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K-Theory of the Leaf Space of Foliations Formed by the Generic K-Orbits of Some Indecomposable MD_5-Group

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Abstract. The paper is a continuation of the authors' work [18]. In [18], we consider foliations formed by the maximal dimensional K-orbits (MD_5-foliations) of connected MD_5-groups such that their Lie algebras have 4-dimensional commutative derived ideals and give a topological classification of the considered foliations. In this paper, we study K-theory of the leaf space of some of these MD_5-foliations and analytically describe and characterize the Connes' C^*-algebras of the considered foliations by the method of K-functors.

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1. Introduction

In the decades 1970s-1980s, works of Diep [4], Rosenberg [10], Kasparov [7], Son and Viet [12],... have seen that K-functors are well adapted to characterize a large class of group C^*-algebras. Kirillov's method of orbits allows to find out the class of Lie groups MD, for which the group C^*-algebras can be characterized by means of suitable K-functors (see [5]). In terms of Diep, an MD-group of dimension n (for short, an MD_n-group) is an n-dimensional solvable real Lie group whose orbits in the co-adjoint representation (i.e., the K- representation) are the orbits of zero or maximal dimension. The Lie algebra of an MD_n-group is called an MD_n-algebra (see [5, Section 4.1]).

In 1982, studying foliated manifolds, Connes [3] introduced the notion of C^*-algebra associated to a measured foliation. In the case of Reeb foliations (see Torpe [14]), the method of K-functors has been proved to be very effective in describing the structure of Connes' C^*-algebras. For every MD-group G, the family of K-orbits of maximal dimension forms a measured foliation in terms of Connes [3]. This foliation is called MD-foliation associated to G.

Combining the methods of Kirillov (see [8, Section 15]) and of Connes (see [3, Section 2, 5]), the first author had studied MD_4-foliations associated with all indecomposable connected MD_4-groups and characterized Connes' C^*-algebras of these foliations in [16]. Recently, Vu and Shum [17] have classified, up to isomorphism, all the MD_5-algebras having commutative derived ideals.

In [18], we have given a topological classification of MD_5-foliations associated to the indecomposable connected and simply connected MD_5-groups, such that MD_5-algebras of them have 4-dimensional commutative derived ideals. There are exactly 3 topological types of the considered MD_5-foliations, denoted by $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$. All MD_5-foliations of type $\mathcal{F}_1$ are the trivial fibrations with connected fibre on 3-dimensional sphere S^3, so Connes' C^*-algebras of them are isomorphic to the C^*-algebra $C(S^3) \otimes \mathcal{K}$ following [3, Section 5], where $\mathcal{K}$ denotes the C^*-algebra of compact operators on an (infinite dimensional separable) Hilbert space.

The purpose of this paper is to study K-theory of the leaf space and to characterize the structure of Connes' C^*-algebras $C^*(V, \mathcal{F})$ of all MD_5-foliations $(V, \mathcal{F})$ of type $\mathcal{F}_2$ by the method of K-functors. Namely, we will express $C^*(V, \mathcal{F})$ by two repeated extensions of the form

$$0 \longrightarrow C_0(X_1) \otimes \mathcal{K} \longrightarrow C^*(V, \mathcal{F}) \longrightarrow B_1 \longrightarrow 0 \ ,$$

$$0 \longrightarrow C_0(X_2) \otimes \mathcal{K} \longrightarrow B_1 \longrightarrow C_0(Y_2) \otimes \mathcal{K} \longrightarrow 0 \ ,$$

then we will compute the invariant system of $C^*(V, \mathcal{F})$ with respect to these extensions. If the given C^*-algebras are isomorphic to the reduced crossed products of the form $C_0(V) \rtimes H$, where H is a Lie group, we can use the Thom-Connes isomorphism to compute the connecting map δ_0, δ_1.

In another paper, we will study the similar problem for all MD_5-foliations of type $\mathcal{F}_3$.

2. The MD_5-Foliations of Type $\mathcal{F}_2$

Originally, we will recall geometry of K-orbits of MD_5-groups which associate with MD_5-foliations of type $\mathcal{F}_2$ (see [18]).

In this section, G will be always a connected and simply connected MD_5-group such that its Lie algebras $\mathcal{G}$ is an indecomposable MD_5-algebra generated by $\{X_1, X_2, X_3, X_4, X_5\}$ with $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}] = \mathbb{R}.X_2 \oplus \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \cong \mathbb{R}^4$, $ad_{X_1} \in \mathrm{End}(\mathcal{G}) \equiv \mathrm{Mat}_4(\mathbb{R})$. Namely, $\mathcal{G}$ will be one of the following Lie algebras which are studied in [17, 18].

$\mathcal{G}_{5,4,11(\lambda_1,\lambda_1,\varphi)}$

$$ad_{X_1} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix};\ \lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}, \lambda_1 \neq \lambda_2, \varphi \in (0, \pi).$$

$\mathcal{G}_{5,4,12(\lambda,\varphi)}$

$$ad_{X_1} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix};\ \lambda \in \mathbb{R}\backslash\{0\}, \varphi \in (0, \pi).$$

$\mathcal{G}_{5,4,13(\lambda,\varphi)}$

$$ad_{X_1} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix};\ \lambda \in \mathbb{R}\backslash\{0\}, \varphi \in (0, \pi).$$

The connected and simply connected Lie groups corresponding to these algebras are denoted by $G_{5,4,11\,(\lambda_1,\lambda_1,\varphi)}$, $G_{5,4,12(\lambda,\varphi)}$, $G_{5,4,13(\lambda,\varphi)}$. All of these Lie groups are MD_5-groups (see [17]) and G is one of them. We now recall the geometric description of the K-orbits of G in the dual space $\mathcal{G}^*$ of $\mathcal{G}$. Let $\{X_1^*, X_2^*, X_3^*, X_4^*, X_5^*\}$ be the basis in $\mathcal{G}^*$ dual to the basis $\{X_1, X_2, X_3, X_4, X_5\}$ in $\mathcal{G}$. Denote by Ω_F the K-orbit of G including $F = (\alpha, \beta + i\gamma, \delta, \sigma)$ in $\mathcal{G}^* \cong \mathbb{R}^5$.

- If $\beta + i\gamma = \delta = \sigma = 0$ then $\Omega_F = \{F\}$ (the 0-dimensional orbit).
- If $|\beta + i\gamma|^2 + \delta^2 + \sigma^2 \neq 0$ then Ω_F is the 2-dimensional orbit as follows

$$\Omega_F = \left[\begin{array}{l} \left\{(x, (\beta + i\gamma).e^{(a.e^{-i\varphi})}, \delta.e^{a\lambda_1}, \sigma.e^{a\lambda_2}),\ x, a \in \mathbb{R}\right\} \\ \qquad \text{when } G = G_{5,4,11(\lambda_1,\lambda_2,\varphi)}, \lambda_1, \lambda_2 \in \mathbb{R}^*, \varphi \in (0;\pi), \\ \left\{(x, (\beta + i\gamma).e^{(a.e^{-i\varphi})}, \delta.e^{a\lambda}, \sigma.e^{a\lambda}),\ x, a \in \mathbb{R}\right\} \\ \qquad \text{when } G = G_{5,4,12(\lambda,\varphi)},\ \lambda \in \mathbb{R}^*,\ \varphi \in (0;\pi), \\ \left\{(x, (\beta + i\gamma)\,.e^{(a.e^{-i\varphi})}, \delta.e^{a\lambda}, \delta.ae^{a\lambda} + \sigma.e^{a\lambda}),\ x, a \in \mathbb{R}\right\} \\ \qquad \text{when } G = G_{5,4,13(\lambda,\varphi)},\ \lambda \in \mathbb{R}^*,\ \varphi \in (0;\pi). \end{array} \right.$$

In [18], we have shown that, the family $\mathcal{F}$ of maximal-dimensional K-orbits of G forms measured foliation in terms of Connes on the open submanifold

$$V = \left\{(x, y, z, t, s) \in G^* : y^2 + z^2 + t^2 + s^2 \neq 0\right\} \cong \mathbb{R} \times \left(\mathbb{R}^4\right)^* (\subset \mathcal{G}^* \equiv \mathbb{R}^5).$$

Furthermore, all foliations $\left(V, \mathcal{F}_{4,11(\lambda_1,\lambda_2,\varphi)}\right)$, $\left(V, \mathcal{F}_{4,12(\lambda,\varphi)}\right)$, $\left(V, \mathcal{F}_{4,13(\lambda,\varphi)}\right)$ are topologically equivalent to each other $(\lambda_1, \lambda_2, \lambda \in \mathbb{R}\setminus\{0\}, \varphi \in (0;\pi))$. Thus, we need only to choose an envoy among them to describe the structure of the C^*-algebra. In this case, we choose the foliation $\left(V, \mathcal{F}_{4,12\left(1,\frac{\pi}{2}\right)}\right)$.

In [18], we have described the foliation $\left(V, \mathcal{F}_{4,12\left(1,\frac{\pi}{2}\right)}\right)$ by a suitable action of $\mathbb{R}^2$. Namely, we have the following result.

Proposition 2.1. *The foliation* $\left(V, \mathcal{F}_{4,12\left(1,\frac{\pi}{2}\right)}\right)$ *can be given by an action of the commutative Lie group* $\mathbb{R}^2$ *on the manifold* V.

Proof. One needs only to verify that the following action λ of $\mathbb{R}^2$ on V gives the foliation $\left(V, \mathcal{F}_{4,12\left(1,\frac{\pi}{2}\right)}\right)$

$$\lambda : \mathbb{R}^2 \times V \to V,$$

$$((r, a), (x, y + iz, t, s)) \mapsto \left(x + r, (y + iz).e^{-ia}, t.e^a, s.e^a\right),$$

where $(r, a) \in \mathbb{R}^2$, $(x, y + iz, t, s) \in V \cong \mathbb{R} \times \left(\mathbb{C} \times \mathbb{R}^2\right)^* \cong \mathbb{R} \times \left(\mathbb{R}^4\right)^*$. Hereafter, for simplicity of notation, we write $(V, \mathcal{F})$ instead of $\left(V, \mathcal{F}_{4,12\left(1,\frac{\pi}{2}\right)}\right)$. ∎

It is easy to see that the graph of $(V, \mathcal{F})$ is indentical with $V \times \mathbb{R}^2$, so by [3, Section 5], it follows from Proposition 2.1 that

Corollary 2.2 (Analytical description of $C^*(V, \mathcal{F})$). *The Connes' C^*-algebra $C^*(V, \mathcal{F})$ can be analytically described by the reduced crossed product of $C_0(V)$ by $\mathbb{R}^2$ as follows*

$$C^*(V, \mathcal{F}) \cong C_0(V) \rtimes_\lambda \mathbb{R}^2.$$

3. $C^*(V, \mathcal{F})$ as Two Repeated Extensions

3.1. Let V_1, W_1, V_2, W_2 be the following submanifolds of V

$$V_1 = \{(x, y, z, t, s) \in V : s \neq 0\} \cong \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^*,$$

$$W_1 = V \setminus V_1 = \{(x, y, z, t, s) \in V : s = 0\} \cong \mathbb{R} \times \left(\mathbb{R}^3\right)^* \times \{0\} \cong \mathbb{R} \times \left(\mathbb{R}^3\right)^*,$$

$$V_2 = \{(x, y, z, t, 0) \in W_1 : t \neq 0\} \cong \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^*,$$

$$W_2 = W_1 \setminus V_2 = \{(x, y, z, t, 0) \in W_1 : t = 0\} \cong \mathbb{R} \times \left(\mathbb{R}^2\right)^*.$$

It is easy to see that the action λ in Proposition 2.1 preserves the subsets V_1, W_1, V_2, W_2. Let i_1, i_2, μ_1, μ_2 be the inclusions and the restrictions

$$\begin{array}{l} i_1 : C_0\left(V_1\right) \to C_0\left(V\right), \quad i_2 : C_0\left(V_2\right) \to C_0\left(W_1\right), \\ \mu_1 : C_0\left(V\right) \to C_0\left(W_1\right), \mu_2 : C_0\left(W_1\right) \to C_0\left(W_2\right), \end{array}$$

where each function of $C_0\left(V_1\right)$ (resp. $C_0\left(V_2\right)$) is extented to the one of $C_0\left(V\right)$ (resp. $C_0\left(W_1\right)$) by taking the value of zero outside V_1 (resp. V_2).

It is known a fact that i_1, i_2, μ_1, μ_2 are λ-equivariant and the following sequences are equivariantly exact:

$$0 \longrightarrow C_0(V_1) \xrightarrow{i_1} C_0(V) \xrightarrow{\mu_1} C_0(W_1) \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow C_0(V_2) \xrightarrow{i_2} C_0(W_1) \xrightarrow{\mu_2} C_0(W_2) \longrightarrow 0\ . \tag{2}$$

3.2. Now we denote by $\left(V_1, \mathcal{F}_1\right), \left(W_1, \mathcal{F}_1\right), \left(V_2, \mathcal{F}_2\right), \left(W_2, \mathcal{F}_2\right)$ restrictions of the foliations $(V, \mathcal{F})$ on V_1, W_1, V_2, W_2, respectively.

Theorem 3.1. *$C^*(V, \mathcal{F})$ admits the following canonical repeated extensions*

$$0 \longrightarrow J_1 \xrightarrow{\widehat{i_1}} C^*(V, F) \xrightarrow{\widehat{\mu_1}} B_1 \longrightarrow 0\ , \tag{γ_1}$$

$$0 \longrightarrow J_2 \xrightarrow{\widehat{i_2}} B_1 \xrightarrow{\widehat{\mu_2}} B_2 \longrightarrow 0\ , \tag{γ_2}$$

where

$$\begin{array}{l} J_1 = C^*\left(V_1, \mathcal{F}_1\right) \cong C_0\left(V_1\right) \rtimes_\lambda \mathbb{R}^2 \cong C_0\left(\mathbb{R}^3 \cup \mathbb{R}^3\right) \otimes K, \\ J_2 = C^*\left(V_2, \mathcal{F}_2\right) \cong C_0\left(V_2\right) \rtimes_\lambda \mathbb{R}^2 \cong C_0\left(\mathbb{R}^2 \cup \mathbb{R}^2\right) \otimes K, \\ B_2 = C^*\left(W_2, \mathcal{F}_2\right) \cong C_0\left(W_2\right) \rtimes_\lambda \mathbb{R}^2 \cong C_0\left(\mathbb{R}_+\right) \otimes K, \\ B_1 = C^*\left(W_1, \mathcal{F}_1\right) \cong C_0\left(W_1\right) \rtimes_\lambda \mathbb{R}^2, \end{array}$$

and the homomorphisms $\widehat{i_1}, \widehat{i_2}, \widehat{\mu_1}, \widehat{\mu_2}$ are defined by

$$\begin{array}{l} \left(\widehat{i_k} f\right)(r, s) = i_k f\left(r, s\right),\ k = 1, 2, \\ \left(\widehat{\mu_k} f\right)(r, s) = \mu_k f\left(r, s\right),\ k = 1, 2. \end{array}$$

Proof. We note that the graph of $\left(V_1, \mathcal{F}_1\right)$ is indentical with $V_1 \times \mathbb{R}^2$, so by [3, Section 5], $J_1 = C^*\left(V_1, \mathcal{F}_1\right) \cong C_0\left(V_1\right) \rtimes_\lambda \mathbb{R}^2$. Similarly, we have

$$\begin{array}{c} B_1 \cong C_0\left(W_1\right) \rtimes_\lambda \mathbb{R}^2, \\ J_2 \cong C_0\left(V_2\right) \rtimes_\lambda \mathbb{R}^2, \\ B_2 \cong C_0\left(W_2\right) \rtimes_\lambda \mathbb{R}^2. \end{array}$$

From the equivariantly exact sequences in 3.1 and by [2, Lemma 1.1] we obtain the repeated extensions (γ_1) and (γ_2).

Furthermore, the foliation $(V_1, \mathcal{F}_1)$ can be derived from the submersion

$$p_1 : V_1 \approx \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}^3 \cup \mathbb{R}^3$$

$$p_1\left(x, y, z, t, s\right) = \left(y, z, t, \text{sign} s\right).$$

Hence, by a result of [3, p. 562], we get $J_1 \cong C_0\left(\mathbb{R}^3 \cup \mathbb{R}^3\right) \otimes K$. The same argument shows that

$$J_2 \cong C_0\left(\mathbb{R}^2 \cup \mathbb{R}^2\right) \otimes K, \;\; B_2 \cong C_0\left(\mathbb{R}_+\right) \otimes K.$$

■

4. Computing the Invariant System of $C^*(V, \mathcal{F})$

Definition 4.1. The set of elements $\{\gamma_1, \gamma_2\}$ corresponding to the repeated extensions (γ_1), (γ_2) in the Kasparov groups $\text{Ext}(B_i, J_i)$, $i = 1, 2$ is called the system of invariants of $C^*(V, \mathcal{F})$ and denoted by Index $C^*(V, \mathcal{F})$.

Remark 4.2. Index $C^*(V, \mathcal{F})$ determines the so-called stable type of $C^*(V, \mathcal{F})$ in the set of all repeated extensions

$$0 \longrightarrow J_1 \longrightarrow E \longrightarrow B_1 \longrightarrow 0\ ,$$

$$0 \longrightarrow J_2 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow 0\ .$$

The main result of the paper is the following.

Theorem 4.3. *Index* $C^*(V, \mathcal{F}) = \{\gamma_1, \gamma_2\}$, *where*

$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ *in the group* $\text{Ext}\left(B_1, J_1\right) = \text{Hom}\left(\mathbb{Z}^2, \mathbb{Z}^2\right)$*;*

$\gamma_2 = (1, 1)$ *in the group* $\text{Ext}\left(B_2, J_2\right) = \text{Hom}\left(\mathbb{Z}, \mathbb{Z}^2\right)$.

To prove this theorem, we need some lemmas as follows.

Lemma 4.4. *Set* $I_2 = C_0(\mathbb{R}^2 \times \mathbb{R}^*)$ *and* $A_2 = C_0\left(\left(\mathbb{R}^2\right)^*\right)$. *The following diagram is commutative*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & K_j(I_2) & \longrightarrow & K_j(C_0(\mathbb{R}^3)^*) & \longrightarrow & K_j(A_2) & \longrightarrow & K_{j+1}(I_2) & \longrightarrow \cdots \\
 & \downarrow \beta_1 & & \downarrow \beta_1 & & \downarrow \beta_1 & & \downarrow \beta_1 & \\
\cdots \longrightarrow & K_{j+1}(C_0(V_2)) & \longrightarrow & K_{j+1}(C_0(W_1)) & \longrightarrow & K_{j+1}(C_0(W_2)) & \longrightarrow & K_j(C_0(V_2)) & \longrightarrow \cdots
\end{array}$$

where β_1 *is the isomorphism defined in [13, Theorem 9.7] or in [2, Corollary VI.3],* $j \in \mathbb{Z}/2\mathbb{Z}$.

Proof. Let

$$k_2 : I_2 = C_0\left(\mathbb{R}^2 \times \mathbb{R}^*\right) \to C_0\left(\left(\mathbb{R}^3\right)^*\right)$$

$$v_2 : C_0\left(\left(\mathbb{R}^3\right)^*\right) \to A_2 = C_0\left(\left(\mathbb{R}^2\right)^*\right)$$

be the inclusion and restriction defined similarly as in 3.1.

One gets the exact sequence

$$0 \longrightarrow I_2 \xrightarrow{k_2} C_0\big((\mathbb{R}^3)^*\big) \xrightarrow{v_2} A_2 \longrightarrow 0 .$$

Note that

$$C_0(V_2) \cong C_0\left(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^*\right) \cong C_0(\mathbb{R}) \otimes I_2,$$
$$C_0(W_2) \cong C_0\left(\mathbb{R} \times \left(\mathbb{R}^2\right)^*\right) \cong C_0(\mathbb{R}) \otimes A_2,$$
$$C_0(W_1) \cong C_0\left(\mathbb{R} \times \left(\mathbb{R}^3\right)^*\right) \cong C_0(\mathbb{R}) \otimes C_0\left(\mathbb{R}^3\right)^* .$$

The extension (2) thus can be identified with the following one

$$0 \longrightarrow C_0(\mathbb{R}) \otimes I_2 \xrightarrow{id \otimes k_2} C_0(\mathbb{R}) \otimes C_0(\mathbb{R}^3)^* \xrightarrow{id \otimes v_2} C_0(\mathbb{R}) \otimes A_2 \longrightarrow 0 .$$

Now, using [13, Theorem 9.7, Corollary 9.8] we obtain the assertion of Lemma 4.4. ■

Lemma 4.5. *Set $I_1 = C_0\left(\mathbb{R}^2 \times \mathbb{R}^*\right)$ and $A_1 = C\left(S^2\right)$. The following diagram is commutative*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & K_j(I_1) & \longrightarrow & K_j\big(C(S^3)\big) & \longrightarrow & K_j(A_1) & \longrightarrow & K_{j+1}(I_1) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_2} & \\
\cdots \longrightarrow & K_j(C_0(V_1)) & \longrightarrow & K_j(C_0(V)) & \longrightarrow & K_j(C_0(W_1)) & \longrightarrow & K_{j+1}(C_0(V_1)) & \longrightarrow \cdots
\end{array}$$

where β_2 is the Bott isomorphism, $j \in \mathbb{Z}/2\mathbb{Z}$.

Proof. The proof is similar to that of Lemma 4.4, by using the exact sequence (1) and diffeomorphisms: $V \cong \mathbb{R} \times \left(\mathbb{R}^4\right)^* \cong \mathbb{R} \times \mathbb{R}_+ \times S^3$, $W_1 \cong \mathbb{R} \times \left(\mathbb{R}^3\right)^* \cong \mathbb{R} \times \mathbb{R}_+ \times S^2$. ■

Before computing the K-groups, we need the following notations. Let $u : \mathbb{R} \to S^1$ be the map

$$u(z) = e^{2\pi i\left(z/\sqrt{1+z^2}\right)},\ z \in \mathbb{R}.$$

Denote by u_+ (resp. u_-) the restriction of u on $\mathbb{R}_+$ (resp. $\mathbb{R}_-$). Note that the class $[u_+]$ (resp. $[u_-]$) is the canonical generator of $K_1\left(C_0\left(\mathbb{R}_+\right)\right) \cong \mathbb{Z}$ (resp. $K_1\left(C_0\left(\mathbb{R}_-\right)\right) \cong \mathbb{Z}$). Let us consider the matrix valued function $p : \left(\mathbb{R}^2\right)^* \cong S^1 \times \mathbb{R}_+ \to M_2(\mathbb{C})$ (resp. $\overline{p} : S^2 \cong D/S^1 \to M_2(\mathbb{C})$) defined by:

$$p\,(x;y)\,(\text{resp. } \overline{p}(x,y)) = \frac{1}{2}\begin{pmatrix} 1-\cos\pi\sqrt{x^2+y^2} & \frac{x+iy}{\sqrt{x^2+y^2}}\sin\pi\sqrt{x^2+y^2} \\ \frac{x-iy}{\sqrt{x^2+y^2}}\sin\pi\sqrt{x^2+y^2} & 1+\cos\pi\sqrt{x^2+y^2} \end{pmatrix}.$$

Then p (resp. $\overline{p}$) is an idempotent of rank 1 for each $(x;y) \in \left(\mathbb{R}^2\right)^*$ (resp. $(x;y) \in D/S^1$). Let $[b] \in K_0\left(C_0\left(\mathbb{R}^2\right)\right)$ be the Bott element, [1] be the generator of $K_0\left(C\left(S^1\right)\right) \cong \mathbb{Z}$.

Lemma 4.6. (See [15, p. 234])

(i) $K_0(B_1) \cong \mathbb{Z}^2,\ K_1(B_1) = 0,$

(ii) $K_0(J_2) \cong \mathbb{Z}^2$ *is generated by* $\varphi_0\beta_1\big([b]\boxtimes[u_+]\big)$ *and* $\varphi_0\beta_1\big([b]\boxtimes[u_-]\big);\ K_1(J_2) = 0,$

(iii) $K_0(B_2) \cong \mathbb{Z}$ *is generated by* $\varphi_0\beta_1\big([1]\boxtimes[u_+]\big);\ K_1(B_2) \cong \mathbb{Z}$ *is generated by* $\varphi_1\beta_1\big([p]-[\varepsilon_1]\big)$, *where* $\varphi_j, j \in \mathbb{Z}/2\mathbb{Z}$, *is the Thom-Connes isomorphism (see* [2]), β_1 *is the isomorphism in Lemma 4.4,* ε_1 *is the constant matrix* $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ *and* $\boxtimes$ *is the external tensor product (see, for example,* [2, VI.2]).

Lemma 4.7. (i) $K_0\big(C^*(V,\mathcal{F})\big) \cong \mathbb{Z},\ K_1\big(C^*(V,\mathcal{F})\big) \cong \mathbb{Z},$

(ii) $K_0(J_1) = 0;\ K_1(J_1) \cong \mathbb{Z}^2$ *is generated by* $\varphi_1\beta_2\big([b]\boxtimes[u_+]\big)$ *and* $\varphi_1\beta_2\big([b]\boxtimes[u_-]\big),$

(iii) $K_1(B_1) = 0;\ K_0(B_1) \cong \mathbb{Z}^2$ *is generated by* $\varphi_0\beta_2[\bar{1}]$ *and* $\varphi_0\beta_2\big([\bar{p}]-[\varepsilon_1]\big)$, *where* $\bar{1}$ *is unit element in* $C(S^2)$, φ_0 *is the Thom-Connes isomorphism,* β_2 *is the Bott isomorphism.*

Proof. (i) $K_i\big(C^*(V,\mathcal{F})\big) \cong K_i\big(C(S^3)\big) \cong \mathbb{Z},\ i = 0,1.$

(ii) The proof is similar to (ii) of Lemma 4.6.

(iii) By [9, p. 206], we have $K_0\big(C(S^2)\big) = \mathbb{Z}[\bar{1}] + \mathbb{Z}[q]$, where $q \in P_2\big(C(S^2)\big)$. Otherwise, in [9, p. 48, 53, 56]; [13, p. 162], one has shown that the map

$$\dim : K_0\big(C(S^2)\big) \to \mathbb{Z}$$

is a surjective group homomorphism which satisfied $\dim[\bar{1}] = 1,\ \ker(\dim) = \mathbb{Z}$ and non-zero element $q \in P_2\big(C(S^2)\big)$ in the kernel of the map dim has the form $[q] = [\bar{p}] - [\varepsilon_1]$. Hence, the result is derived straight away because β_2 and φ_0 are isomorphisms. ∎

Proof of Theorem 4.3. (i) Computation of (γ_1). Recall that the extension (γ_1) in Theorem 3.1 gives rise to a six-term exact sequence

$$\begin{array}{ccccc}
0 = K_0(J_1) & \longrightarrow & K_0\big(C^*(V,F)\big) & \longrightarrow & K_0(B_1) \\
\uparrow{\scriptstyle \delta_1} & & & & \downarrow{\scriptstyle \delta_0} \\
0 = K_1(B_1) & \longleftarrow & K_1\big(C^*(V,F)\big) & \longleftarrow & K_1(J_1)
\end{array}$$

By [11, Theorem 4.14], the isomorphisms

$$\mathrm{Ext}(B_1, J_1) \cong \mathrm{Hom}\big((K_0(B_1), K_1(J_1)\big) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$$

associates the invariant $\gamma_1 \in \mathrm{Ext}(B_1, J_1)$ to the connecting map $\delta_0 : K_0(B_1) \to K_1(J_1)$.

Since the Thom-Connes isomorphism commutes with $K-$theoretical exact sequence (see [14, Lemma 3.4.3]), we have the following commutative diagram ($j \in \mathbb{Z}/2\mathbb{Z}$):

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & K_j(J_1) & \longrightarrow & K_j\big(C^*(V,F)\big) & \longrightarrow & K_j(B_1) & \longrightarrow & K_{j+1}(J_1) & \longrightarrow & \cdots \\
& & \uparrow{\scriptstyle \varphi_j} & & \uparrow{\scriptstyle \varphi_j} & & \uparrow{\scriptstyle \varphi_j} & & \uparrow{\scriptstyle \varphi_{j+1}} & & \\
\cdots & \longrightarrow & K_j\big(C_0(V_1)\big) & \longrightarrow & K_j\big(C_0(V)\big) & \longrightarrow & K_j\big(C_0(W_1)\big) & \longrightarrow & K_{j+1}\big(C_0(V_1)\big) & \longrightarrow & \cdots
\end{array}$$

In view of Lemma 4.5, the following diagram is commutative

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & K_j\big(C_0(V_1)\big) & \longrightarrow & K_j\big(C_0(V)\big) & \longrightarrow & K_j\big(C_0(W_1)\big) & \longrightarrow & K_{j+1}\big(C_1(V_1)\big) & \longrightarrow & \cdots \\
& & \uparrow{\scriptstyle \beta_2} & & \uparrow{\scriptstyle \beta_2} & & \uparrow{\scriptstyle \beta_2} & & \uparrow{\scriptstyle \beta_2} & & \\
\cdots & \longrightarrow & K_j(I_1) & \longrightarrow & K_j\big(C(S^3)\big) & \longrightarrow & K_j(A_1) & \longrightarrow & K_{j+1}(I_1) & \longrightarrow & \cdots
\end{array}$$

Consequently, instead of computing $\delta_0 : K_0(B_1) \to K_1(J_1)$, it is sufficient to compute $\delta_0 : K_0(A_1) \to K_1(I_1)$. Thus, by the proof of Lemma 4.7, we have to define $\delta_0\big([\bar{p}] - [\varepsilon_1]\big) = \delta_0\big([\bar{p}]\big)$ (because $\delta_0\big([\varepsilon_1]\big) = (0;0)$ and $\delta_0\big([\bar{1}]\big) = (0;0)$). By the usual definition (see [13, p. 170]), for $[\bar{p}] \in K_0(A_1)$, $\delta_0\big([\bar{p}]\big) = \big[e^{2\pi i\tilde{p}}\big] \in K_1(I_1)$, where $\tilde{p}$ is a preimage of $\bar{p}$ in (a matrix algebra over) $C(S^3)$, i.e. $v_1\tilde{p} = \bar{p}$.

We can choose $\tilde{p}(x,y,z) = \dfrac{z}{\sqrt{1+z^2}}\bar{p}(x,y)$, $(x,y,z) \in S^3$.

Let $\tilde{p}_+$ (resp. $\tilde{p}_-$) be the restriction of $\tilde{p}$ on $\mathbb{R}^2 \times \mathbb{R}_+$ (resp. $\mathbb{R}^2 \times \mathbb{R}_-$). Then we have

$$\delta_0\big([\bar{p}]\big) = \big[e^{2\pi i\tilde{p}}\big] = \big[e^{2\pi i\tilde{p}_+}\big] + \big[e^{2\pi i\tilde{p}_-}\big] \in K_1\Big(C_0\big(\mathbb{R}^2\big) \otimes C_0\big(\mathbb{R}_+\big)\Big) \oplus K_1\Big(C_0\big(\mathbb{R}^2\big) \otimes C_0\big(\mathbb{R}_-\big)\Big) = K_1(I_1).$$

By [13, Section 4], for each function $f : \mathbb{R}_\pm \to Q_n\widetilde{C_0\big(\mathbb{R}^2\big)}$ such that $\lim\limits_{x\to\pm 0} f(t) = \lim\limits_{x\to\pm\infty} f(t)$, where $Q_n\widetilde{C_0\big(\mathbb{R}^2\big)} = \Big\{a \in M_n\widetilde{C_0\big(\mathbb{R}^2\big)}, e^{2\pi i a} = Id\Big\}$, the class $[f] \in K_1\big(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}_\pm)\big)$ can be determined by $[f] = W_f.[b] \boxtimes [u_\pm]$, where $W_f = \dfrac{1}{2\pi i}\displaystyle\int_{\mathbb{R}_\pm} Tr\big(f'(z)f^{-1}(z)\big)dz$ is the winding number of f.

By simple computation, we get $\delta_0\big([p]\big) = [b]\boxtimes[u_+] + [b]\boxtimes[u_-]$. Thus $\gamma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{Z}^2)$.

(ii) Computation of (γ_2). The extension (γ_2) gives rise to a six-term exact sequence

$$\begin{array}{ccccc} K_0(J_2) & \longrightarrow & K_0(B_1) & \longrightarrow & K_0(B_2) \\ \uparrow{\scriptstyle \delta_1} & & & & \downarrow{\scriptstyle \delta_0} \\ K_1(B_2) & \longleftarrow & K_1(B_1) & \longleftarrow & K_1(J_2) = 0 \end{array}$$

By [11, Theorem 4.14], $\gamma_2 = \delta_1 \in \operatorname{Hom}\big(K_1(B_2), K_0(J_2)\big) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}^2)$. Similarly to part (i), taking account of Lemmas 4.4 and 4.6, we have the following commutative diagram ($j \in \mathbb{Z}/2\mathbb{Z}$)

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & K_j(J_2) & \longrightarrow & K_j(B_1) & \longrightarrow & K_j(B_2) & \longrightarrow & K_{j+1}(J_2) & \longrightarrow & \cdots \\ & & \uparrow{\scriptstyle \varphi_j} & & \uparrow{\scriptstyle \varphi_j} & & \uparrow{\scriptstyle \varphi_j} & & \uparrow{\scriptstyle \varphi_{j+1}} & & \\ \cdots & \longrightarrow & K_j\big(C_0(V_2)\big) & \longrightarrow & K_j\big(C_0(W_1)\big) & \longrightarrow & K_j\big(C_0(W_2)\big) & \longrightarrow & K_{j+1}\big(C_0(V_2)\big) & \longrightarrow & \cdots \\ & & \uparrow{\scriptstyle \beta_1} & & \uparrow{\scriptstyle \beta_1} & & \uparrow{\scriptstyle \beta_1} & & \uparrow{\scriptstyle \beta_1} & & \\ \cdots & \longrightarrow & K_{j-1}(I_2) & \longrightarrow & K_{j-1}\big(C_0(\mathbb{R}^3)^*\big) & \longrightarrow & K_{j-1}(A_2) & \longrightarrow & K_j(I_2) & \longrightarrow & \cdots \end{array}$$

Thus we can compute $\delta_0 : K_0(A_2) \to K_1(I_2)$ instead of $\delta_1 : K_1(B_2) \to K_0(J_2)$. By the proof of Lemma 4.6, we have to define $\delta_0\big([p] - [\epsilon_1]\big) = \delta_0\big([p]\big)$ (because $\delta_0\big([\epsilon_1]\big) = (0,0)$). Using the same argument as above, we get $\delta_0\big([p]\big) = [b]\boxtimes[u_+] + [b]\boxtimes[u_-]$. Thus $\gamma_2 = (1,1) \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}^2) \cong \mathbb{Z}^2$. The proof is complete. ■

References

1. L. G. Brown, R. G. Douglas, and P. A. Fillmore, Extension of C^*-algebra and K-homology, *Ann. Math.* **105** (1977), 265–324.
2. A. Connes, An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\mathbb{R}$, *Adv. In Math.* **39** (1981), 31–55.
3. A. Connes, A survey of foliations and operator algebras, *Proc. Sympos. Pure Math.* **38** (1982), 521–628.
4. D. N. Diep, Structure of the group C^*-algebra of the group of affine transformations of the line (Russian), *Funktsional. Anal. I Prilozhen* **9** (1975), 63–64.
5. D. N. Diep, *Method of Noncommutative Geometry for Group C^*-algebras*, Reseach Notes in Mathematics Series, Vol. 416, Cambridge: Chapman and Hall-CRC Press, 1999.
6. M. Karoubi, *K-theory: An introduction, Grund. der Math. Wiss.* Vol. 226, Springer-Verlag, Berlin-Heidelberg-New York, 1978.
7. G. G. Kasparov, The operator K-functor and extensions of C^*-algebras, *Math. USSR Izvestija* **16** (3) (1981), 513–572.
8. A. A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, Berlin-Heidenberg-New York, 1976.
9. M. Rordam, F. Larsen, and N. Laustsen, *An Introduction to K-Theory for C^*-Algebras*, Cambridge University Press, United Kingdom, 2000.

10. J. Rosenberg, The C^*-algebras of some real p-adic solvable groups, *Pacific J. Math.* **65** (1) (1976), 175–192.
11. J. Rosenberg, Homological invariants of extension of C^*-algebras, *Proc. Sympos. Pure Math.* **38** (1982), AMS Providence R.I., 35–75.
12. V. M. Son and H. H. Viet, Sur la structure des C^*-algebres dúne classe de groupes de Lie, *J. Operator Theory* **11** (1984), 77–90.
13. J. L. Taylor, Banach algebras and topology, in: *Algebras in Analysis*, Academic Press, New York, 1975, pp. 118–186.
14. A. M. Torpe, K-theory for the leaf space of foliations by Reeb component, *J. Func. Anal.* **61** (1985), 15–71.
15. L. A. Vu, On the structure of the C^*-algebra of the foliation formed by the orbits of maximal dimendion of the real diamond group, *J. Operator Theory* **24** (1990), 227–238.
16. L. A. Vu, The foliation formed by the K-orbits of maximal dimension of the MD_4-group, PhD Thesis, Ha Noi, 1990 (in Vietnamese).
17. L. A. Vu and K. P. Shum, Classification of 5-dimensional MD-algebra having commutative derived ideals, *Adv. Algebr. Comb.* World Scientific Publishing Co., pp. 353–371.
18. L. A. Vu and D. Q. Hoa, The topology of foliations formed by the generic K-orbits of a subclass of the indecomposable MD_5-groups, *Science in China, series A: Mathemmatics* **52** (2) (2009), 351–360.