal{H}$ be the set of all z-extensions H of G. If the (Weak) Corestriction Principle holds for the image of $ab^p_{H,et}$ for $p = 0$ (resp. $p = 1$) for all $H \in \mathcal{H}$, then the same holds for G, i.e. the (Weak) Corestriction Principle holds for the image of $ab^p_{G,et}$. In particular, if $b(p)$ holds for reductive R-groups with simply connected semisimple parts then $b(p)$ holds itself.*[2] ∎

4.4. *Assume that for all R-morphisms $G \to T$ of reductive R-groups, with T a R-torus, the (Weak) Corestriction Principle holds for the image of the induced connecting map $\mathrm{H}^p_{et}(R,G) \to \mathrm{H}^p_{et}(R,T)$, and for some fixed p, $0 \le p \le 1$. Then, for all reductive R-groups G, the same holds for $ab^p_{G,et} : \mathrm{H}^p_{et}(R,G) \to \mathrm{H}^p_{ab,et}(R,G)$, i.e., $a(p,p) \Rightarrow b(p)$. In particular, if* a) *holds, then* b) *holds.* ∎

4.5. *If the statement* a) *holds for p=q=0 (resp. $p = q = 1$) then it also holds for p=0, q=1 (resp. p=1, q=2), therefore we have $a(0,0) \Rightarrow a(0,1)$, $a(1,1) \Rightarrow a(1,2)$.* ∎

4.6. a) *Let $\tilde{G}$ be the simply connected covering of a semisimple R-group G with fundamental group $F = Ker(\tilde{G} \to G)$, $\tilde{F} = Cent(\tilde{G})$ (the center) and let F_1 be a R-subgroup scheme of $\tilde{F}$, $G_1 := \tilde{G}/F_1$. If for some p, $0 \le p \le 1$, the (Weak) Corestriction Principle holds for the image of the coboundary map $\delta : \mathrm{H}^p_{et}(R,G) \to \mathrm{H}^{p+1}_{et}(R,F)$ then the same holds for the image of $\delta_1 : \mathrm{H}^p_{et}(R,G) \to \mathrm{H}^{p+1}_{et}(R,F_1)$.*
b) *Let $\bar{G}$ be an adjoint semisimple R-group and let F be a R-subgroup scheme of $\tilde{F}$. If for some p, $0 \le p \le 1$, the (Weak) Corestriction Principle holds for the image of the coboundary map $\delta : \mathrm{H}^p_{et}(R,\bar{G}) \to \mathrm{H}^{p+1}_{et}(R,\tilde{F})$ then the same holds for the image of $\delta_1 : \mathrm{H}^p_{et}(R,\bar{G}) \to \mathrm{H}^{p+1}_{et}(R,F)$. In particular, we have $e(p) \Leftrightarrow f(p)$.*

Proof. We give only proof of a). We have the following commutative diagram with exact lines

$$\begin{array}{ccccc} \mathrm{H}^p_{et}(R,\tilde{G}) & \to & \mathrm{H}^p_{et}(R,G) & \to & \mathrm{H}^{p+1}_{et}(R,F) \\ \downarrow & & \downarrow = & & \downarrow \\ \mathrm{H}^p_{et}(R,G_1) & \to & \mathrm{H}^p_{et}(R,G) & \to & \mathrm{H}^{p+1}_{et}(R,F_1) \end{array}$$

[2] The similar statement in [30] should be precised in the same way.

Since the coboundary map $\mathrm{H}^p_{et}(R,G) \to \mathrm{H}^{p+1}_{et}(R,F_1)$ factors through the map $\mathrm{H}^p_{et}(R,G) \to \mathrm{H}^{p+1}_{et}(R,F)$, the assertion follows. ■

4.7. *Assume that $a(p,q)$ holds for all G with simply connected semisimple part G'. Then $a(p,q)$ holds itself.*

4.8. *Let $1 \to T \to G_1 \xrightarrow{\pi} G \to 1$ be an exact sequence of R-groups, where G is semisimple and T is a central smooth subgroup of a reductive R-group G_1. Let $G_1 = G'_1 S_1$, where $G'_1 = [G_1, G_1]$, S_1 is a central torus and $F := S_1 \cap G'_1$ is the schematic intersection. If for some p, $0 \leq p \leq 1$, the (Weak) Corestriction Principle holds for the image of the coboundary map $\delta : \mathrm{H}^p_{et}(R,G) \to \mathrm{H}^{p+1}_{et}(R,F)$, then the same holds for the image of the coboundary map $\mathrm{H}^p_{et}(R,G) \to \mathrm{H}^{p+1}_{et}(R,T)$. In particular, $d(p) \Rightarrow c(p)$, i.e.* c) $\Leftrightarrow$ d).

4.9. *Assume that for some p, $0 \leq p \leq 1$, the (Weak) Corestriction Principle with respect to the extension R'/R holds for the image of any connecting map $\mathrm{H}^p_{et}(R,H) \to \mathrm{H}^p_{et}(R,T)$ for reductive R-groups H, T, where T is a R-torus. Assume also that Theorem Hilbert-90 holds for induced tori in $\mathcal{C}_R$, i.e. $\mathrm{H}^1_{et}(R',T_0) = 0$ for any induced R'-torus T_0 and for any $R' \in \mathcal{C}_R$. Then the same principle holds for the image of any coboundary map $\mathrm{H}^p_{et}(R,G) \to \mathrm{H}^{p+1}_{et}(R,F)$ deduced from an isogeny $1 \to F \to G_1 \to G \to 1$ with smooth kernel F and reductive G, i.e. $a(p,p) \Rightarrow d(p)$. In particular, under above condition, if* a) *holds then* d) *holds.*

Proof. The idea of the proof is the same as in the case of fields, but there are some cocycle computations that we need to proceed explicitly in order to facilitate the reading. First, by the statement 4.6, a), we may assume that G_1 is simply connected. Take an embedding $F \hookrightarrow T_1$, where T_1 is an R-induced torus.We embed F into $T_1 \times G_1$ as follows $f \mapsto (f, f^{-1})$. Thus, by taking $H := (T_1 \times G_1)/F$ (with respect to the embedding of F above) we have a z-extension

$$1 \to T_1 \to H \to G \to 1$$

of G. It fits into the following analog of Ono's crossed diagram ([18]), which allows one to embed an exact sequence with finite kernel of multiplicative type (i.e. isogeny) into another with induced (i.e. quasi-split) torus as a kernel. In particular, we have an embedding $G_1 \hookrightarrow H$. (This argument should also be included at the beginning of the proof of 2.8 of [33], p. 231, too.) We will denote all maps in the following diagrams (over R and its extension R') by the same symbols:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & F & \xrightarrow{i} & G_1 & \xrightarrow{\lambda} & G & \to & 1 \\
 & & \downarrow j & & \downarrow j & & \downarrow = & & \\
1 & \to & T_1 & \xrightarrow{i} & H & \xrightarrow{\alpha} & G & \to & 1 \\
 & & \gamma\downarrow & & \downarrow\gamma & & & & \\
 & & T & = & T & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \qquad (*)$$

where T_1 is an induced (i.e. quasi-split) R-torus. From this diagram we derive the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & T(R) \\
 & & \downarrow & & \downarrow & & & & \downarrow \delta \\
1 & \to & F(R) & \xrightarrow{i} & G_1(R) & \xrightarrow{\lambda} & G(R) & \xrightarrow{\beta} & \mathrm{H}^1_{et}(R,F) \\
 & & \downarrow j & & \downarrow j & & \downarrow = & & \downarrow \theta \\
1 & \to & T_1(R) & \xrightarrow{i} & H(R) & \xrightarrow{\alpha} & G(R) & \to & 1 \\
 & & \downarrow \gamma & & \downarrow \gamma & & & & \\
 & & T(R) & = & T(R) & & & & \\
 & & \downarrow \delta & & \downarrow \zeta & & & & \\
 & & & & & & & &
\end{array} \qquad (*')$$

$$G(R) \xrightarrow{\beta} \mathrm{H}^1_{et}(R,F) \xrightarrow{i_*} \mathrm{H}^1_{et}(R,G_1).$$

Here j denotes the inclusion, $j(x) = x$ for all $x \in G_1(R)$. We need the following lemma, which is an analog of a lemma due to Merkurjev (see [15]) in the case of fields. It is valid for any cross-diagram $(*)$ above, where T is not necessarily an induced torus.

Lemma 4.1.1 *With the cross-diagram as above, we have the following anti-commutative diagram*

$$\begin{array}{ccc} H(R) & \stackrel{\alpha}{\to} & G(R) \\ \gamma \downarrow & & \downarrow \beta \\ T(R) & \stackrel{\delta}{\to} & \mathrm{H}^1_{et}(R,F) \end{array}$$

for any commutative ring R, i.e. for any $h \in H(R)$, we have $\delta(\gamma(h)) = -\beta(\alpha(h))$.

Proof. In our case, it is well-known by a theorem of Artin, that the Čech cohomology agrees with the étale cohomology (see [1], Corollary 4.2, or [17], Chap. III, Theorem 2.17), so we may and shall replace $\mathrm{H}^1_{et}(R,F)$ by $\check{\mathrm{H}}^1_{et}(R,F)$. For any covering S/R, we set $S^{\otimes n} := S \otimes_R \cdots \otimes_R S$ (n-times). Let

$$e_i : S^{\otimes n} \to S^{\otimes(n+1)}$$

be the map $s_1 \otimes \cdots \otimes s_{i-1} \otimes s_i \cdots \otimes s_n \mapsto s_1 \otimes \cdots \otimes s_{i-1} \otimes 1 \otimes s_i \otimes \cdots s_n$, and consider the following Čech - Amitsur complex related with the extension S/R (see , e.g., [11], Exp. 190, [17]. Chap. III, Sec. 2, or [27], Exp. V)

$$1 \to F(R) \to F(S) \stackrel{d_0}{\to} F(S^{\otimes 2}) \stackrel{d_1}{\to} F(S^{\otimes 3}) \stackrel{d_2}{\to} F(S^{\otimes 4}) \to \cdots, \qquad (1)$$

where F is considered as a covariant functor from the category $Com.Alg_R$ of commutative R-algebras to the category Gr of groups and the differential d_i are given by the formula (written additively in the commutative case, for simplicity)

$$d_i = F(e_1) - F(e_2) + \cdots + (-1)^{i+1} F(e_{i+2}).$$

In particular, we have $d_0(f) = f_1 - f_2$, for all $f \in F(S)$, and for $f \in F(S)$, $f \in \mathrm{Im}\ (F(R) \to F(S))$ if and only if $f \in \mathrm{Ker}\ (d_0)$. By convention, for $x \in F(S^{\otimes n})$, we denote

$$x_{i_1 \ldots i_t} := F(e_{i_t}) \circ F(e_{i_{t-1}}) \circ \cdots \circ F(e_{i_1})(x).$$

Denote $Z^1(S/R,F) := \mathrm{Ker}\ (d_1) \subset F(S^{\otimes 2})$ the set of 1-cocycles, $Z^2(S/R,F) = \mathrm{Ker}\ (d_2) \subset H(S^{\otimes 3})$ the set of 2-cocycles, etc... Then

$$\check{\mathrm{H}}^1_{et}(S/R,F) = \mathrm{Ker}\ d_1/\mathrm{Im}\ d_0 = Z^1(S/R,F)/\mathrm{Im}\ d_0,$$
$$\check{\mathrm{H}}^2_{et}(S/R,F) = \mathrm{Ker}\ d_2/\mathrm{Im}\ d_1 = Z^2(S/R,F)/\mathrm{Im}\ d_1 ...$$
$$\check{\mathrm{H}}^p_{et}(R,F) := \lim_{\to S/R} \check{\mathrm{H}}^p_{et}(S/R,F), p \geq 0.$$

Take any element $h \in H(R)$, $g = \alpha(h)$. Consider the top exact sequence in the diagram $(*')$. To describe the image $\beta(g) \in \check{\mathrm{H}}^1_{et}(R, F)$, we take a faithfully flat étale extension S/R such that $g \in \mathrm{Im}\ (G_1(S) \to G(S))$, say $g = \lambda(g') = \alpha(j(g'))$. Hence there exists $t' \in T_1(S)$ such that $h = g't'$. We know that the element $f := g_2'^{-1} g_1' \in F(S^{\otimes 2})$ is a 1-cocycle in $Z^1(S/R, F)$, i.e. $f_2 = f_3 f_1$. The image of f in $\check{\mathrm{H}}^1_{et}(R, F)$ will be $\beta(g)$. On the other hand, since $h = g't'$, it follows that $\gamma(h) = \gamma(t') = t \in T(R)$. Since $t' \in T_1(S)$, the image $\delta(t)$ of t will be the image of the 1-cocycle $f' = t_2'^{-1} t_1'$ in $\check{\mathrm{H}}^1_{et}(R, F)$. Therefore the element ff' has as its image the element $\beta(g)\delta(t) \in \check{\mathrm{H}}^1_{et}(R, F)$. One checks that ff' satisfies the following equalities

$$ff' = g_2'^{-1} g_1' t_2'^{-1} t_1' = (g't')_2^{-1} (g't')_1 = 1$$

since T_1 is a central torus of H, and we have $h_1 = h_2$ due to the faithfully flat descent, or by considering the Čech - Amitsur complex

$$1 \to H(R) \to H(S) \overset{d_{H,0}}{\to} H(S^{\otimes 2})$$

associated to *noncommutative* group scheme H, since for $h \in H(R)$, we have $d_0(h) = 1$ if and only if $h_1 = h_2$. (We thank the referee for indicating this last argument.) The lemma is therefore proved. ■

We continue the proof of 4.8 and we assume first that $p = 0$. Note that in this case, the Weak Corestriction Principle is just the usual Corestriction Principle. We denote the corestriction map with respect to extension R'/R for F by

$$\mathrm{Cores}_{R'/R,F} : \mathrm{H}^i_{et}(R', F) \to \mathrm{H}^i_{et}(R, F).$$

Since T_1 is an induced torus, by assumption, we have $\mathrm{H}^1_{et}(R', T_1) = 0$, so we have $\alpha(H(R')) = G(R')$. Now for any $g' \in G(R')$, let $h' \in H(R')$ such that $\alpha(h') = g'$, and denote $t' = \gamma(h')$, $f' = \beta(g')$. Since the diagram in Lemma 4.1.1 is anti-commutative, we have

$$\delta(t') = \delta(\gamma(h')) = -\beta(\alpha(h'))$$

$$= -f'.$$

Let $f := \mathrm{Cores}_{R'/R,F}(f')$. We need to show that $f \in \mathrm{Im}\ (\beta)$. By assumption, the Corestriction Principle holds for the image of $H(R) \to T(R)$, so there is $h \in H(R)$ such that

$$\gamma(h) = t := N_{R'/R}(t').$$

Set $g = \alpha(h)$. Then

$$\delta(t) = \delta(\gamma(h))$$

$$= -\beta(\alpha(h))$$

$$
\begin{aligned}
&= -\beta(g) \\
&= \delta(N_{R'/R}(t')) \\
&= \mathrm{Cores}_{R'/R,F}(\delta(t')).
\end{aligned}
$$

Since $\delta(t') = -f'$ (see above), we get $\mathrm{Cores}_{R'/R,F}(\delta(t')) = -f$. Therefore $f = \beta(g)$ and $f \in \langle \mathrm{Im}\ (\beta) \rangle$ and the case $p = 0$ is proved.

Now let $p = 1$. For any extension $R' \in \mathcal{C}_R$ of R, g' any element from $\mathrm{H}^1_{et}(R', G)$, by Lemma 2.2.1 we may choose a g'-lifting z-extension

$$1 \to T_1 \to H \to G \to 1,$$

defined over R, such that there is an embeding $F \hookrightarrow T_1$. In fact, choose an R-embedding $F \hookrightarrow Z_1$, and assume that

$$1 \to Z_2 \to H_2 \to G \to 1$$

is a g'-lifting z-extension over R, where $Z_i, i = 1, 2$ are induced R-tori (Lemma 2.2.1, a), c)). Set $H := Z_1 \times_R H_2, T_1 := Z_1 \times_R Z_2$. We have the following exact sequence of R-group schemes

$$1 \to Z_1 \to H \to H_2 \to 1.$$

Since Hilbert's Theorem-90 holds for induced tori defined over Spec $R', R' \in \mathcal{C}_R$, it follows that we have a surjective map $f : \mathrm{H}^1_{et}(R', H) \to \mathrm{H}^1_{et}(R', H_2)$. By considering the exact sequence of R-group schemes

$$1 \to T_1 \to H \to G \to 1,$$

it follows from the definition that T_1 is also an R-induced torus and it is clear that we have a z-extension of G. Since g' is lifted to an element of $\mathrm{H}^1_{et}(R', H_0)$, and since f is surjective, it follows that g' is also lifted to an element of $\mathrm{H}^1_{et}(R', H)$. Thus we have a z-extension of G

$$1 \to T_1 \to H \to G \to 1$$

which is g'-lifting such that $F \hookrightarrow T_1$. We consider the following diagram, which is similar to the one we have just considered, with the only difference that the dimension is shifted, and by abuse of the notation, we use the same symbol for maps induced on cohomologies.

$$\begin{array}{ccccccc} & & & & & & \mathrm{H}^1_{et}(R',T) \\ & & & & & & \downarrow \Delta \\ \mathrm{H}^1_{et}(R',F) & \xrightarrow{i} & \mathrm{H}^1_{et}(R',G_1) & \xrightarrow{\lambda} & \mathrm{H}^1_{et}(R',G) & \xrightarrow{\beta} & \mathrm{H}^2_{et}(R',F) \\ \downarrow j & & \downarrow j & & \downarrow = & & \downarrow \theta \\ \mathrm{H}^1_{et}(R',T_1) & \xrightarrow{i} & \mathrm{H}^1_{et}(R',H) & \xrightarrow{\alpha} & \mathrm{H}^1_{et}(R',G) & \to & \mathrm{H}^2_{et}(R',T_1) \\ \downarrow \pi & & \downarrow \gamma & & & & \\ \mathrm{H}^1_{et}(R',T) & = & \mathrm{H}^1_{et}(R',T) & & & & \\ \downarrow \Delta & & & & & & \end{array} \quad (*'')$$

$$\mathrm{H}^1_{et}(R',G) \xrightarrow{\beta} \mathrm{H}^2_{et}(R',F)$$

We need the following analogue of Lemma 4.1.1 for higher dimension.

Lemma 4.1.2 *We have the following anti-commutative diagram for any ring R and for any cross-diagram $(*)$*

$$\begin{array}{ccc} \mathrm{H}^1_{et}(R,H) & \xrightarrow{\alpha} & \mathrm{H}^1_{et}(R,G) \\ \Big\downarrow \gamma & & \Big\downarrow \beta \\ \mathrm{H}^1_{et}(R,T) & \xrightarrow{\Delta} & \mathrm{H}^2_{et}(R,F) \end{array}$$

Proof. As above, we may and shall replace $\mathrm{H}^1_{et}(R,F)$ by $\check{\mathrm{H}}^1_{et}(R,F)$, $\mathrm{H}^2_{et}(R,F)$ by $\check{\mathrm{H}}^2_{et}(R,F)$, $\mathrm{H}^1_{et}(R,H)$ by $\check{\mathrm{H}}^1_{et}(R,H)$, and $\mathrm{H}^1_{et}(R,G)$ by $\check{\mathrm{H}}^1_{et}(R,G)$. We consider the following Čech - Amitsur (non-abelian) complexes for a faithfully flat extension S/R

$$1 \to H(R) \to H(S) \overset{d_{H,0}}{\to} H(S^{\otimes 2}) \overset{d_{H,1}}{\to} H(S^{\otimes 3}), \quad (2)$$

$$1 \to G(R) \to G(S) \overset{d_{G,0}}{\to} G(S^{\otimes 2}) \overset{d_{G,1}}{\to} G(S^{\otimes 3}), \quad (3)$$

where the differentials $d_{H,i}$ are given by the formulae (written multiplicatively)

$$d_{H,0} = H(e_1)H(e_2)^{-1}, d_{H,1} = H(e_1)H(e_2)^{-1}H(e_3),$$

and the same for the differentials $d_{G,i}$. One defines $Z^1(S/R,H) = \text{Ker } d_{H,1} \subset H(S^{\otimes 2})$, and for $a, b \in Z^1(S/R,H)$, $a \sim b$ in $Z^1(S/R,H)$ if $a = c_1^{-1}bc_2$ for some $c \in H(S)$, and let

$$\check{\mathrm{H}}^1_{et}(S/R,H) = Z^1(S/R,H)/\sim .$$

Similarly, we define corresponding objects for G.

Take any element $\hat{h} \in \mathrm{H}^1_{et}(R,H)$. We may choose a covering S/R such that $\hat{h}$ is coming from S, i.e. due to the relation between Čech and étale cohomology said above, we may assume that $\hat{h} \in \check{\mathrm{H}}^1_{et}(S/R,H)$, which is the cohomology class $[h]$

of $h \in Z^1(S/R, H)$. Let $\hat{g} = \alpha(\hat{h}) \in \check{\mathrm{H}}^1_{et}(S/R, G)$. Then $\hat{g} = [g]$, where $g = \alpha(h)$, which is clearly a 1-cocycle with values in $S^{\otimes 2}$. We may take a faithfully flat étale extension $U/S^{\otimes 2}$, such that $g \in \mathrm{Im}\ (G_1(U) \to G(U))$, say $g = \lambda(y), y \in G_1(U)$. We need the following special case of a theorem of Artin [1], Sec. 2 (see also [12], Sec. 9).

Theorem 4.1.3 (Cf. [1], Theorem 4.1) *Let X be a noetherian scheme, such that any finite subset of points of which is contained in an affine open subset of X. Let $E \to X$ be an étale morphism of finite type, and let $\alpha : W \to E^{(n)} := E \times_X \cdots \times_X E$ ($n-$times) be an étale cover. Then there exists an étale surjective morphism $\beta : E' \to E$ such that $\beta^{(n)} : E'^{(n)} \to E^{(n)}$ factors through α.*

By this theorem, for $n = 2$, there is a faithfully flat étale extension S'/S such that $U \subset S' \otimes_R S'$. Thus by replacing S by S', we may assume from the very beginning, that there is $y \in G_1(S^{\otimes 2})$ such that $\lambda(y) = \alpha(y) = g = \alpha(h)$. Thus $y^{-1}h \in \mathrm{Ker}\ (\alpha) = \mathrm{Im}\ (i)$, and so $y^{-1}h = z$, or $h = yz$, with $z \in T_1(S^{\otimes 2})$. Since g is a 1-cocycle, we have

$$g_1 g_2^{-1} g_3 = 1,$$

i.e. $\lambda(y_1 y_2^{-1} y_3) = 1$, thus $y_1 y_2^{-1} y_3 \in \mathrm{Ker}\ (\lambda) = \mathrm{Im}\ (i : F \to G_1)$, say $y_1 y_2^{-1} y_3 = f \in F(S^{\otimes 3})$, which is a 2-cocycle in $Z^2(S/R, F)$ with value in $S^{\otimes 3}$. Its cohomological class $[f]$ is, as well-known, exactly the image $\beta([g])$. On the other hand, $t := \gamma(h) = \gamma(z) \in T(S^{\otimes 2})$ is a 1-cocycle with value in $S^{\otimes 2}$. As above, since $t_1 t_2^{-1} t_3 = 1$, it follows that $z_1 z_2^{-1} z_3 \in \mathrm{Ker}\ (\pi) = \mathrm{Im}\ (j)$, thus the cohomological class of $f' := z_1 z_2^{-1} z_3 \in F(S^{\otimes 3})$ is exactly the image $\Delta([t])$ in $\check{\mathrm{H}}^2_{et}(S/R, F)$. The product of these two elements in $\check{\mathrm{H}}^2_{et}(S/R, F)$ is the cohomological class of

$$f' f = (z_1 z_2^{-1} z_3)(y_1 y_2^{-1} y_3).$$

Since $h = yz \in Z^1(S/R, H)$, we have $h_1 h_2^{-1} h_3 = 1$. Due to the fact that T_1 is a central subgroup in H, we have

$$\begin{aligned} 1 &= (yz)_1 (yz)_2^{-1} (yz)_3 \\ &= (y_1 y_2^{-1} y_3)(z_1 z_2^{-1} z_3), \end{aligned}$$

hence $\Delta([t])\beta([g])$ is the trivial class in $\check{\mathrm{H}}^2_{et}(S/R, F)$ as required and the lemma is proven. ∎

With $g' \in \mathrm{H}^1_{et}(R', G)$ as above, let $h' \in \mathrm{H}^1_{et}(R', H)$ such that $g' = \alpha(h')$. (Recall that H is a g'-lifting z-extension.) Then

$$\beta(g') = -\Delta(\gamma(h'))$$

by the lemma above. Therefore

$$\begin{aligned} \mathrm{Cores}_{R'/R,F}(\beta(g')) &= -\mathrm{Cores}_{R'/R,F}(\Delta(\gamma(h'))) \\ &= -\Delta(\mathrm{Cores}_{R'/R,T}(\gamma(h'))). \end{aligned}$$

By assumption, we have $\mathrm{Cores}_{R'/R,T}(\gamma(h')) = \Sigma_i \epsilon_i \gamma(h_i)$ for $\epsilon_i = \pm 1$, and some $h_i \in \check{\mathrm{H}}^1_{et}(S/R, H)$. Let $g_i = \alpha(h_i) \in \check{\mathrm{H}}^1_{et}(S/R, G)$. Then from above we have

$$\begin{aligned}\mathrm{Cores}_{R'/R,F}(\beta(g')) &= -\Delta(\mathrm{Cores}_{R'/R,T}(\gamma(h'))) \\ &= -\Delta(\Sigma_i \epsilon_i \gamma(h_i)) \\ &= -\Sigma_i \epsilon_i \Delta(\gamma(h_i)) \\ &= \Sigma_i \epsilon_i \beta(\alpha(h_i)) \text{ (by } (**)) \\ &= \Sigma_i \epsilon_i \beta(g_i) \in \langle \mathrm{Im}\ (\beta) \rangle\end{aligned}$$

as required. (Note that, since F is a finite commutative group scheme over R, the group $\check{\mathrm{H}}^2_{et}(S/R, F)$, as well-known, is a torsion group, so the above (possibly alternated) sum can be written actually as a sum (with all $\epsilon_i = 1$). We thank the referee for this remark). For a R-reductive group scheme G we denote by $\bar{G}$ the adjoint group scheme of G, $\bar{G} = G/Cent(G)$, $\tilde{G}$ the simply connected covering of $\bar{G}$.

Proposition 4.10. *Let $\bar{G}_0$ be an adjoint semisimple R-group scheme and $\tilde{F} :=$ Ker $(\tilde{G}_0 \to \bar{G}_0)$.*
a) ($p = 0$) *Assume that Weak Corestriction Principle holds for the image of the coboundary map $\delta : \mathrm{H}^0_{et}(R, \bar{G}_0) \to \mathrm{H}^1_{et}(R, \tilde{F})$. Then the same holds for any connecting map $\alpha : \mathrm{H}^0_{et}(R, G) \to \mathrm{H}^0_{et}(R, T)$ for all connected reductive R-groups G, T, with T a torus such that $\bar{G} = \bar{G}_0$. In particular, if $f(0)$ holds then $a(0, 0)$ holds.*
b) ($p = 1$) *Assume that Corestriction Principle holds for the image of the coboundary map $\Delta : \mathrm{H}^1_{et}(R, \bar{G}_0) \to \mathrm{H}^2_{et}(R, \tilde{F})$. Then the same holds for any connecting map $\mathrm{H}^1_{et}(R, G) \to \mathrm{H}^1_{et}(R, T)$ for connected reductive groups G, T with a torus T such that $\bar{G} = \bar{G}_0$. In particular, if $f(1)$ holds then $a(1, 1)$ also holds.*

Proof. a) Notice that in the case $p = 0$, the Weak Corestriction Principle is just the Corestriction Principle. Assume that we are given an exact sequence $1 \to G_1 \to G \to T \to 1$, of connected reductive R-groups with T a torus. Let $G' = [G, G]$,S be the radical of G, which is a central torus of G. Then we have a surjective and flat R-isogeny $G' \times_{Spec\ R} S \to G$ (see [26], Exp. 22, 6.2.3), given by the product. Thus we may write $G = G'.S$. Denote $F' = Cent(G')$, $F = G' \cap S$, which are finite central subgroup schemes of G'. From Proposition 3.5 and its proof it follows that the Weak Corestriction Principle holds for the connecting map $\delta : \mathrm{H}^0_{et}(R, \bar{G}_0) \to \mathrm{H}^1_{et}(R, F')$.
Consider the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & G' & \to & G'.S & \to & S/F & \to & 1 \\ & & \downarrow & & \downarrow = & & \downarrow & & \\ 1 & \to & G_1 & \to & G'.S & \to & T & \to & 1 \end{array}$$

Since T is a torus, $G' \hookrightarrow G_1$ as a closed subgroup. Since the morphism $G \to T$ factors through the morphism $G \to S/F$, it is clear that the assertion follows if we can prove it for S/F. Thus in the sequel we may assume that $T = S/F$. Consider the following commutative diagram (notice that $\bar{G} = \bar{G}_0$ by assumption)

$$\begin{array}{ccccccccc} 1 & \to & F' & \to & G'.S & \to & \bar{G} \times S/F & \to & 1 \\ & & \downarrow & & \downarrow = & & \downarrow & & \\ 1 & \to & G' & \to & G'.S & \to & S/F & \to & 1 \end{array} \qquad (*''')$$

and also the following commutative diagram

$$\begin{array}{ccccc} G(R) & \xrightarrow{\beta} & \bar{G}(R) \times (S/F)(R) & \xrightarrow{\delta} & \mathrm{H}^1_{et}(R, F') \\ \downarrow = & & \downarrow p & & \downarrow q \\ G(R) & \xrightarrow{\alpha} & (S/F)(R) & \xrightarrow{\delta_1} & \mathrm{H}^1_{et}(R, G') \end{array}$$

and also the same diagram with R replaced by R'. From our assumption it follows that the Corestriction Principle holds for the image of δ', since it holds when restricting δ' to $\bar{G}(= \bar{G}_0)$ and to S/F. We claim that the composition of the maps

$$\bar{G}(R') \times (S/F)(R') \xrightarrow{p'} (S/F)(R') \xrightarrow{\mathrm{Cores}_{R'/R,S/F}} (S/F)(R) \xrightarrow{\delta_1} \mathrm{H}^1_{et}(R, G'),$$

and that of the maps

$$\bar{G}(R') \times (S/F)(R') \xrightarrow{\delta'} \mathrm{H}^1_{et}(R', F') \xrightarrow{\mathrm{Cores}_{R'/R,F'}} \mathrm{H}^1_{et}(R, F') \xrightarrow{q} \mathrm{H}^1_{et}(R, G')$$

are the same. Indeed, denote by p and q the maps similar to p' and q', by considering the rings R and R' interchanged. Then for $x = (g', s') \in \bar{G}(R') \times (S/F)(R')$ and $s = \mathrm{Cores}_{R'/R,S/F}(s') \in (S/F)(R)$ we have

$$\delta_1(\mathrm{Cores}_{R'/R,S/F}(p'(x))) = \delta_1(\mathrm{Cores}_{R'/R,S/F}(s')) = \delta_1(s).$$

Consider the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & F' & \to & G' & \to & \bar{G} & \to & 1 \\ & & \downarrow = & & \downarrow & & \downarrow & & \\ 1 & \to & F' & \to & G'.S & \to & \bar{G} \times S/F & \to & 1 \end{array}$$

where the second and third vertical arrows are natural inclusions. We derive from this the following commutative diagram

$$\begin{array}{ccccccc} F'(R) \to & G'(R) & \to & \bar{G}(R) & \stackrel{\delta}{\to} & \mathrm{H}^1_{et}(R, F') \\ \downarrow = & \downarrow & & \downarrow & & \downarrow = \\ F'(R) \to & (G'S)(R) & \to & \bar{G}(R) \times (S/F)(R) & \stackrel{\delta}{\to} & \mathrm{H}^1_{et}(R, F') \end{array}$$

and also the same diagram with R replaced by R'. By abuse of notation, since $\bar{G} \hookrightarrow \bar{G} \times S/F$, we denote the restriction of δ (resp. δ') to $\bar{G}(R)$ (resp. $\bar{G}(R')$) also by δ (resp. δ'). Then by assumption , there is $g \in \bar{G}(R)$ such that

$$\mathrm{Cores}_{R'/R,F'}(\delta'(g')) = \delta(g),$$

hence

$$\mathrm{Cores}_{R'/R,F'}(\delta'(g', s')) = \delta(g, s),$$

so it follows that for $y = (g, s) \in \bar{G}(R) \times (S/F)(R)$ we have

$$\begin{aligned} \delta_1(\mathrm{Cores}_{R'/R,S/F}(p'(x)) &= \delta_1(s) \\ &= \delta_1(p(g, s)) \\ &= q(\delta(g, s)) \\ &= q(\mathrm{Cores}_{R'/R,F'}(\delta'(g', s'))) \\ &= q(\mathrm{Cores}_{R'/R,F'}(\delta'(x))) \end{aligned}$$

as claimed. Now the assertion of the theorem follows from the equality $\alpha' = p'\beta'$. Indeed, let $x' \in G(R')$, $x'' = \beta'(x') = (g', s')$, $y' = \alpha'(x')$, $y = \mathrm{Cores}_{R'/R,S/F}(y')$. Then $\alpha'(x') = p'\beta'(x') = p'(x'')$ hence

$$\begin{aligned} \delta_1(\mathrm{Cores}_{R'/R,S/F}(p'(x''))) &= q(\mathrm{Cores}_{R'/R,F'}(\delta'(x''))) \\ &= q(\mathrm{Cores}_{R'/R,F'}(\delta'(\beta'(x')))) \\ &= 1, \end{aligned}$$

since $\delta'\beta' = 0$. Therefore

$$\mathrm{Cores}_{R'/R,S/F}(p'(x'')) \in \mathrm{Ker}\ (\delta_1) = \mathrm{Im}\ (\alpha).$$

b) Now consider the case $p = 1$. Let us be given an exact sequence

$$1 \to G_1 \to G \to T \to 1$$

of connected reductive k-groups with T a torus such that $\bar{G} = \bar{G}_0$. As in case a), we may assume that $T = S/F$. From Proposition 4.6 and its proof it follows that the Corestriction Principle holds for $\mathrm{H}^1_{et}(R, \bar{G}) \to \mathrm{H}^2_{et}(R, F')$. We may use the above notation and consider the following exact sequences

$$1 \to F' \to \quad G' \quad \to \quad \bar{G} \quad \to 1,$$

$$1 \to F' \to F'.S \to S/F \to 1$$

and the following derived exact sequences of cohomology sets:

$$\mathrm{H}^1_{et}(R,F') \to \mathrm{H}^1_{et}(R,G') \to \mathrm{H}^1_{et}(R,\bar{G}) \stackrel{\Delta_1}{\to} \mathrm{H}^2(R,F'),$$

$$\mathrm{H}^1_{et}(R,F') \to \mathrm{H}^1_{et}(R,SF') \to \mathrm{H}^1_{et}(R,S/F) \stackrel{\Delta_2}{\to} \mathrm{H}^2_{et}(R,F'),$$

also similar sequences when R is replaced by R' (where we put "$'$" on the corresponding maps). The following diagram, derived from $(*''')$, is commutative with exact lines

$$\begin{array}{ccccc} \mathrm{H}^1_{et}(R',G) & \stackrel{\beta'}{\to} & \mathrm{H}^1_{et}(R',\bar{G}) \times \mathrm{H}^1_{et}(R',S/F) & \stackrel{\Delta'}{\to} & \mathrm{H}^2(R',F') \\ \downarrow = & & \downarrow & & \\ \mathrm{H}^1_{et}(R',G) & \stackrel{\alpha'}{\to} & \mathrm{H}^1_{et}(R',S/F) & & \end{array}$$

Here one can check that the map Δ' is such that

$$\Delta'(g,s) = \Delta'_1(g) + \Delta'_2(s),$$

and the "+" is taken in $\mathrm{H}^2(R',F')$. Let $x' \in \mathrm{H}^1_{et}(R',G)$, $\beta'(x') = (g',s')$, $g' \in \mathrm{H}^1_{et}(R',\bar{G})$, $s' \in \mathrm{H}^1_{et}(R',(S/F))$, $s = \mathrm{Cores}_{R'/R,S/F}(s') \in \mathrm{H}^1_{et}(R,(S/F))$. Then $\Delta'(g',s') = 0$, so $\Delta'_2(s') = -\Delta'_1(g')$. By assumption, the Corestriction Principle holds for the image of Δ_1 so via corestriction map we have

$$\begin{aligned} \Delta_2(s) &= \mathrm{Cores}_{R'/R,S/F}(\Delta'_2(s')) \\ &= \mathrm{Cores}_{R'/R,S/F}(-\Delta'_1(g')) \\ &= -\mathrm{Cores}_{R'/R,S/F}(\Delta'_1(g')) \\ &= -\Delta_1(\bar{g}) \end{aligned}$$

for some $\bar{g} \in \mathrm{H}^1_{et}(R,\bar{G})$. Therefore $\Delta(\bar{g},s) = 0$, i.e. $(\bar{g},s) = \beta(g), g \in \mathrm{H}^1_{et}(R,G)$, or equivalently $s = \alpha(g)$. ∎

Remark. The same proof of Proposition 4.10, b) works also in the case $p = 0$, so we have another proof of Proposition 4.10, a).

Finally by summing up the results we proved above we obtain the following theorem which is the main result of this section.

Theorem 4.11. 1) *Assuming* (Hyp_R)*, then for Weak Corestriction Principle we have the following equivalent statements:*

$$\text{a)} \Leftrightarrow \text{b)},\ \text{c)} \Leftrightarrow \text{d)},\ \text{e)} \Leftrightarrow \text{f)}.$$

2) *If* R *is a ring such that* $\mathrm{H}^1_{et}(R', T) = 0$ *for any induced* R'*-torus* T*,* $R' \in \mathcal{C}_R$ *then we have the following interdependence between the statements* a) - f) *with particular values of* p *and* q
a) *For low dimension:*

$$a(0,1) \Leftarrow a(0,0) \Leftrightarrow b(0) \Leftrightarrow c(0) \Leftrightarrow d(0) \Leftrightarrow e(0) \Leftrightarrow f(0)$$

b) *For higher dimension:*

$$a(1,2) \Leftarrow a(1,1) \Leftrightarrow b(1)$$

$$\Downarrow$$

$$c(1) \Leftrightarrow d(1)$$

$$\Downarrow$$

$$e(1) \Leftrightarrow f(1)$$

In general, without assuming (Hyp_R)*, by ignoring* $b(i)$*, all above implications without* $b(i)$ *involving, hold true.*

Proof. It follows from results proved above . We just indicate the logical dependence of these statements ; the rest follows from this.
b) $\Rightarrow$ a) : see Proposition 4.2. a) $\Rightarrow$ b) : see Proposition 4.4. $a(0,0) \Rightarrow a(0,1)$, $a(1,1) \Rightarrow a(1,2)$: see Proposition 4.5. e) $\Leftrightarrow$ f) : see Proposition 4.6.c) $\Leftrightarrow$ d) : see Proposition 4.7.a) $\Rightarrow$ d) : see Proposition 4.8.f(0) $\Rightarrow a(0,0)$: see Proposition 4.9, a). The first part of b) also follows from this.
The other equivalent relations follow from above ones. ∎

If we consider the Corestriction Principle instead of Weak Corestriction Principle, then Theorem 4.11 still holds and we have a stronger assertion as follows.

Theorem 4.12. 1) *Assuming* (Hyp_R)*, then for Corestriction principle there are the following equivalence relations*

$$\text{a)} \Leftrightarrow \text{b)},\quad \text{c)} \Leftrightarrow \text{d)},\quad \text{e)} \Leftrightarrow \text{f)}.$$

2) *If* R *is a ring such that* $\mathrm{H}^1_{et}(R', T) = 0$ *for any induced* R'*-torus* T*,* $R' \in \mathcal{C}_R$*, then the following relations between above statements for certain values of* p, q

hold.
For low dimension we have

$$a(0,1) \Leftarrow a(0,0) \Leftrightarrow b(0) \Leftrightarrow c(0) \Leftrightarrow d(0) \Leftrightarrow e(0) \Leftrightarrow f(0).$$

For higher dimension we have

$$a(1,2) \Leftarrow a(1,1) \Leftrightarrow b(1) \Leftrightarrow c(1) \Leftrightarrow d(1) \Leftrightarrow e(1) \Leftrightarrow f(1)$$

3) *In general, without assuming* (Hyp_R)*, by ignoring* $b(i)$*, all above implications without* $b(i)$ *involving, hold true.*

Proof. The proof follows from above, by using Proposition 4.10, b). ∎

Remark. Notice that in the case of spectrum of a field, all condition related with smoothness can be omitted, and we can consider flat cohomology instead of étale cohomology (see [32]).

Corollary 4.13. *The Corestriction Principle in higher dimension* (*i.e. the statements* $a(1,1), b(1), c(1), d(1), e(1), f(1)$) *does not hold in general.*

Proof. The example given by Rosset and Tate shows that the Corestriction Principle for connecting map $\mathrm{H}^1(k, \mathrm{PGL}_n) \to \mathrm{H}^2(k, \mu_n)$ does not hold true in general. So by Theorem 3.2, b), in general, this is neither true for connecting maps $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, T)$. where G, T are reductive groups and T is a diagonalisable group, i.e. $a(1,1)$ does not hold. The other cases follow in the same way. ∎

5. A reduction to quasi-split case

Let G and T be reductive group schemes defined over a ring R, where T is commutative. In this section we are interested in Weak Corestriction Principle for the image of the coboundary map

$$\Delta : \mathrm{H}^1_{et}(R, G) \to \mathrm{H}^2_{et}(R, T).$$

This map is induced from the exact sequence of R-group schemes

$$1 \to T \to G_1 \xrightarrow{\pi} G \to 1,$$

where we assume G_1 is connected (and reductive) and T is a central closed R-subgroup scheme of G_1. Our goal is to reduce the proof of the Weak Corestriction Principle to the case of quasi-split group schemes.

We now show how to reduce the statement $a(1,2)$ of Sec. 3.3, that the Weak Corestriction Principle holds for the image of the coboundary map $\mathrm{H}^1_{et}(R, G) \to \mathrm{H}^2_{et}(R, T)$, where G is a connected reductive group, T is a central diagonalizable group, to the same statement in the quasi-split case. It is well-known that G_1 becomes split over a finite étale extension S/R (which follows from [26], Exp. XXII, Corollaire 2.3). Denote by G^q the unique quasi-split R-form of G, (see loc.cit, Exp. XXIV, Corollaire 3.12) such that G can be obtained from G^q by inner twisting, with an 1-cocycle $g \in Z^1(S/R, G^q)$. This twisting does not affect the group T, so we have the coboundary maps $\Delta : \mathrm{H}^1_{et}(R, G) \to \mathrm{H}^2_{et}(R, T)$, $\Delta_q : \mathrm{H}^1_{et}(R, G^q) \to \mathrm{H}^2_{et}(R, T)$.

Proposition 5.1. *With above notation, the image of Δ and that of Δ_q generate the same subgroup of $\mathrm{H}^2_{et}(R, T)$. Therefore, if the Weak Corestriction Principle holds for the image of Δ_q then it also holds for that of Δ.*

Corollary 5.2. *If the Weak Corestriction Principle holds for the image of coboundary maps $\Delta_q : \mathrm{H}^1_{et}(R, G^q) \to \mathrm{H}^2_{et}(R, T)$ for all quasi-split groups G^q and diagonalizable group T, then it also holds in general.*

Proof of Proposition 5.1. We have the following commutative diagram, where all vertical maps are bijections (the "translation maps") (see [23], Chap. I, Prop. 44 and [10], Chap. IV, Sec. 4, Prop. 4.3.4).

$$\begin{array}{ccc} \mathrm{H}^1_{et}(R, G) & \stackrel{\Delta}{\to} & \mathrm{H}^2_{et}(R, T) \\ \downarrow & & \downarrow \\ \mathrm{H}^1_{et}(R, G^q) & \stackrel{\Delta_q}{\to} & \mathrm{H}^2_{et}(R, T), \end{array}$$

and similar diagram over any étale finite extension of R. Since the image of Δ and Δ_q contains 0 (the neutral element of the group $\mathrm{H}^2_{et}(R, T)$), and the right vertical map is just the translation by the element $-\Delta_q(g)$, we see that for any element $g' \in \mathrm{H}^1(R, G)$, we have

$$\Delta(g') \in D_q(R) := \langle \mathrm{Im}\ (\Delta_q) \rangle.$$

By symmetry, for any element $g'' \in \mathrm{H}^1(R, G^q)$, we have

$$\Delta_q(g'') \in D(R) := \langle \mathrm{Im}\ (\Delta) \rangle.$$

Therefore $D(R) = D_q(R)$ and it is also true for any ring extension of R and the assertion follows. ∎

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