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Dynamics of the Stochastic Equation of Cooperative Population

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Abstract. In this paper, we consider the equation system of population dynamics for two species in cooperation with a stochastic perturbation of environmental variation that is proportional to the number of individuals. We study the asymptotic behavior of the solution according to different relations between the coefficients.

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1. Introduction

The stable cooperative Lotka-Volterra equation

$$\begin{cases} \dot{x}_t = (a_1 + b_{12}y_t - b_{11}x_t)x_t, \\ \dot{y}_t = (a_2 + b_{21}x_t - b_{22}y_t)y_t, \end{cases} \tag{1}$$

where a_i, b_{ij} $(i, j = 1, 2)$ are positive constants, has attracted a lot attentions of works. For the stochastic Lotka-Volterra equations of two species, there are some but not too much in mathematical literature, and almost nothing in statistical inference. Here, we mention one of the first attempts in this direction, a very interesting paper of Arnold et al. [1] where the authors used the theory of Brownian motion processes and the related white noise models to study the sample paths of the equation

$$\begin{cases} dX_t = (a_1 + b_{12}Y_t - b_{11}X_t)X_t dt + \sigma X_t dW_t, \\ dY_t = (a_2 + b_{21}X_t - b_{22}Y_t)Y_t dt + \rho Y_t dW_t, \end{cases} \tag{2}$$

where the stochastic processes X_t and Y_t represent the quantities of two species; $a_i, b_{ij}, \sigma, \rho$ are constants, $b_{ii} > 0$ $(i, j = 1, 2)$; σ, ρ are the coefficients of the effects of environmental stochastic perturbations on the populations and W_t is a standard Wiener processes. This model is called a competitive model if $a_i > 0,\ b_{ij} < 0,\ i \neq j$, a prey-predator model if $a_1 > 0,\ b_{12} < 0,\ a_2 < 0,\ b_{21} > 0$, a cooperative model if $a_i > 0,\ b_{ij} > 0$. In these cases, the random factor makes influences on the intrinsic growth rates of species.

Recently, this model has been considered by Mao et al. [4], Du et al. [2] due to the sample path of the solutions. In these works, the authors showed the upper growth rate and lower growth rate of the solutions of equation (2). By research on long-time behaviour of densities of the distributions of the solutions, Rudnicki [7] has been considered a prey-predator model and Yashima et al. [10] has been considered a competitive model and they showed the densities can converge in L^1 to an invariant density or can converge weakly to a singular measure. Then in [8], Rudnicki assumed that the environmental influence on both populations is described by stochastic perturbations and it is proportional to the number of individuals.

To continue those researches, in this paper, we consider a cooperative model. It means that we study the following system

$$\begin{cases} dX_t = (a_1 + b_{12}Y_t - b_{11}X_t)X_t dt + X_t(\sigma_1 dW_t^1 + \sigma_2 dW_t^2), \\ dY_t = (a_2 + b_{21}X_t - b_{22}Y_t)Y_t dt + Y_t(\rho_1 dW_t^1 + \rho_2 dW_t^2), \end{cases} \tag{3}$$

where $a_i,\ b_{ij}$ are positive constants, σ_i, ρ_i are nonnegative constants $(i, j = 1, 2)$ and $W_t^1,\ W_t^2$ are two independent standard Wiener processes. For ecological reason, we will consider only solutions of the system with $X_0 > 0$ and $Y_0 > 0$. Through this paper we assume that the solution with such initial values are unique and globally positive.

Generally, we will assume that our system is not deterministic (at least one coefficient σ_1, σ_2, ρ_1 or ρ_2 is not equal to zero). There are two kinds of stochastic perturbations: weak correlation $(\sigma_1\rho_2 \neq \sigma_2\rho_1)$ and strong correlation $(\sigma_1\rho_2 = \sigma_2\rho_1)$.

By putting

$$X_t = e^{\xi_t}, \quad Y_t = e^{\eta_t}, \quad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2}, \quad \rho = \sqrt{\rho_1^2 + \rho_2^2}, \quad \sigma^* = \sigma_1 + \sigma_2,$$
$$\rho^* = \rho_1 + \rho_2, \quad c_1 = a_1 - \frac{\sigma^2}{2}, \quad c_2 = a_2 - \frac{\rho^2}{2}.$$

and substituting this transformation into equation (3) we obtain

$$\begin{cases} d\xi_t = (c_1 + b_{12}e^{\eta_t} - b_{11}e^{\xi_t})dt + \sigma_1 dW_t^1 + \sigma_2 dW_t^2, \\ d\eta_t = (c_2 + b_{21}e^{\xi_t} - b_{22}e^{\eta_t})dt + \rho_1 dW_t^1 + \rho_2 dW_t^2. \end{cases} \tag{4}$$

We have 6 cases in total:

[I] $\sigma_1\rho_2 \neq \sigma_2\rho_1,$

[II] $\sigma = 0,\ \rho > 0,\ b_{11}b_{22} - b_{12}b_{21} \neq 0,$

[III] $\sigma > 0,\ \rho = 0,\ b_{11}b_{22} - b_{12}b_{21} \neq 0,$

[IV] $\sigma = 0,\ \rho > 0,\ b_{11}b_{22} - b_{12}b_{21} = 0,$

[V] $\sigma > 0,\ \rho = 0,\ b_{11}b_{22} - b_{12}b_{21} = 0,$

[VI] $\sigma_1\rho_2 = \sigma_2\rho_1,\ \sigma > 0,\ \rho > 0.$

The aim of this paper is to study further the asymptotic behaviour of system (4) by considering the convergence of the density of the solution in all cases of the coefficients. The study of such a problem is motivated by the question of influence of stochastic perturbations on time evolution of population sizes.

The paper is organized as follows. In Sec. 2 we show some properties of Markov semigroup and Fokker-Planck equation associated with system (4). In Sec. 3, we study asymptotic properties of the semigroup including asymptotic stability and sweeping. The theorem concerning with asymptotic stability and sweeping allow us to formulate the Forguel alternative. This alternative says that under suitable conditions a Markov semigroup is asymptotically stable or sweeping. By virtue of the Hasminskii function, it excludes sweeping and we obtain asymptotic stability.

2. Markov Semigroups and Fokker-Planck Equation

In this section we recall some properties of Markov semigroups and Fokker-Planck equation that have been dealt with in [7] and use them to study the asymptotic stability of (4).

Let the triple (X, Σ, m) be a σ-finite measure space. Denote by D the subset of the space $L^1 = L^1(X, \Sigma, m)$, consisting of the densities, i.e. $D = \{f \in L^1 : f \geq 0, \|f\| = 1\}$. A linear mapping $P : L^1 \to L^1$ is called a Markov operator if $P(D) \subset D$. The Markov operator P is called an integral or kernel operator if there exists a measurable function $k : X \times X \to [0, \infty)$ such that $Pf(x) = \int_X k(x, y)f(y)m(dy)$ for every density f. Let $\{P(t)\}_{t\geq 0}$ be a semigroup of linear operators on L^1. The family $\{P(t)\}_{t\geq 0}$ is said to be a Markov semigroup if $P(t)$ is a semigroup and for every $t \geq 0$, $P(t)$ is a Markov operator. $\{P(t)\}_{t\geq 0}$ is called integral, if for each $t > 0$, the operator $P(t)$ is an integral Markov operator.

The semigroup $\{P(t)\}_{t\geq 0}$ is called asymptotically stable if there is an invariant density f_* such that $\lim_{t\to\infty}\|P(t)f - f_*\| = 0$ for $f \in D$ (a density f_* is called invariant under the semigroup $\{P(t)\}_{t\geq 0}$ if $P(t)f_* = f_*$ for each $t \geq 0$). The Makov semigroup $\{P(t)\}_{t\geq 0}$ is called sweeping with respect to a set $A \in \Sigma$ if for every $f \in D$,

$$\lim_{t\to\infty}\int_A P(t)f(x)m(dx) = 0.$$

If the semigroup is either asymptotically stable or "sweeping" with respect to compact sets then we say that the semigroup has the "Foguel alternative".

Consider the space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), m)$, where $\mathcal{B}(\mathbb{R}^2)$ is the σ-algebra of Borel subsets of $\mathbb{R}^2$ and m is the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Let (ξ_t, η_t) be a solution of (4). The density of the random variable (ξ_t, η_t), if it exists and is smooth, can be found from the Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\rho^2\frac{\partial^2 v}{\partial y^2} + (\sigma_1\rho_1 + \sigma_2\rho_2)\frac{\partial^2 v}{\partial x\partial y} - \frac{\partial(f_1 v)}{\partial x} - \frac{\partial(f_2 v)}{\partial y} \tag{5}$$

and the infinitessimal operator of equation (4) is

$$Lv = \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\rho^2\frac{\partial^2 v}{\partial y^2} + (\sigma_1\rho_1 + \sigma_2\rho_2)\frac{\partial^2 v}{\partial x\partial y} + f_1\frac{\partial v}{\partial x} + f_2\frac{\partial v}{\partial y}, \tag{6}$$

where

$$f_1(x, y) = c_1 - b_{11}e^x + b_{12}e^y,$$
$$f_2(x, y) = c_2 + b_{21}e^x - b_{22}e^y.$$

Let $\mathcal{P}(t, x_0, y_0, A)$ be the transition probability function of the Markov diffusion process (ξ_t, η_t), i.e., $\mathcal{P}(t, x_0, y_0, A) = \text{Prob}\{(\xi_t, \eta_t) \in A\}$ with the initial condition $\xi_0 = x_0$, $\eta_0 = y_0$ and $k(t, x, y, x_0, y_0)$ be the density of $\mathcal{P}(t, x_0, y_0, \cdot)$ (if it exists). It is known that if the initial random variable (ξ_0, η_0) has the density $v(x, y)$, then for any $t > 0$, (ξ_t, η_t) has the density $u(t, x, y)$ given by

$$u(t, x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(t, x, y; \xi, \eta)v(\xi, \eta)d\xi d\eta.$$

For any $t \geq 0$, we consider the operator $P(t)$ defined by

$$P(t)v(x, y) = u(t, x, y); \quad v(x, y) \in D, \quad (x, y) \in \mathbb{R}^2.$$

By the contractivity of the operator $P(t)$ on D, we can extend it to a contraction one on $L^1(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), m)$. For more detail, we refer to [7].

3. Asymptotic Stability

In this section, we prove the main results of our paper. To prove the main theorems, we need some lemmas

Lemma 3.1. *If one of three conditions* [I], [II], [III] *holds, then the transition probability function* $\mathcal{P}(t, x_0, y_0, \cdot)$ *has a density* $k(t, x, y, x_0, y_0)$ *with respect to* m *and* $k \in C^\infty((0,\infty) \times \mathbb{R}^2 \times \mathbb{R}^2)$.

Proof. We apply Hormander Theorem on the existence of smooth densities of the transition probability function for the degenerate diffusion processes described by system (4).

If $X(x) = (X_1, ..., X_d)$ and $Y(x) = (Y_1, ..., Y_d)$ are vector fields on $\mathbb{R}^d$ then the Lie bracket $[X, Y]$ is a vector field given by

$$[X, Y]_j(x) = \sum_{k=1}^{d} \left(X_k \frac{\partial Y_j}{\partial x_k}(x) - Y_k \frac{\partial X_j}{\partial x_k}(x) \right), \quad j = 1, \ldots, d.$$

Let

$$a_0(x, y) = \begin{bmatrix} c_1 - b_{11}e^x + b_{12}e^y \\ c_2 + b_{21}e^x - b_{22}e^y \end{bmatrix}, \quad a_1 = \begin{bmatrix} \sigma_1 \\ \rho_1 \end{bmatrix} \text{ and } a_2 = \begin{bmatrix} \sigma_2 \\ \rho_2 \end{bmatrix}.$$

It is easy to get that

$$[a_0, a_i] = \begin{bmatrix} \sigma_i b_{11} e^x - \rho_i b_{12} e^y \\ \rho_i b_{22} e^y - \sigma_i b_{21} e^x \end{bmatrix},$$

$$[a_0, [a_0, a_i]] = \begin{bmatrix} \sigma_i c_1 b_{11} e^x - \rho_i c_2 b_{12} e^y + (\sigma_i - \rho_i) b_{12}(b_{21} + b_{11}) e^{x+y} \\ -\sigma_i c_1 b_{21} e^x + \rho_i c_2 b_{22} e^y - (\sigma_i - \rho_i) b_{21}(b_{12} + b_{22}) e^{x+y} \end{bmatrix},$$

$$[a_1, [a_0, a_i]] = \begin{bmatrix} \sigma_i^2 b_{11} e^x - \rho_i^2 b_{12} e^y \\ \rho_i^2 b_{22} e^y - \sigma_i^2 b_{21} e^x \end{bmatrix}, \; i = 1, 2.$$

First, we consider the case [I] holds, then two vectors $a_1,\ a_2$ span the space $\mathbb{R}^2$. Second, if [II] holds, then we can assume that $\rho_1 \neq 0$. Thus it is easy to see that $[a_0, a_1]$ and $[a_1, [a_0, a_1]]$ span the space $\mathbb{R}^2$. Finally, the case [III] is similar to the case [II] and we get the Hormander condition:

(H) *For every* $(\xi, \eta) \in \mathbb{R}^2$ *vectors*

$$a_1, a_2, [a_i, a_j](\xi, \eta)_{0 \le i,j \le 2}, [a_i, [a_j, a_k]](\xi, \eta)_{0 \le i,j,k \le 2}, \cdots$$

span the space $\mathbb{R}^2$.

Under the condition (H), the transition probability function $\mathcal{P}(t, x_0, y_0, \cdot)$ has a density $k(t, x, y, x_0, y_0)$ and $k \in C^\infty((0,\infty) \times \mathbb{R}^2 \times \mathbb{R}^2)$ (see [5]). ■

By Lemma 3.1, we see that $\{P(t)\}_{t \ge 0}$ is an integral Markov semigroup with a continuous kerner k. Now we follow the method dealt with in [7] to check the positivity of k. Fix a point $(x_0, y_0) \in \mathbb{R}^2$ and a function

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in C([0,T];\mathbb{R}^2).$$

Consider the following system of differential equations

$$\begin{cases} x'_\phi = \sigma_1\phi_1(t) + \sigma_2\phi_2(t) + c_1 + b_{12}e^{y_\phi} - b_{11}e^{x_\phi}, \\ y'_\phi = \rho_1\phi_1(t) + \rho_2\phi_2(t) + c_2 + b_{21}e^{x_\phi} - b_{22}e^{y_\phi}. \end{cases} \tag{7}$$

Put

$$\bar{f}_1(x,y) = \sigma_1\phi_1(t) + \sigma_2\phi_2(t) + c_1 + b_{12}e^{y} - b_{11}e^{x},$$
$$\bar{f}_2(x,y) = \rho_1\phi_1(t) + \rho_2\phi_2(t) + c_2 + b_{21}e^{x} - b_{22}e^{y},$$

we can rewrite (7) under the form

$$\begin{cases} x'_\phi(t) = \bar{f}_1(x_\phi(t), y_\phi(t)), \\ y'_\phi(t) = \bar{f}_2(x_\phi(t), y_\phi(t)). \end{cases}$$

Let $\mathbf{x}_\phi = \begin{bmatrix} x_\phi \\ y_\phi \end{bmatrix}$ be the solution of (7) with the initial condition $x_\phi(0) = x_0;\ y_\phi(0) = y_0$. Let $F : C([0,T], \mathbb{R}^2) \to \mathbb{R}^2$ be a map defined by $F(h) = \mathbf{x}_{\phi+h}(T)$. We calculate the Frechet derivative $D_{x_0,y_0,\phi}$ of F at ϕ. Let $\mathbf{f} = (\bar{f}_1, \bar{f}_2)$ and $\Lambda(t) = \mathbf{f}'(x_\phi(t), y_\phi(t))$. Denote $Q(t,s)$, for $0 \le s \le t \le T$, the fundamental solution matrix of the equation

$$\dot{Y} = \Lambda(t)Y,$$

that is $\frac{\partial Q(t,s)}{\partial t} = \Lambda(t)Q(t,s)$ and $Q(s,s) = I$. Then

$$D_{x_0,y_0,\phi}h = \int_0^T Q(T,s)\, q\, h(s)ds, \quad \text{where} \quad q = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \rho_1 & \rho_2 \end{bmatrix}.$$

Let $\varepsilon \in (0,T)$ and $h(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$, where

$$h_1(t) = h_2(t) = \begin{cases} 0 \text{ if } 0 \le t \le T - \varepsilon, \\ \frac{1}{\varepsilon}(t - T + \varepsilon) \text{ if } T - \varepsilon \le t \le T. \end{cases}$$

By Taylor formula we have $Q(T,s) = I - \Lambda(T)(T-s) + o(T-s)$ as $s \to T$. Thus,

$$D_{x_0,y_0,\phi}h = \left(\int\limits_{T-\varepsilon}^{T} \frac{s-T+\varepsilon}{\varepsilon} ds\right) \begin{bmatrix}\sigma^*\\ \rho^*\end{bmatrix} + \\ + \left(\int\limits_{T-\varepsilon}^{T} \frac{s-T+\varepsilon}{\varepsilon}(s-T)ds\right) \Lambda(T) \begin{bmatrix}\sigma^*\\ \rho^*\end{bmatrix} + o(\varepsilon^2) \\ = \frac{\varepsilon}{2}\begin{bmatrix}\sigma^*\\ \rho^*\end{bmatrix} - \frac{\varepsilon^2}{6}\Lambda(T)\begin{bmatrix}\sigma^*\\ \rho^*\end{bmatrix} + o(\varepsilon^2).$$

Put $\bar{x} = x_\phi(T),\ \bar{y} = y_\phi(T),\ \bar{c} = \phi(T)$. By a direct calculation we obtain

$$\Lambda(T) = \begin{bmatrix}-b_{11}e^{\bar{x}} & b_{12}e^{\bar{y}}\\ b_{21}e^{\bar{x}} & -b_{22}e^{\bar{y}}\end{bmatrix},$$

and

$$\Lambda(T)\begin{bmatrix}\sigma^*\\ \rho^*\end{bmatrix} = \begin{bmatrix}-b_{11}\sigma^* e^{\bar{x}} + b_{12}\rho^* e^{\bar{y}}\\ b_{21}\sigma^* e^{\bar{x}} - b_{22}\rho^* e^{\bar{y}}\end{bmatrix} = e^{\bar{x}}\begin{bmatrix}-b_{11}\sigma^* + b_{12}\rho^* e^{\bar{y}-\bar{x}}\\ b_{21}\sigma^* - b_{22}\rho^* e^{\bar{y}-\bar{x}}\end{bmatrix}.$$

It is easy to see that there is a constant c such that the vectors $\begin{bmatrix}\sigma^*\\ \rho^*\end{bmatrix}$ and $\Lambda(T)\begin{bmatrix}\sigma^*\\ \rho^*\end{bmatrix}$ are linearly independent if $\bar{y} - \bar{x} \neq c$. Thus rank $D_{x_0,y_0,\phi} = 2$.

Summing up, we have

Lemma 3.2. *There exists a constant c such that* $\operatorname{rank}(D_{x_0,y_0,\phi}) = 2$ *if* $y_\phi(T) \neq x_\phi(T) + c$.

We now consider the controlability of system (7). We have the following lemma

Lemma 3.3. *If one of the conditions* [I], [II], [III] *holds, then system* (7) *is controlable in* $\mathbb{R}^2$ *i.e., for all* $(x_0, y_0),\ (x_1, y_1) \in \mathbb{R}^2$*, there exist a piecewise-continuous control function* $\phi = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$ *and* $T > 0$ *such that* $x_\phi(0) = x_0, y_\phi(0) = y_0, x_\phi(T) = x_1, y_\phi(T) = y_1$.

Proof. First, we consider the case [I]. It follows that $\sigma.\rho \neq 0$. Then, we can assume that $\sigma_1 \neq 0$. Using the substitution

$$z_\phi(t) = y_\phi(t) - \frac{\rho_1}{\sigma_1}x_\phi(t), \tag{8}$$

system (7) becomes

$$\begin{cases} x'_\phi = \sigma_1\phi_1 + g_1(x_\phi, z_\phi),\\ z'_\phi = g_2(x_\phi, z_\phi), \end{cases} \tag{9}$$

where

$$g_1(x,z) = c_1 + \sigma_2\phi_2 - b_{11}e^x + b_{12}e^z e^{rx},\ g_2(x,z) = A + Be^x - Ce^z e^{rx},$$

where $A = c_2 - rc_1 + (\rho_2 - r\sigma_2)\phi_2$, $B = rb_{11} + b_{21}$, $C = rb_{12} + b_{22}$, $r = \rho_1{\sigma_1}^{-1}$.

We have the following claims

Claim 1. Fix $z_0, z_1 \in \mathbb{R}$ and $z_1 < z_0$. Since $\rho_2 - r\sigma_2 \neq 0$, then we can choose $\phi_2(t) =$ constant such that $A < 0$. Since $\lim\limits_{x\to-\infty} g_2(x,z) = A$ uniformly in $z \in [z_0, z_1]$, it follows that there exists x_0 (sufficiently small) such that $g_2(x_0, z) \leq \frac{A}{2} < 0$ for any $z \in [z_1, z_0]$. Thus, we can find $\phi_1(t)$ and $T > 0$ such that system (9) has the solution $(x_\phi(t), z_\phi(t))$ satisfying $x_\phi \equiv x_0$ and $z_\phi(0) = z_0$, $z_\phi(T) = z_1$.

Claim 2. Fix $z_0, z_1 \in \mathbb{R}$ and $z_0 < z_1$. It is similar to Claim 1 that we can also choose $\phi_2(t) =$ constant such that $A > 0$ and then we can find $\phi_1(t)$ and $T > 0$ such that system (9) has the solution $(x_\phi(t), z_\phi(t))$ satisfying $x_\phi \equiv x_0$ and $z_\phi(0) = z_0, z_\phi(T) = z_1$.

Claim 3. Fix $x_0 \in \mathbb{R}, L > 0, A_1 > A_0$ and $\varepsilon > 0$ such that $\varepsilon < \min\left\{\frac{L}{4}, \frac{A_1 - A_0}{4}\right\}$. Let $\phi_2(t) \equiv 0$,

$$m = \max\{|g_1(x,z)| + |g_2(x,z)| : x \in [x_0, x_0 + L], z \in [A_0, A_1]\},$$

and $t_0 = \frac{\varepsilon}{m}, \phi_1(t) \equiv \frac{3mL}{4\sigma_1\varepsilon}$. For every $z_0 \in [A_0 + \varepsilon, A_1 - \varepsilon]$, we see that the solution of system (9) with $x_\phi(0) = x_0,\ z_\phi(0) = z_0$ satisfies

$$x_\phi(t_0) \in \left(x_0 + \frac{L}{2}, x_0 + L\right) \text{ and } z_\phi(t) \in [z_0 - \varepsilon, z_0 + \varepsilon], \quad \text{for all } t \in [0, t_0]. \quad (10)$$

Indeed, from system (9), we have

$$x'_\phi(t) \geq \sigma_1\phi_1 - m > \frac{mL}{2\varepsilon} > 0 \text{and } x'_\phi(t) \leq \sigma_1\phi_1 + m = m\left(\frac{3L}{4\varepsilon} + 1\right).$$

Then

$$x_\phi(t_0) \geq x_0 + \frac{mL}{2\varepsilon}t_0 = x_0 + \frac{L}{2},\ x_\phi(t_0) > x_\phi(t) > x_0$$

and

$$x_\phi(t_0) \leq mt_0\left(\frac{3L}{4\varepsilon} + 1\right) + x_0 < x_0 + L$$

for all $t \in [0, t_0]$.

Further,

$$|z_\phi(t) - z_\phi(0)| = \left|\int_0^t g_2(x_\phi(s), z_\phi(s))ds\right| \leq \int_0^{t_0} mds = mt_0 = \varepsilon \quad \forall t \in [0, t_0].$$

From (10) it follows that for $(x_1, z_1) \in (x_0\ , x_0 + \frac{L}{2}] \times [A_0 + 2\varepsilon, A_1 - 2\varepsilon]$ there exists $z_0 \in [z_1 - \varepsilon, z_1 + \varepsilon]$ and $T \in (0, t_0)$ such that $x_\phi(T) = x_1$ and $z_\phi(T) = z_1$. The same proof works for $x_1 \in (x_0 - \frac{L}{2}, x_0]$. Then we get that for all $(x_1, z_1) \in$

$\left(x_0 - \frac{L}{2}, x_0 + \frac{L}{2}\right) \times [A_0 + 2\varepsilon, A_1 - 2\varepsilon]$, there exist $z_0 \in [z_1 - \varepsilon, z_1 + \varepsilon]$, $T \in (0, t_0)$ and a piecewise-continuous control function ϕ such that $x_\phi(T) = x_1, z_\phi(T) = z_1$. By Claims 1-3, the lemma is true for the case [I].

Second, consider the case [II]. We choose $\phi_1(t) = \phi_2(t)$, system (7) becomes

$$\begin{cases} x'_\phi(t) = f_1(x_\phi(t), y_\phi(t)), \\ y'_\phi(t) = f_2(x_\phi(t), y_\phi(t)) + \rho^* \phi_1(t). \end{cases} \tag{11}$$

Since for every $y \in \mathbb{R}$, $\lim\limits_{x\to\infty} f_1(x, y) = -\infty$, and for sufficiently large $y \in \mathbb{R}$, $\lim\limits_{x\to-\infty} f_1(x, y) = c_1 + b_{12}e^y > 0$ and by the same arguments as Claims 1-3, the lemma is also true for the case [II]. Similarly, the lemma holds for the case [III]. ■

By the continuity with respect to the initial condition, we conclude that instead of using piecewise-continuous controls we can use continuous controls, i.e., $\phi \in C(0, T; \mathbb{R}^2)$.

It is known that (see [7]), if $\phi \in C([0, T]; \mathbb{R}^2)$ such that the derivative $D_{x_0,y_0,\phi}$ has the rank 2 then $k(T, \bar{x}, \bar{y}, x_0, y_0) > 0$ for $\bar{x} = x_\phi(T), \bar{y} = y_\phi(T)$. Therefore, it follows from Lemmas 3.1-3.3 that

Lemma 3.4. *If one of the conditions* [I], [II], [III] *holds, then for any point* (x_0, y_0) *and for almost* (x, y) *in* $\mathbb{R}^2$*, there exists* $T > 0$ *such that* $k(T, x, y, x_0, y_0) > 0$.

Theorem 3.5. *Let* (ξ_t, η_t) *be a solution of system* (4) *under one of the conditions* [I], [II], [III]. *Then for every* $t > 0$ *the distribution of* (ξ_t, η_t) *has a density* $u(t, x, y)$ *which satisfies* (5). *Further, the semigroup* $\{P(t)\}_{t\geq 0}$ *is asymptotically stable on* $\mathbb{R}^2$*, i.e., there exists a unique stationary density* $u_*(x, y)$ *of* (5) *such that*

$$\lim_{t\to\infty} \iint\limits_{\mathbb{R}^2} |u(t, x, y) - u_*(x, y)| dxdy = 0.$$

Proof. By virtue of Lemma 3.1 it follows that $\{P(t)\}_{t\geq 0}$ is an integral Markov semigroup with a continuous kernel $k(t, x, y, x_0, y_0)$ for $t > 0$. Then, the distribution of (ξ_t, η_t) has a density $u(t, x, y)$ which satisfies (5). According to Lemma 3.4, corresponding to the assumptions of the theorem, for every $f \in D$ we have

$$\int_0^\infty P(t) f dt > 0 \text{ a.e. on } E. \tag{12}$$

From Corollary 1 in [7] it follows immediately that the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or is sweeping with respect to compact sets. Now, we will exclude the sweeping of the semigroup by construction a Khasminskii function. Such a function is a nonnegative C^2−function V such that

$$\sup_{\|(x,y)\|>R} LV(x, y) < 0 \quad \text{for some } R > 0, \text{ i.e.,}$$

$$\sup_{\|(x,y)\|>R}\left\{\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}+\frac{1}{2}\rho^2\frac{\partial^2 V}{\partial y^2}+(\sigma_1\rho_1+\sigma_2\rho_2)\frac{\partial^2 V}{\partial x\partial y}+f_1\frac{\partial V}{\partial x}+f_2\frac{\partial V}{\partial y}\right\}<0. \tag{13}$$

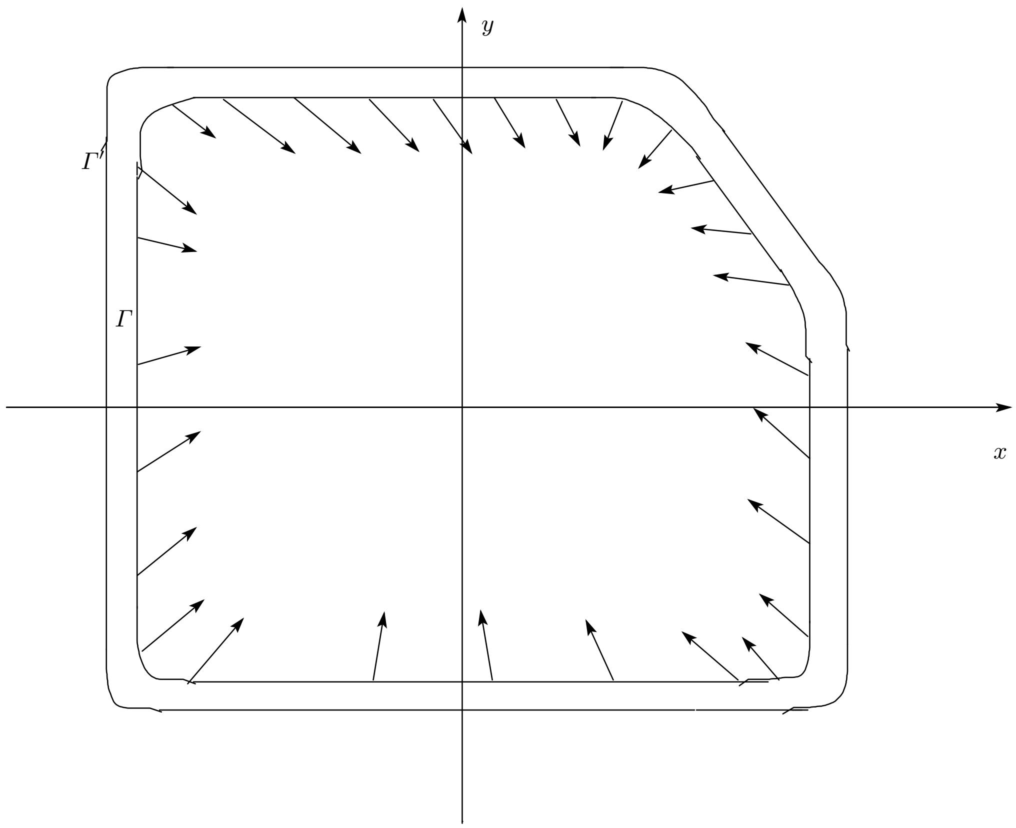

Fig. 1

The function $V(x,y)$ can be constructed similarly as one that is constructed graphically by Rudnicki [7]. We recall that method. First, for sufficiently large x^2+y^2, let Γ, Γ' be two curves constructed from line segments and from segments of circles with a constant and sufficiently large radius r. The line segments and segments of circles get in touch with other at intersection points and satisfy that the distance d between the parallel segments of Γ and Γ' is constant. Then, the function $V(x,y)$ is constant on Γ and Γ' (see Fig. 1) and $V(x_1,y_1)-V(x,y)=d$ for $(x,y)\in\Gamma$ and $(x_1,y_1)\in\Gamma'$. Fig. 2 shows the graphs of Γ and Γ' near a "vertice". For sufficiently large x^2+y^2, the vectors $[f_1(x,y), f_2(x,y)]$ direct inside the domains bounded by the curves Γ and Γ'. Then there exist constants $C_0>0$ and $R_0>0$ such that

$$f_1\frac{\partial V}{\partial x} + f_2\frac{\partial V}{\partial y} \le -C_0 \text{ for all } (x,y) \text{ such that } x^2+y^2 > R_0^2.$$

We have

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\rho^2\frac{\partial^2 V}{\partial y^2} + (\sigma_1\rho_1+\sigma_2\rho_2)\frac{\partial^2 V}{\partial x\partial y} = O\left(\frac{1}{r}\right)$$

for points from segments of circles of Γ and

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\rho^2\frac{\partial^2 V}{\partial y^2} + (\sigma_1\rho_1+\sigma_2\rho_2)\frac{\partial^2 V}{\partial x\partial y} = 0$$

for other points. Then, choose r sufficiently large, there exists $R > R_0$ such that $LV(x,y) \le -\frac{C_0}{2}$ when $x^2+y^2 > R^2$.

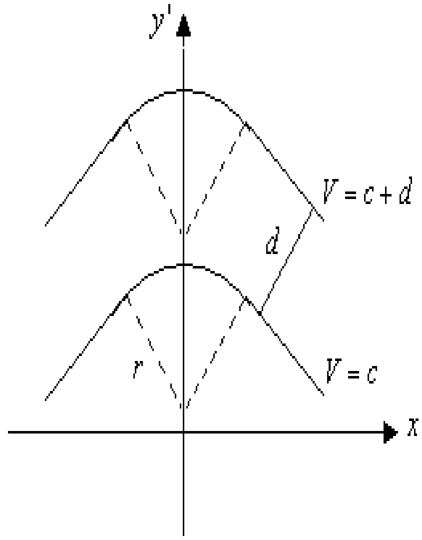

Fig. 2

Using a similar argument as in [6] one can check that the existence of a Khasminskii function ensures that the semigroup is not sweeping from the ball $B_R = \{(x,y) : x^2+y^2 \le R\}$. Thus, the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable on E, i.e., there exists a unique stationary density $u_*(x,y)$ of (5) and

$$\lim_{t\to\infty}\iint\limits_{\mathbb{R}^2} |u(t,x,y)-u_*(x,y)|dxdy = 0.$$

We complete the proof. ■

Next, we consider the case [VI]. Then we can transfer system (4) to the following system

$$\begin{cases} d\xi_t = (c_1 + b_{12}e^{\eta_t} - b_{11}e^{\xi_t})dt + \sigma dW_t, \\ d\eta_t = (c_2 + b_{21}e^{\xi_t} - b_{22}e^{\eta_t})dt + \rho dW_t, \end{cases} \tag{14}$$

where $W_t = \frac{\sigma_1}{\sigma}W_t^1 + \frac{\sigma_2}{\sigma}W_t^2 = \frac{\rho_1}{\rho}W_t^1 + \frac{\rho_2}{\rho}W_t^2$ is a standard Wiener process. Since system (14) has been studied by Ton [9], we will only recall the results of this case.

Theorem 3.6 ([9]). *Let (ξ_t, η_t) be a solution of system (4) under the condition [VI]. Then for every $t > 0$ the distribution of (ξ_t, η_t) has a density $u(t, x, y)$ which satisfies the Fokker-Planck equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\rho^2\frac{\partial^2 v}{\partial y^2} + \sigma\rho\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial(f_1 v)}{\partial x} - \frac{\partial(f_2 v)}{\partial y}\cdot$$

And the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable on E, where

$$E := \begin{cases} \mathbb{R}^2 \text{ if } \rho c_1 > \sigma c_2 \text{ and } \sigma > \rho \text{ or } \rho c_1 < \sigma c_2 \text{ and } \sigma < \rho, \\ \left\{(x,y) \in \mathbb{R}^2 : y \leq \frac{\rho}{\sigma}x + M_0\right\} \text{ if } \rho c_1 > \sigma c_2 \text{ and } \sigma \leq \rho, \\ \left\{(x,y) \in \mathbb{R}^2 : x \leq \frac{\sigma}{\rho}y + M_1\right\} \text{ if } \rho c_1 < \sigma c_2 \text{ and } \rho \leq \sigma, \end{cases}$$

i.e., there exists a unique stationary density $u_*(x, y)$ of the Fokker-Planck equation

$$\frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\rho^2\frac{\partial^2 v}{\partial y^2} + \sigma\rho\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial(f_1 v)}{\partial x} - \frac{\partial(f_2 v)}{\partial y} = 0,$$

such that

$$\lim_{t\to\infty} \iint_E |u(t,x,y) - u_*(x,y)|dxdy = 0.$$

Further, if $\frac{\sigma}{\rho} = \frac{c_2}{c_1} = 1$ then $\lim\limits_{t\to\infty}(\xi_t - eta_t) = \ln\frac{b_{12}+b_{22}}{b_{11}+b_{21}}$ a.s. and

(a) *If $(b_{21} - b_{11})\sqrt{\frac{b_{12}+b_{22}}{b_{11}+b_{21}}} + (b_{12} - b_{22})\sqrt{\frac{b_{11}+b_{21}}{b_{12}+b_{22}}} \geq 0$ then*

$$\lim_{t\to\infty}(\xi_t + \eta_t) = \infty \quad a.s..$$

(b)$(b_{21} - b_{11})\sqrt{\frac{b_{12}+b_{22}}{b_{11}+b_{21}}} + (b_{12} - b_{22})\sqrt{\frac{b_{11}+b_{21}}{b_{12}+b_{22}}} < 0$ *then*

$$\overline{\lim}_{t\to\infty}(\xi_t + \eta_t) = \infty, \quad \underline{\lim}_{t\to\infty}(\xi_t + \eta_t) = -\infty \quad a.s..$$

Next, we consider the case [IV]. Put $W_t = \frac{\rho_1}{\rho}W_t^1 + \frac{\rho_2}{\rho}W_t^2$, then W_t is a standard Wiener process and system (4) becomes

$$\begin{cases} d\xi_t = (c_1 + b_{12}e^{\eta_t} - b_{11}e^{\xi_t})dt, \\ d\eta_t = (c_2 + b_{21}e^{\xi_t} - b_{22}e^{\eta_t})dt + \rho dW_t. \end{cases} \tag{15}$$

We have

Theorem 3.7. *If the condition* [IV] *holds, then for every* $t \geq 0$*,*

$$b_{11}\eta_t + b_{21}\xi_t = b_{11}\eta_0 + b_{21}\xi_0 + (c_2b_{11} + b_{21}c_1)t + \rho b_{11}W_t.$$

Proof. It follows from $\dfrac{b_{21}}{b_{11}} = \dfrac{b_{22}}{b_{12}}$ and system (15) that

$$d\left(\eta_t + \frac{b_{21}}{b_{11}}\xi_t\right) = \left(c_2 + \frac{b_{21}}{b_{11}}c_1\right)dt + \rho dW_t.$$

Thus $\eta_t + \dfrac{b_{21}}{b_{11}}\xi_t = \eta_0 + \dfrac{b_{21}}{b_{11}}\xi_0 + \left(c_2 + \dfrac{b_{21}}{b_{11}}c_1\right)t + \rho W_t$. So, we complete the proof. ■

Similarly, we have

Theorem 3.8. *If the condition* [V] *holds, then for every* $t \geq 0$*,*

$$b_{22}\xi_t + b_{12}\eta_t = b_{22}\xi_0 + b_{12}\eta_0 + (c_1b_{22} + b_{12}c_2)t + \sigma b_{22}W_t^*,$$

where the standard Wiener process $W_t^* = \dfrac{\rho_1}{\rho}W_t^1 + \dfrac{\rho_2}{\rho}W_t^2$.

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