Vietnam Journal of Mathematics **38**:1(2010) 9-16

Vietnam Journal
of
MATHEMATICS

On Ideals of Subtraction Semigroups

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Received September 17, 2007
Revised November 19, 2009

Abstract. In this paper we have derived equivalent conditions for an ideal to be prime in a subtraction semigroup X. We have also shown that if I is an ideal of X with $I \cap M = \phi$, where M is an m-system then I is contained in a prime ideal P with $P \cap M = \phi$.

2000 Mathematics Subject Classification: 06F35.

Key words: subtraction semigroup, prime ideal, regular, strongly regular.

1. Introduction

Schein [4] considered systems of the form $(\phi; \circ, \backslash)$ where ϕ is a set of functions closed under the composition " $\circ$ " of functions (and hence $(\phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence$(\phi; \backslash)$ is a subtraction algebra in the sense of [1]). Zelinka [6] discussed a problem proposed by Schein concerning the structure of multiplication in a subtraction semigroup. Eun Hwan Roh, Kyung Ho Kim and Jong Geol Lee [2] obtained significant results in subtraction semigroups. We have defined an ideal which is different from that of Lee [2] and showed that if P is prime then for every $a,\ b \in X,\ aXb \subseteq P \Rightarrow a \in P$ or $b \in P$. The theory of subtraction semigroups runs almost parallel to the theory of rings. But in ring theory it has been shown that if R is a strongly regular ring then every prime ideal is maximal. But this is not true in subtraction semigroups. We have shown that if P is a prime k-bound ideal then P is maximal in a strongly regular subtraction semigroup X.

2. Preliminaries

Definition 2.1. A non empty set X together with a binary operation " $-$ " is said to be a subtraction algebra if it satisfies the following identities:

1. $x - (y - x) = x$.
2. $x - (x - y) = y - (y - x)$.
3. $(x - y) - z = (x - z) - y$, for every $x,\ y,\ z \in X$.

The subtraction determines an order relation on X : $a \leq b \Leftrightarrow a - b \leq 0$, where $a - a = 0$ is an element that does not depend on the choice of $a \in X$.

Example 2.2. Let A be any non empty set. Then $(P(A), \setminus)$ is a subtraction algebra, where "$P(A)$" denotes the power set of A and "$\setminus$" denotes the set theoretic subtraction.

Let X be a subtraction algebra. Then for every $x,\ y \in X$ we have:

1. $x - 0 = x$ and $0 - x = 0$.
2. $(x - y) - x = 0$.
3. $(x - y) - y = x - y$.
4. $(x - y) - (y - x) = x - y$.

Definition 2.3. Let X and $X^{'}$ be subtraction algebras. Then $f : X \to X^{'}$ is said to be a homomorphism if $f(x - y) = f(x) - f(y)$ for every $x,\ y \in X$.

Following [3], we have the following definition of subtraction semigroup.

Definition 2.4. A nonempty set X together with two binary operations " $-$ " and "." is said to be a subtraction semigroup if it satisfies the following:

1. $(X; -)$ is a subtraction algebra.
2. $(X; \cdot)$ is a semigroup.
3. $x(y - z) = xy - xz$ and $(x - y)z = xz - yz$ for every $x,\ y,\ z \in X$.

Example 2.5. Let $\mathit{\Gamma}$ be a subtraction algebra. Then the set $M_h(\mathit{\Gamma})$ of all homomorphisms of $\mathit{\Gamma}$ into $\mathit{\Gamma}$ is a subtraction semigroup under pointwise subtraction and composition of mappings.

Unless stated otherwise throughout this paper X stands for a subtraction semigroup not necessarily with identity.

3. Ideals of subtraction semigroups

Definition 3.1. Let $(X, -, \cdot)$ be a subtraction semigroup. A nonempty subset I of X is called a left (right) ideal if $x - y \in I$, for every $x \in I,\ y \in X$ and $XI \subseteq I$ $(IX \subseteq I)$. If I is both a left and right ideal then I is an ideal, denoted by $I \trianglelefteq X$.

Remark 3.2. Let $I \subseteq X$. Then the following are equivalent:
(i) $(\forall x \in I,\ y \in X)\ x - y \in I$.
(ii) $x \leq y$ and $y \in I \Rightarrow x \in I$.

Definition 3.3. An ideal I is said to be a k-ideal if $x - y \in I$ and $y \in I$ implies $x \in I$.

Example 3.4. Consider the following subtraction semigroup

$-$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	3	4	3	1
2	2	5	0	2	5	4
3	3	0	3	0	3	3
4	4	0	0	4	0	4
5	5	5	0	5	5	0

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	4	3	4	0
2	0	4	2	0	4	5
3	0	3	0	3	0	0
4	0	4	4	0	4	0
5	0	0	5	0	0	5

Here $\{0,\ 1,\ 3,\ 4\}$ is a k-ideal. $\{0,\ 3,\ 4,\ 5\}$ is an ideal but not a k-ideal, since $2 - 4 = 5 \in \{0,\ 3,\ 4,\ 5\}$ but $2 \notin \{0,\ 3,\ 4,\ 5\}$.

Definition 3.5. An ideal I is said to be k-bound if for every ideal $J \supset I,\ J$ is a k-ideal.

Example 3.6. Consider the subtraction semigroup given in Example 3.4. Here $\{0,\ 2,\ 3,\ 4,\ 5\}$ is a k-bound ideal.

Theorem 3.7. *Let $a \in X$. Then*
$< a >= \{x \in X | \, x - a = 0$ *or* $x - ar = 0$ *or* $x - sa = 0$ *or* $x - r_1 a r_2 = 0,\ r,\ s,\ r_1,\ r_2 \in X\}$ *is the principal ideal generated by a.*

Proof. Let $i \in < a >$ and $x \in X$. Then $i - a = 0$ or $i - ar = 0$ or $i - sa = 0$ or $i - r_1 a r_2 = 0$, for some $r,\ s,\ r_1,\ r_2 \in X$. Suppose $i - a = 0$. Then $(i - x) - a = (i - a) - x = 0$ and hence $i - x \in < a >$. Similarly for other cases also we can show that $i - x \in < a >$. Similarly $ix \in < a >$ and $xi \in < a >$ for every $x \in X$. Clearly $a \in < a >$. Thus $< a >$ is an ideal containing a. Let A be an ideal of X containing a. Let $x \in < a >$. Then $x - a = 0$ or $x - ar = 0$ or $x - sa = 0$ or $x - r_1 a r_2 = 0$, for some $r,\ s,\ r_1,\ r_2 \in X$. Since A is an ideal containing a, we have $ar,\ sa,\ r_1 a r_2 \in A$. Since $x \leq a$ or $x \leq ar$ or $x \leq sa$ or $x \leq r_1 a r_2$, for some $r,\ s,\ r_1,\ r_2 \in X$, by Remark 3.2, we have $x \in A$. Hence $< a >$ is the smallest ideal of X containing a. ■

Definition 3.8. An ideal P is said to be prime if for ideals $A,\ B$ of $X,\ AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Example 3.9. Consider the following subtraction semigroup

−	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

.	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

Here $\{0,\ b,\ c\}$ is a prime ideal. But $\{0,\ a\}$ is not prime.

A subset A of X is said to be right (left) invariant if $AX \subseteq A(XA \subseteq A)$. For A, $B \subseteq X$, $(A : B)_l = \{x \in X\,|xB \subseteq A\}$ and $(A : B)_r = \{x \in X\,|Bx \subseteq A\}$. If A is an ideal and B is right (left) invariant then $(A : B)_r$ $((A : B)_l)$ is an ideal. For $A,\ B \subseteq X,\ AB = \{ab\,|a \in A,\ b \in B\}$.

Theorem 3.10. *Let P be an ideal of X. Then the following are equivalent:*

1. *P is a prime ideal.*
2. *If $a,\ b \in X$ such that $aXb \subseteq P$, then $a \in P$ or $b \in P$.*
3. *If $a,\ b \in X$ such that $<a><b> \subseteq P$, then $a \in P$ or $b \in P$.*
4. *$\forall\ I,\ J \trianglelefteq X:\ I \supset P$ and $J \supset P \Rightarrow IJ \nsubseteq P$.*
5. *$\forall\ I,\ J \trianglelefteq X:\ I \nsubseteq P$ and $J \nsubseteq P \Rightarrow IJ \nsubseteq P$.*
6. *If U and V are right ideals in X such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.*
7. *If U and V are left ideals in X such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.*

Proof. (1)⇒(2) Let $a,\ b \in X$ such that $aXb \subseteq P$. Then $XaXbX \subseteq P$ and hence $(XaX)(XbX) \subseteq P$. Thus $<XaX><XbX> \subseteq P$. Since P is prime $<XaX> \subseteq P$ or $<XbX> \subseteq P$. Suppose that $<XaX> \subseteq P$. Let $x \in X<a>X$. Then $x = y_1a'y_2,\ y_1,\ y_2 \in X$, where $a' - a = 0$ or $a' - ar = 0$ or $a' - sa = 0$ or $a' - r_1ar_2 = 0,\ r,\ s,\ r_1,\ r_2 \in X$. Thus $x \in <XaX>$. Now $<a>^3 \subseteq X<a>X \subseteq <XaX> \subseteq P$. Hence $a \in P$. Similarly if $XbX \subseteq P$, it follows that $b \in P$.

(2)⇒(3) Let $a,\ b \in X$ such that $<a><b> \subseteq P$. Now $aXb \subseteq <a><b> \subseteq P$. Hence $a \in P$ or $b \in P$.

(3)⇒(4) Let $I,\ J \trianglelefteq X:\ I \supset P$ and $J \supset P$. Take $i \in I \backslash P$ and $j \in J \backslash P$. Then $<i><j> \nsubseteq P$. Hence $IJ \nsubseteq P$.

(4)⇒(5) Let $I,\ J \trianglelefteq X:\ I \nsubseteq P$ and $J \nsubseteq P$. Take $i \in I \backslash P$ and $j \in J \backslash P$. Then $<i> \cup P \supset P$ and $<j> \cup P \supset P$. Hence $(<i> \cup P)(<j> \cup P) \nsubseteq P$. Since P is an ideal $<i><j> \nsubseteq P$. Thus $IJ \nsubseteq P$.

(5)⇒(1) is obvious.

(3)⇒(6) Suppose that U and V are right ideals in X such that $UV \subseteq P$. Let us assume that $U \nsubseteq P$. Let $u \in U \backslash P$ and $v \in V$. Since $<u><v> \subseteq P,\ v \in P$.

Similarly (3)⇒(7) can be shown. It is trivial that (6)⇒(1) and (7)⇒(1). ■

Definition 3.11. A set M of elements of X is said to be an m-system if for every $a,\ b \in M$, there exists $x \in X$ such that $axb \in M$.

Theorem 3.12. *Let $M \subseteq X$ be an m-system in X and I be an ideal of X with $I \cap M = \phi$. Then I is contained in a prime ideal $P \neq X$ with $P \cap M = \phi$.*

Proof. Let $\mathcal{I} = \{J \trianglelefteq X :\ J \cap M = \phi\}$. Clearly I is non empty. By Zorn's lemma, $\mathcal{I}$ contains a maximal element P. P is an ideal $\neq X$. Let $J_1,\ J_2 \trianglelefteq X :\ J_1 \supset P$ and $J_2 \supset P$. Take some $j_1 \in J_1 \cap M$ and $j_2 \in J_2 \cap M$. Hence there exists an $x \in X$ such that $j_1 x j_2 \in M$. Since $< j_1 >< j_2 > \subseteq J_1 J_2,\ J_1 J_2 \nsubseteq P$. Hence P is prime by Theorem 3. 10. ■

Definition 3.13. An ideal Q is said to be semiprime if for every ideal I of X such that $I^2 \subseteq Q \Rightarrow I \subseteq Q$.

Example 3.14. Consider the following subtraction semigroup

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

.	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
c	0	0	0	c

Here $\{0,\ a\}$ is not a semiprime ideal. Consider the subtraction semigroup given in Example 3.9. Here $\{0, b\}$ is a semiprime ideal which is not prime.

Definition 3.15. If $I \trianglelefteq X$, then $P(I)$ is the intersection of all prime ideals containing I. Clearly $P(I)$ is a semiprime ideal containing I.

Theorem 3.16. *Let X be a subtraction semigroup. If $x \in P(I)$, then there exists $k \in \mathbb{N}$ such that $x^k \in I$.*

Proof. Let $I \trianglelefteq X$ and $x \in P(I)$. Let $M = \{x,\ x^2,\ x^3, \ldots\}$. If $I \cap M = \phi$ then by Theorem 3.12, there is some prime ideal $P \supseteq I$ with $P \cap M = \phi$, a contradiction to $x \in P(I)$. Hence $I \cap M \neq \phi$. Thus there exists $k \in \mathbb{N} :\ x^k \in I$. ■

Theorem 3.17. *Let Q be an ideal of X. Then the following are equivalent:*

1. *Q is a semiprime ideal.*
2. *If $a \in X$ such that $aXa \subseteq Q$, then $a \in Q$.*
3. *If $a \in X$ such that $< a >^2 \subseteq Q$, then $a \in Q$.*
4. *If U is a right ideal in X such that $U^2 \subseteq Q$, then $U \subseteq Q$.*
5. *If U is a left ideal in X such that $U^2 \subseteq Q$, then $U \subseteq Q$.*

Proof. The proof is similar to that of Theorem 3.10. ■

4. Regular subtraction semigroups

A subtraction semigroup X is said to be regular if given $a \in X$, there is $x \in X$ such that $axa = a$. X is called strongly regular when for each $a \in X,\ a = xa^2$, for some $x \in X$. A subtraction semigroup X is said to have IFP (insertion of factors property) if for $a,\ b$ in X if $ab = 0$ implies $axb = 0$ for all $x \in X$. An element $x \in X$ is said to be nilpotent if there exists a positive integer n such that $x^n = 0$. An ideal I is said to be nil if every element of I is nilpotent. X is said to be reduced if X has no non zero nilpotent elements. An element $e \in X$ is said to be idempotent if $e^2 = e$. An element $a \in X$ is said to be central if $ax = xa$ for every $x \in X$. For $C \subseteq X$, we denote the set $\{x \in X \mid xC = 0\}$ by $l(C)$ and $\{x \in X \mid Cx = 0\}$ by $r(C)$. For $A,\ B \subseteq X$ we define $A - B = \{a - b \mid a \in A,\ b \in B\}$. If A and B are ideals then $A - B$ is also an ideal.

Example 4.1. The subtraction semigroup given in Example 3.9 is a regular subtraction semigroup. But the subtraction semigroup given in 3.14 is not regular since there is no $x \in X$ such that $a = axa$.

Theorem 4.2. *Let X be a subtraction semigroup. Then $N = \{a \in X \mid < a > \text{ is a nil ideal in } X\}$ is the largest nil ideal in X.*

Proof. Let $a \in N$ and $b \in X$. Then $< a >$ is a nil ideal in X. Let $t \in < a - b >$. Then $t - (a - b) = 0$ or $t - (a - b)r = 0$ or $t - s(a - b) = 0$ or $t - r_1(a - b)r_2 = 0,\ r,\ s,\ r_1,\ r_2 \in X$. Thus $t = a_1 - b_1,\ a_1 \in < a >,\ b_1 \in < b >$. Since a_1 is nilpotent there exists a positive integer n such that ${a_1}^n = 0$. Hence $t^n = 0$. Thus $a - b \in N$. Let $r \in X$. Then $< ar > \subseteq < a >$ and hence $ar \in N$. Similarly $ra \in N$. Clearly N is a nil ideal in X. Let R be a nil ideal and $s \in R$. Hence $< s >$ is a nil ideal and so $s \in N$. ■

Lemma 4.3. *If X is reduced then the idempotents are central.*

Proof. Now $ab = 0$ implies $ba = 0$ since $(ba)^2 = b(ab)a$. Again for any x in $X,\ (axb)^2 = ax(ba)xb = 0$. Hence $axb = 0$. Thus X has IFP. For $e^2 = e,\ x \in X,\ e(xe - exe) = 0$. Hence $xe(xe - exe) = 0$ and $exe(xe - exe) = 0$. Thus $(xe - exe)^2 = 0$, showing that $xe - exe = 0$. Also $e(exe - xe) = 0$ implies $exe(exe - xe) = 0$ and $xe(exe - xe) = 0$. Thus $exe - xe = 0$. Hence $xe \leq exe \leq xe$. Thus $xe = exe$. Similarly $ex = exe$ and hence $xe = ex$. ■

Lemma 4.4. *If X has IFP, then for each subset S of X, $l(S)$ and $r(S)$ are k-ideals of X.*

Proof. Let $x \in l(S)$ and $y \in X$. Then $xS = 0$. Now for $s \in S,\ (x - y)s = xs - ys = 0$, showing that $x - y \in l(S)$. Since X has IFP for $r \in X,\ xrs = 0$ for every $s \in S$. Hence $xr \in l(S)$. Clearly $rx \in l(S)$. Let $x - y \in l(S)$ and $y \in l(S)$. For $s \in S,\ 0 = (x - y)s = xs - ys = xs$ showing that $x \in l(S)$. Similarly it can be verified for $r(S)$. ■

Remark 4.5. If X is reduced then $ab = 0$ implies $ba = 0$ and hence $l(S) = r(S)$, denoted by $A(S)$.

Theorem 4.6. *A subtraction semigroup X is strongly regular if and only if it is regular and without nonzero nilpotent elements.*

Proof. Let X be strongly regular. Suppose $a \in X$ such that $a^2 = 0$. Since X is strongly regular there exists some $x \in X$ such that $a = xa^2 = 0$. Thus $a^2 = 0$ implies $a = 0$ for every a in X. Hence X is without nonzero nilpotent elements. Let $a \in X$. Then $a = xa^2$, for some $x \in X$. Hence $(a - axa)a = 0$. Since X is without nonzero nilpotent elements, we have $ab = 0$ implies $ba = 0$ and therefore X has IFP. Hence $a - axa = 0$. Similarly we have $axa - a = 0$ and hence $a = axa$.

Conversely, let X be a regular subtraction semigroup without nonzero nilpotent elements. Let $a \in X$. Since X is regular $a = aya$, for some $y \in X$. Since ya is an idempotent, by Lemma 4.3, $a = aya = ya^2$. Thus X is strongly regular. ■

It is well known that if R is a strongly regular ring, then every prime ideal is maximal. But this is not true in subtraction semigroups. Consider the subtraction semigroup given in Example 3.4, which is strongly regular. Here $\{0, 3, 5\}$ is a prime ideal which is not maximal since $\{0, 3, 5\} \subset \{0, 3, 4, 5\}$.

Theorem 4.7. *Let X be a strongly regular subtraction semigroup. Then*
(a) *Xa is a k-ideal for all $a \in X$.*
(b) *Every k-bound prime ideal is maximal.*
(c) *Every ideal I of X fulfills $I = I^2$.*

Proof. (a) Let X be a strongly regular subtraction semigroup and let $a \in X$. Then $a = xa^2$ for some $x \in X$. Hence $a = axa$ by Theorem 4.6. Let $xa = e$.Then e is an idempotent and $Xa = Xe$. Denoting the set $\{n - ne | n \in X\}$ by S we claim that $A(S) = Xe$. Since $(n - ne)e = 0$ for any $n \in X$ using IFP $(n - ne)Xe = 0$. Hence $Xe \subseteq A(S)$. Suppose $y \in A(S)$. Since X is strongly regular there exists some $z \in X$ such that $y = zy^2$. Now $(zy - zye)y = 0$. Thus $y - ye = 0$ by Lemma 4.4. Also $((ye - y) - (ye - y)e)y = 0$, so that $(ye - y)y = 0$. Then $y(ye - y) = 0$ and $ye(ye - y) = 0$. Now $(ye - y)^2 = 0$ and hence $ye - y = 0$. Thus $y = ye \in Xe$ and it follows that $Xa = Xe = A(S)$. Since $A(S)$ is a k-ideal, Xa is a k-ideal.
(b) Let P be a prime k -bound ideal and suppose $P \subset M$ for some ideal M of X. Let $a \in M \backslash P$. Now $a = xa^2$ for some $x \in X$. For any $n \in X$, $na = nxa^2$. Hence $(n - nxa)a = 0$. Since X has IFP, $X(n - nxa)Xa = 0$. Thus $Xa \subseteq P$ or $X(n - nxa) \subseteq P$. Suppose $Xa \subseteq P$. Since $a = xa^2 \in Xa$, we have $a \in P$ a contradiction. Suppose $X(n-nxa) \subseteq P$. Since X is strongly regular, $(n-nxa) = x(n - nxa)^2 \in X(n - nxa) \subseteq P \subset M$. Then $(n - nxa) \in P \subset M$. Since $a \in M$ and P is k-bound, $n \in M$. Thus $M = X$.
(c) The proof is obvious. ■

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