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On a Class of Lifting Modules

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Abstract. In this paper, we introduce principally δ-lifting modules which are analogous to δ-lifting modules and principally δ-semiperfect modules as a generalization of δ-semiperfect modules and investigate their properties.

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1. Introduction

Throughout this paper all rings have an identity, all modules considered are unital right modules. Let M be a module and N, P be submodules of M. We call P *a supplement* of N in M if $M = P + N$ and $P \cap N$ is small in P. A module M is called *supplemented* if every submodule of M has a supplement in M. A module M is called *lifting* if, for all $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in M. Supplemented and lifting modules have been discussed by several authors (see [2, 5, 3, 6]) and these modules are useful in characterizing semiperfect and right perfect rings (see [5, 8]).

In this note, we study and investigate principally δ-lifting modules and principally δ-semiperfect modules. A module M is called *principally δ-lifting* if for each cyclic submodule has the *δ-lifting property*, i.e., for each $m \in M$, M has a decomposition $M = A \oplus B$ with $A \leq mR$ and $mR \cap B$ is δ-small in B, where B is called *a δ-supplement* of mR. A module M is called *principally δ-semiperfect* if, for each $m \in M$, M/mR has a projective δ-cover. We prove that if M_1 is semisimple, M_2 is principally δ-lifting, M_1 and M_2 are relatively projective, then $M = M_1 \oplus M_2$ is a principally δ-lifting module. Among others we also prove that for a principally δ-semiperfect module M, M is principally δ-supplemented, each factor module of M is principally δ-semiperfect, hence any homomorphic image and any direct summand of M is principally δ-semiperfect. As an application, for a projective module M, it is shown that M is principally δ-semiperfect if and only if it is principally δ-lifting, and therefore a ring R is principally δ-semiperfect if and only if it is principally δ-lifting.

In Sec. 2, we give some properties of δ-small submodules that we use in the paper, and in Sec. 3, principally δ-lifting modules are introduced and various properties of principally δ-lifting and δ-supplemented modules are obtained. In Sec. 4, principally δ-semiperfect modules are defined and characterized in terms of principally δ-lifting modules.

In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/\mathbb{Z}n$ we denote, respectively, integers, rational numbers, the ring of integers and the $\mathbb{Z}$-module of integers modulo n. For unexplained concepts and notations, we refer the reader to [1, 5].

2. δ-Small Submodules

Following Zhou [10], a submodule N of a module M is called a *δ-small submodule* if, whenever $M = N + X$ with M/X singular, we have $M = X$. We begin by stating the next lemma which is contained in [10, Lemmas 1.2 and 1.3].

Lemma 2.1. *Let M be a module. Then we have the following.*

(1) *If N is δ-small in M and $M = X + N$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \leq N$.*

(2) *If K is δ-small in M and $f : M \to N$ is a homomorphism, then $f(K)$ is δ-small in N. In particular, if K is δ-small in $M \leq N$, then K is δ-small in N.*

(3) *Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is δ-small in $M_1 \oplus M_2$ if and only if K_1 is δ-small in M_1 and K_2 is δ-small in M_2.*

(4) *Let N, K be submodules of M with K is δ-small in M and $N \leq K$. Then N is also δ-small in M.*

Lemma 2.2. *Let M be a module and $m \in M$. Then the following are equivalent.*

(1) *mR is not δ-small in M.*

(2) *There is a maximal submodule N of M such that $m \notin N$ and M/N singular.*

Proof. (1) $\Rightarrow$ (2) Let $\Gamma := \{B \leq M \mid B \neq M, mR + B = M, M/B \text{ singular}\}$. Since mR is not δ-small in M, there exists a proper submodule B of M such that

$mR + B = M$ and M/B is singular. So Γ is non empty. Let Λ be a nonempty totally ordered subset of Γ and $B_0 := \cup_{B\in\Lambda} B$. If m is in B_0 then there is a $B \in \Lambda$ with $m \in B$. Then $B = mR + B = M$ which is a contradiction. So we have $m \notin B_0$ and $B_0 \neq M$. Since $mR + B_0 = M$ and M/B_0 is singular, B_0 is an upper bound in Γ. By Zorn's Lemma, Γ has a maximal element, say N. If N is a maximal submodule of M there is nothing to do. Assume that there exists a submodule K containing N properly. Since N is maximal in Γ, K is not in Γ. Since $M = mR + N$ and $N \leq K$, so $M = mR + K$. M/K is singular as a homomorphic image of the singular module M/N. Hence K must belong to the Γ. This is the required contradiction.

(2) $\Rightarrow$ (1) Let N be a maximal submodule with $m \in M\backslash N$ and M/N singular. We have $M = mR + N$. Then mR is not δ-small in M. ∎

Let A and B be submodules of M with $A \leq B$. A is called a *δ-cosmall submodule* of B in M if B/A is δ-small in M/A. Let A be a submodule of M. A is called a *δ-coclosed submodule* in M if A has no proper δ-cosmall submodules in M. A submodule A is called *δ-coclosure* of B in M if A is δ-coclosed submodule of M and it is δ-cosmall submodule of B. Equivalently, for any submodule $C \leq A$ with A/C δ-small in M/C implies $C = A$ and B/A is δ-small in M/A. Note that δ-coclosed submodules need not always exist.

Lemma 2.3. *Let A and B be submodules of M with $A \leq B$. Then we have:*

(1) *A is δ-cosmall submodule of B in M if and only if $M = A + L$ for any submodule L of M with $M = B + L$ and M/L singular.*

(2) *If A is δ-small and B is δ-coclosed in M, then A is δ-small in B.*

Proof. (1) Necessity: Let $M = B + L$ and M/L be singular. We have $M/A = B/A + (L + A)/A$ and $M/(L + A)$ is singular as homomorphic image of the singular module M/L. Since B/A is δ-small, $M/A = (L + A)/A$ or $M = L + A$. Sufficiency: Let $M/A = B/A + K/A$ and M/K be singular. Then $M = B + K$. By hypothesis, $M = A + K$ and so $M = K$. Hence A is a δ-cosmall submodule of B in M.

(2) Assume that A is a δ-small submodule of M and B is δ-coclosed in M. Let $B = A + K$ with B/K singular. Since B is δ-coclosed in M, to complete the proof, by part (1) it suffices to show that K is a δ-small submodule of B in M. Let $M = B + L$ with M/L singular. By assumption, $M = A + K + L = K + L$ since $M/(K + L)$ is singular. By (1), K is a δ-small submodule of B in M. ∎

Lemma 2.4. *Let A, B and C be submodules of M with $M = A + C$ and $A \leq B$. If $B \cap C$ is a δ-small submodule of M, then A is a δ-cosmall submodule of B in M.*

Proof. Let $M/A = B/A + L/A$ with M/L singular. We have $M = B + L$ and $B = A + (B \cap C)$. Then $M = A + (B \cap C) + L = (B \cap C) + L$. Hence $M = L$ since $B \cap K$ is δ-small in M and M/L is singular. Hence B/A is δ-small in M/A. Thus A is a δ-cosmall submodule of B in M. ∎

3. Principally δ-Lifting Modules

In this section, we study and investigate some properties of principally δ-lifting modules. The following definition is motivated by [10, Lemma 3.4] and Lemma 3.4.

Definition 3.1. A module M is called *finitely δ-lifting* if for any finitely generated submodule A of M has the *δ-lifting property*, that is, there is a decomposition $M = N \oplus S$ with $N \leq A$ and $A \cap S$ is δ-small in S. In this case $A \cap S$ is δ-small in S if and only if $A \cap S$ is δ-small in M. A module M is called *principally δ-lifting* if for each cyclic submodule has the *principally δ-lifting property*, i.e., for each $m \in M$, M has a decomposition $M = A \oplus B$ with $A \leq mR$ and $mR \cap B$ is δ-small in B.

Example 3.2. Every submodule of any semisimple module satisfies principally δ-lifting property.

Example 3.3. Let p be a prime integer and n any positive integer. Then the $\mathbb{Z}$-module $M = \mathbb{Z}/\mathbb{Z}p^n$ is a principally δ-lifting module.

Lemma 3.4. *The following are equivalent for a module M:*

(1) *M is finitely δ-lifting.*

(2) *M is principally δ-lifting.*

Proof. See [8] and [10]. ■

Let M be a module and N a submodule of M. A submodule L is called a *δ-supplement* of N in M if $M = N + L$ and $N \cap L$ is δ-small in L (therefore in M).

Proposition 3.5. *Let M be a principally δ-lifting module. Then we have:*

(1) *Every direct summand of M is a principally δ-lifting module.*

(2) *Every cyclic submodule C of M has a δ-supplement S which is a direct summand, and C contains a complementary summand of S in M.*

Proof. (1) Let K be a direct summand of M and $k \in K$. Then M has a decomposition $M = N \oplus S$ with $N \leq kR$ and $kR \cap S$ is δ-small in M. It follows that $K = N \oplus (K \cap S)$, and $kR \cap (K \cap S) \leq kR \cap S$ is δ-small in M and so $kR \cap (K \cap S)$ is δ-small in K. Therefore K is a principally δ-lifting module.

(2) Assume that M is a principally δ-lifting module and C is a cyclic submodule of M. Then we have $M = N \oplus S$, where $N \leq C$ and $C \cap S$ is δ-small in M. Hence $M = N + S \leq C + S \leq M$, we have $M = C + S$. Since S is a direct summand and $C \cap S$ is δ-small in M, $C \cap S$ is δ-small in S. Therefore S is a δ-supplement of C in M. ■

Theorem 3.6. *The following are equivalent for a module M:*

(1) *M is a principally δ-lifting module.*

(2) *Every cyclic submodule C of M can be written as $C = N \oplus S$, where N is a direct summand and S is δ-small in M.*

(3) *For every cyclic submodule C of M, there is a direct summand A of M with $A \leq C$ and C/A δ-small in M/A.*

(4) *Every cyclic submodule C of M has a δ-supplement K in M such that $C \cap K$ is a direct summand in C.*

(5) *For every cyclic submodule C of M, there is an idempotent $e \in \mathrm{End}(M)$ with $eM \leq C$ and $(1-e)C$ δ-small in $(1-e)M$.*

(6) *For each $m \in M$, there exist ideals I and J of R such that $mR = mI \oplus mJ$, where mI is a direct summand of M and mJ is δ-small in M.*

Proof. (1)⇒(2) Let C be a cyclic submodule of M. By hypothesis there exist N and S submodules of M such that $N \leq C$, $C \cap S$ is δ-small in M and $M = N \oplus S$. Then we have $C = N \oplus (C \cap S)$.

(2) ⇒ (3) Let C be a cyclic submodule of M. By hypothesis, $C = N \oplus S$, where N is a direct summand and S is δ-small in M. Let $\pi : M \to M/N$ be the natural projection. Since S is δ-small in M, we have $\pi(S)$ is δ-small in M/N. Since $\pi(S) \cong S \cong C/N$, C/N is δ-small in M/N.

(3) ⇒ (4) Let C be a cyclic submodule of M. By hypothesis, there is a direct summand $A \leq M$ with $A \leq C$ and C/A δ-small in M/A. Let $M = A \oplus A'$. Hence $C = A \oplus (A' \cap C)$. Let $\sigma : M/A \to A'$ denote the obvious isomorphism. Then $\sigma(C/A) = A' \cap C$ is δ-small in A'.

(4) ⇒ (5) Let C be any cyclic submodule of M and $K \leq M$ such that $C \cap K$ is a direct summand of C, $M = C + K$ and $C \cap K$ is δ-small in K. So $C = (C \cap K) \oplus X$ for some $X \leq C$. Then $M = X + (C \cap K) + K = X \oplus K$. Let $e : M \to X$; $e(x + k) = x$ and $(1 - e) : M \to K$; $e(x + k) = k$ be projection maps. $e(M) \leq X \leq C$ and $(1 - e)C = C \cap (1 - e)M = C \cap K$ is δ-small in $(1 - e)M$.

(5) ⇒ (6) Let mR be any cyclic submodule of M. By hypothesis, there exists an idempotent $e \in \mathrm{End}(M)$ such that $eM \leq mR$, $M = eM \oplus (1 - e)M$ and $(1 - e)mR$ is δ-small in $(1 - e)M$. Note that $(mR) \cap ((1 - e)M) = (1 - e)mR$ (for if $m = em_1 + y$, where $em_1 \in eM$, $y \in (mR) \cap ((1 - e)M)$. Then $(1 - e)m = em_1 + (1 - e)y = y$ and so $(1 - e)mR \leq (mR) \cap ((1 - e)M)$. Let $mr = (1 - e)m' \in (mR) \cap ((1 - e)M)$. Then $mr = (1 - e)mr \in (1 - e)mR$. So $(mR) \cap ((1 - e)M) \leq (1 - e)mR$. Thus $(mR) \cap ((1 - e)M) = (1 - e)mR$). So $mR = eM \oplus (1 - e)mR$. Let $I = \{r \in R : mr \in eM\}$ and $J = \{t \in R : mt \in (1 - e)mR\}$. Then $mR = mI \oplus mJ$, $mI = eM$ and $mJ = (1 - e)mR$ is δ-small in $(1 - e)M$.

(6) ⇒ (1) Let $m \in M$. By hypothesis, there exist ideals I and J of R such that $mR = mI \oplus mJ$, where mI is a direct summand and mJ is δ-small in M. Let $M = mI \oplus K$ for some submodule K. Since $K \cap mR \cong mJ$ and mJ is δ-small in M, M is principally δ-lifting. ■

Note that every lifting module is principally δ-lifting. There are principally δ-lifting modules but not lifting.

Example 3.7. Let M be the $\mathbb{Z}$-module $\mathbb{Q}$ and $m \in M$. It is well known that every cyclic submodule mR of M is small, therefore δ-small in M. Hence M is a principally δ-lifting $\mathbb{Z}$-module. If N is a nonsmall proper submodule of M, then N is neither a direct summand nor contains a direct summand of M. It follows that M is not a lifting $\mathbb{Z}$-module.

It is clear that every δ-lifting module is principally δ-lifting. However the converse is not true.

Example 3.8. Let R and T denote the rings in [10, Example 4.1], where

$$R = \sum_{i=1}^{\infty} \bigoplus \mathbb{Z}_2 + \mathbb{Z}_2.1 = \left\{ (f_1, f_2, \ldots, f_n, f, f, \ldots) \in \prod_{i=1}^{\infty} \mathbb{Z}_2 \right\}$$

and

$$T = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \ : \ x \in R, \ y \in \mathrm{Soc}(R) \right\}.$$

Then $\mathrm{Rad}_\delta(T) = \begin{bmatrix} 0 & \mathrm{Soc}(R) \\ 0 & 0 \end{bmatrix}$ and $T/\mathrm{Rad}_\delta(T)$ is not semisimple as isomorphic to R. So T is not δ-semiperfect by [10, Theorem 3.6]. Hence T is not a δ-lifting module over T. It is easy to show that $T/\mathrm{Rad}_\delta(T)$ lifts to idempotents of T, so T is a semiregular ring. Since T is a δ-semiregular ring, every finitely generated right ideal H of T can be written as $H = aT \oplus S$, where $a^2 = a \in T$ and $S \leq \mathrm{Rad}_\delta(T)$ by [10, Theorem 3.5]. Hence T is a principally δ-lifting module.

Proposition 3.9. *Let M be a principally δ-lifting module. If $M = M_1 + M_2$ such that $M_1 \cap M_2$ is cyclic, then M_2 contains a δ-supplement of M_1 in M.*

Proof. Assume that $M = M_1 + M_2$ and $M_1 \cap M_2$ is cyclic. Then we have $M_1 \cap M_2 = N \oplus S$, where N is a direct summand of M and S is δ-small in M. Let $M = N \oplus N'$ and $M_2 = N \oplus (M_2 \cap N')$. It follows that $M_1 \cap M_2 = N \oplus (M_1 \cap M_2 \cap N') = N \oplus S$. Let $\pi : M_2 = N \oplus (M_2 \cap N') \to N'$ be the natural projection. It follows that $\pi(M_1 \cap M_2 \cap N') = M_1 \cap M_2 \cap N' = \pi(S)$. Since S is δ-small in M, it is δ-small in N' by Lemma 2.1. Hence $M = M_1 + (M_2 \cap N')$, $M_2 \cap N' \leq M_2$ and $M_1 \cap (M_2 \cap N')$ is δ-small in $M_2 \cap N'$. $M_2 \cap N'$ is contained in M_2 and a δ-supplement of M_1 in M_2. This completes the proof. ■

Let M be a module. A submodule N is called *fully invariant* if for each endomorphism f of M, $f(N) \leq N$. Let $S = \mathrm{End}(M_R)$, the ring of R-endomorphisms of M. Then M is a left S-, right R-bimodule and a principal submodule N of the right R-module M is fully invariant if and only if N is a sub-bimodule of M. Clearly 0 and M are fully invariant submodules of M. The right R-module M is called a *duo module* provided every submodule of M is fully invariant. For the readers' convenience we state and prove Lemma 3.10 which is proved in [7].

Lemma 3.10. *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules M_i $(i \in I)$ and let N be a fully invariant submodule of M. Then $N = \bigoplus_{i \in I} (N \cap M_i)$.*

Proof. For each $j \in I$, let $p_j : M \to M_j$ denote the canonical projection and let $i_j : M_j \to M$ denote inclusion. Then $i_j p_j$ is an endomorphism of M and hence $i_j p_j(N) \subseteq N$ for each $j \in I$. It follows that $N \subseteq \bigoplus_{j \in I} i_j p_j(N) \subseteq \bigoplus_{j \in I} (N \cap M_j) \subseteq N$, so that $N = \bigoplus_{j \in I} (N \cap M_j)$. ∎

One may suspect that if M_1 and M_2 are principally δ-lifting modules, then $M_1 \oplus M_2$ is also principally δ-lifting. But this is not the case.

Example 3.11. Consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}/\mathbb{Z}2$ and $M_2 = \mathbb{Z}/\mathbb{Z}8$. It is clear that M_1 and M_2 are principally δ-lifting. Let $M = M_1 \oplus M_2$. Then M is not a principally δ-lifting $\mathbb{Z}$-module. Let $N_1 = (\overline{1}, \overline{2})\mathbb{Z}$ and $N_2 = (\overline{1}, \overline{1})\mathbb{Z}$. Then $M = N_1 + N_2$, N_1 is not a direct summand of M and does not contain any nonzero direct summand of M. For any proper submodule N of M, M/N is a singular $\mathbb{Z}$-module. Hence the principal submodule does not satisfy δ-lifting property. It follows that M is not a principally δ-lifting $\mathbb{Z}$-module. By the same reasoning, for any prime integer p, the $\mathbb{Z}$-module $M = (\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ is not principally δ-lifting.

We have already observed by the preceding example that the direct sum of principally δ-lifting modules need not be principally δ-lifting. Note the following fact.

Proposition 3.12. *Let $M = M_1 \oplus M_2$ be a decomposition of M with M_1 and M_2 principally δ-lifting modules. If M is a duo module, then M is principally δ-lifting.*

Proof. Let $M = M_1 \oplus M_2$ be a duo module and mR be a submodule of M. By Lemma 3.10, $mR = ((mR) \cap M_1) \oplus ((mR) \cap M_2)$. Since $(mR) \cap M_1$ and $(mR) \cap M_2$ are principal submodules of M_1 and M_2 respectively, there exist $A_1, B_1 \leq M_1$ such that $A_1 \leq (mR) \cap M_1 \leq M_1 = A_1 \oplus B_1$, $B_1 \cap ((mR) \cap M_1) = B_1 \cap (mR)$ is δ-small in B_1, and $A_2, B_2 \leq M_2$ such that $A_2 \leq (mR) \cap M_2 \leq M_2 = A_2 \oplus B_2$, $B_2 \cap ((mR) \cap M_2) = B_2 \cap (mR)$ is δ-small in B_2. Then $M = A_1 \oplus A_2 \oplus B_1 \oplus B_2$, $A_1 \oplus A_2 \leq N$ and $(mR) \cap (B_1 \oplus B_2) = ((mR) \cap B_1) \oplus ((mR) \cap B_2)$ is δ-small in $M_1 \oplus M_2$. ∎

Lemma 3.13. *The following are equivalent for a module $M = M' \oplus M''$.*

(1) *M' is M''-projective.*

(2) *For each submodule N of M with $M = N + M''$, there exists a submodule $N' \leq N$ such that $M = N' \oplus M''$.*

Proof. See [8, 41.14] ∎

Theorem 3.14. *Let M_1 be a semisimple module and M_2 a principally δ-lifting module. Assume that M_1 and M_2 are relatively projective. Then $M = M_1 \oplus M_2$ is principally δ-lifting.*

Proof. Let $0 \neq m \in M$ and let $K = M_1 \cap ((mR) + M_2)$. We divide the proof into two cases:

Case (i): $K \neq 0$. Then $M_1 = K \oplus K_1$ for some submodule K_1 of M_1 and so $M = K \oplus K_1 \oplus M_2 = (mR) + (M_2 \oplus K_1)$. Hence K is $M_2 \oplus K_1$-projective. By Lemma 3.13, there exists a submodule N of mR such that $M = N \oplus (M_2 \oplus K_1)$. We may assume $(mR) \cap (M_2 \oplus K_1) \neq 0$. Note that for any submodule L of M_2, we have $(mR) \cap (L + K_1) = L \cap ((mR) + K_1)$. In particular $(mR) \cap (M_2 + K_1) = M_2 \cap (mR + K_1)$. Then $mR = N \oplus (mR) \cap (K_1 \oplus M_2)$. There exist $n \in N$ and $m' \in (mR) \cap (K_1 \oplus M_2)$ such that $m = n + m'$. Then $nR = N$ and $m'R = (mR) \cap (K_1 \oplus M_2)$. Since $(mR) \cap (M_2 + K_1) = M_2 \cap ((mR) + K_1)$, $M_2 \cap ((mR) + K_1)$ is a principal submodule of M_2 and M_2 is principally δ-lifting, there exists a submodule X of $M_2 \cap ((mR) + K_1) = (mR) \cap (M_2 \oplus K_1)$ such that $M_2 = X \oplus Y$ and $Y \cap M_2 \cap ((mR) + K_1) = Y \cap ((mR) + K_1)$ is δ-small in $M_2 \cap ((mR) + K_1)$ and in M_2. Hence $M = (N \oplus X) \oplus (Y \oplus K_1)$. Since $N \oplus X \leq mR$ and $(mR) \cap (Y \oplus K_1) = Y \cap ((mR) + K_1)$, $(mR) \cap (Y \oplus K_1) = Y \cap ((mR) + K_1)$ is δ-small in $Y \oplus K_1$. So M is δ-lifting.

Case (ii): $K = 0$. Then $mR \leq M_2$. Since M_2 is δ-lifting, there exists a submodule X of mR such that $M_2 = X \oplus Y$ and $(mR) \cap Y$ is δ-small in Y for some submodule Y of M_2. Hence $M = X \oplus (M_1 \oplus Y)$. Since $(mR) \cap (M_1 \oplus Y) = (mR) \cap Y$ and $(mR) \cap (M_1 \oplus Y) = (mR) \cap Y$ is δ-small in Y. By Lemma 2.1 (3), $(mR) \cap (M_1 \oplus Y)$ is δ-small in $M_1 \oplus Y$. It follows that M is δ-lifting. ■

A module M is said to be *a principally semisimple* if every cyclic submodule is a direct summand of M. Tuganbayev calls a principally semisimple module as a regular module in [4]. Every semisimple module is principally semisimple. Every principally semisimple module is principally δ-lifting. For a module M, we write $\mathrm{Rad}_\delta(M) = \sum\{L \mid L \text{ is a } \delta\text{-small submodule of } M\}$.

Lemma 3.15. *Let M be a principally δ-lifting module. Then $M/\mathrm{Rad}_\delta(M)$ is a principally semisimple module.*

Proof. Let $m \in M$. There exists $M_1 \leq mR$ such that $M = M_1 \oplus M_2$ and $(mR) \cap M_2$ is δ-small in M_2. So $(mR) \cap M_2$ is δ-small in M. Then

$$M/\mathrm{Rad}_\delta(M) = [(mR + \mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M)] \oplus [(M_2 + \mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M)]$$

because $(mR + \mathrm{Rad}_\delta(M)) \cap (M_2 + \mathrm{Rad}_\delta(M)) = \mathrm{Rad}_\delta(M)$. Hence every principal submodule of $M/\mathrm{Rad}_\delta(M)$ is a direct summand. ■

Proposition 3.16. *Let M be a principally δ-lifting module. Then $M = M_1 \oplus M_2$, where M_1 is a principally semisimple module and M_2 is a module with $\mathrm{Rad}_\delta(M)$ essential in M_2.*

Proof. Let M_1 be a submodule of M such that $\mathrm{Rad}_\delta(M) \oplus M_1$ is essential in M and $m \in M_1$. Since M is principally δ-lifting, there exists a direct summand M_2 of M such that $M_2 \leq mR$, $M = M_2 \oplus M_2'$ and $mR \cap M_2'$ is δ-small in M. Hence $mR \cap M_2'$ is a submodule of $\mathrm{Rad}_\delta(M)$ and so $mR \cap M_2' = 0$. Then $m \in M_2$ and $mR = M_2$. Since $M_2 \cap$ $\mathrm{Rad}_\delta(M) = 0$, M_2 is isomorphic to a submodule of $M/\mathrm{Rad}_\delta(M)$. By Lemma 3.15, $M/\mathrm{Rad}_\delta(M)$ is principally semisimple, M_2 is principally semisimple. On the other hand, $\mathrm{Rad}_\delta(M)$ $=\mathrm{Rad}_\delta(M_2')$ is essential in M_2 that it is clear from the construction of M_2'. ■

A nonzero module M is called *δ-hollow* if every proper submodule is δ-small in M, and M is *principally δ-hollow* if every proper cyclic submodule is δ-small in M, and M is *finitely δ-hollow* if every proper finitely generated submodule is δ-small in M. Since finite direct sum of δ-small submodules is δ-small, M is principally δ-hollow if and only if it is finitely δ-hollow.

Lemma 3.17. *The following are equivalent for an indecomposable module M.*

(1) *M is a principally δ-lifting module.*

(2) *M is a principally δ-hollow module.*

Proof. (1)⇒(2) Let $m \in M$. Since M is a principally δ-lifting module, there exist N and S submodules of M such that $N \leq mR$, $mR \cap S$ is δ-small in M and $M = N \oplus S$. By hypothesis, $N = 0$ and $S = M$. So that $mR \cap S = mR$ is δ-small in M.

(2)⇒(1) Let $m \in M$. Then $mR = (mR) \oplus (0)$. By (2) mR is δ-small and (0) is a direct summand in M. Hence M is a principally δ-lifting module. ■

Lemma 3.18. *Let M be a module, then we have*

(1) *If M is principally δ-hollow, then every factor module is principally δ-hollow.*

(2) *If K is a δ-small submodule of M and M/K is principally δ-hollow, then M is principally δ-hollow.*

(3) *M is principally δ-hollow if and only if M is local or* $\mathrm{Rad}_\delta(M) = M$.

Proof. (1) Assume that M is principally δ-hollow and N a submodule of M. Let $m + N \in M/N$ and $(mR + N)/N + K/N = M/N$. Suppose that M/K is singular. We have $mR + K = M$. Since M/K is singular and M is principally δ-hollow, $M = K$.

(2) Let $m \in M$. Assume that $mR + N = M$ for some submodule N with M/N singular. Then $(m + K)R = (mR + K)/K$ is a cyclic submodule of M/K and $(mR + K)/K + (N + K)/K = M/K$ and $M/(N + K)$ is singular as a homomorphic image of M/N. Hence $(N + K)/K = M/K$ or $N + K = M$. By hypothesis $N = M$.

(3) Suppose that M is principally δ-hollow and it is not local. Let N and K be two distinct maximal submodules of M and $k \in K \setminus N$. Then $M = kR + N$ and M/N is a simple module, and so M/N is a singular or projective module.

If M/N is singular, then $M = N$ since kR is δ-small. But this is not possible since N is maximal. So M/N is projective. Hence N is a direct summand. So $M = N \oplus N'$ for some nonzero submodule N' of M, that is, N and kR are proper submodules of M. Since every proper submodule of M is contained in $\mathrm{Rad}_\delta(M)$, $M = \mathrm{Rad}_\delta(M)$. The converse is clear. ■

Proposition 3.19. *Let M be a module. Then the following are equivalent.*

(1) *M is principally δ-hollow.*

(2) *If N is submodule with M/N cyclic, then N is a δ-small submodule of M.*

Proof. (1) $\Rightarrow$ (2) Assume that N is a submodule with M/N cyclic. Lemma 2.1 implies that M/N is principally δ-hollow since being δ-small is preserved under homomorphisms. Since M/N has maximal submodules, and by Lemma 3.18, M/N is local. There exists a unique maximal submodule N_1 containing N. Hence N is small, therefore it is δ-small.

(2) $\Rightarrow$ (1) We prove that every cyclic submodule is δ-small in M. So let $m \in M$ and $M = mR + N$ with M/N singular. Then M/N is cyclic. By hypothesis, N is a δ-small submodule of M. By Lemma 2.1, there exists a projective semisimple submodule Y of N such that $M = (mR) \oplus Y$. Let $Y = \bigoplus_{i \in I} N_i$ where each N_i is simple. Now we write $(mR) \oplus (\bigoplus_{i \neq j} N_j)$. Then $M/((mR) \oplus (\bigoplus_{i \neq j} N_j))$ is a cyclic module as it is isomorphic to simple module N_i. By hypothesis, $((mR) \oplus (\bigoplus_{i \neq j} N_j))$ is δ-small in M. Again by Lemma 2.1, there exists a projective semisimple submodule Z of $((mR) \oplus (\bigoplus_{i \neq j} N_j))$ such that $M = Z \oplus N_i$. Hence M is a projective semisimple module. So $M = N \oplus N'$ for some submodule N'. Then N' is projective. M/N is projective as it is isomorphic to N'. Hence M/N is a both singular and projective module. Thus $M = N$. ■

4. Applications

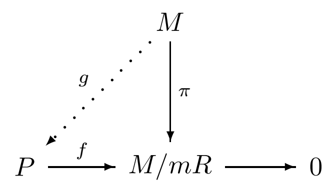
In this section, we introduce and study some properties of principally δ-semiperfect modules. By [10], a projective module P is called *a projective δ-cover* of a module M if there exists an epimorphism $f : P \longrightarrow M$ with $\mathrm{Ker} f$ δ-small in P, and a ring is called *δ-perfect* (or *δ-semiperfect*) if every R-module (or every simple R-module) has a projective δ-cover. For more detailed discussion on δ-small submodules, δ-perfect and δ-semiperfect rings, we refer to [10]. A module M is called *principally δ-semiperfect* if every factor module of M by a cyclic submodule has a projective δ-cover. A ring R is called *principally δ-semiperfect* in case the right R-module R is principally δ-semiperfect. Every δ-semiperfect module is principally δ-semiperfect. In [10], a ring R is called *δ-semiregular* if every cyclically presented R-module has a projective δ-cover.

Theorem 4.1. *Let M be a projective module. Then the following are equivalent.*

(1) *M is principally δ-semiperfect.*

(2) *M is principally δ-lifting.*

Proof. (1)$\Rightarrow$ (2) Let $m \in M$ and $P \xrightarrow{f} M/mR$ be a projective δ-cover and $M \xrightarrow{\pi} M/mR$ the natural epimorphism.



Then there exists a map $M \xrightarrow{g} P$ such that $fg = \pi$. Then $P = g(M) + \mathrm{Ker}(f)$. Since $\mathrm{Ker}(f)$ is δ-small, by Lemma 2.1, there exists a projective semisimple submodule Y of $\mathrm{Ker}(f)$ such that $P = g(M) \oplus Y$. So $g(M)$ is projective. Hence $M = K \oplus \mathrm{Ker}(g)$ for some submodule K of M. It is easy to see that $g(K \cap mR) = g(K) \cap \mathrm{Ker}(f)$ and $\mathrm{Ker}(g) \leq mR$. Hence $M = K + mR$. Next we prove $K \cap (mR)$ is δ-small in K. Since $\mathrm{Ker}(f)$ is δ-small in P, $g(K) \cap \mathrm{Ker}(f) = g(K \cap mR)$ is δ-small in P by Lemma 2.1 (4). Hence $K \cap (mR)$ is δ-small in K since g^{-1} is an isomorphism from $g(M)$ onto K.

(2)$\Rightarrow$ (1) Assume that M is a principally δ-lifting module. Let $m \in M$. There exist direct summands N and K of M such that $M = N \oplus K$, $N \leq mR$ and $mR \cap K$ is δ-small in K. Let $K \xrightarrow{\pi} M/mR$ denote the natural epimorphism defined by $\pi(k) = k + mR$ where $k \in K$, $k + mR \in M/mR$. It is obvious that $\mathrm{Ker}(\pi) = mR \cap K$. It follows that K is a projective δ-cover of M/mR. So M is principally δ-semiperfect. ■

Corollary 4.2. *Let R be a ring. Then the following are equivalent.*

(1) *R is principally δ-semiperfect.*

(2) *R is principally δ-lifting.*

(3) *R is δ-semiregular.*

Proof. (1) $\Leftrightarrow$ (2) Clear by Theorem 4.1.

(2) $\Leftrightarrow$ (3) By Theorem 3.6 (2), R is principally δ-lifting if and only if for every principal right ideal I of R can be written as $I = N \oplus S$, where N is a direct summand and S is δ-small in R. This is equivalent to being R δ-semiregular since for any ring R, $\mathrm{Rad}_\delta(R)$ is δ-small in R and each submodule of a δ-small submodule is δ-small. ■

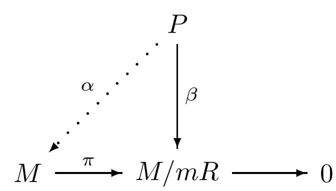
The module M is called *principally δ-supplemented* if every cyclic submodule of M has a δ-supplement in M. Clearly, every δ-supplemented module is principally δ-supplemented. Every principally δ-lifting module is principally δ-supplemented. In a subsequent paper we investigate principally δ-supplemented modules in detail. Now we prove:

Theorem 4.3. *Let M be a principally δ-semiperfect module. Then*

(1) *M is principally δ-supplemented.*

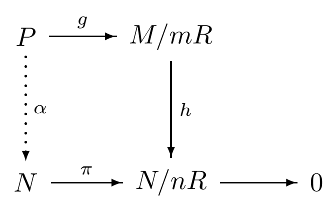
(2) *Each factor module of M is principally δ-semiperfect, hence any homomorphic image and any direct summand of M is principally δ-semiperfect.*

Proof. (1) Let $m \in M$. Then M/mR has a projective δ-cover $P \overset{\beta}{\to} M/mR$. There exists $P \overset{\alpha}{\to} M$ such that the following diagram is commutative, $\beta = \pi\alpha$, where $M \overset{\pi}{\to} M/mR$ is the natural epimorphism.



Then $M = \alpha(P) + mR$, and $\alpha(P) \cap mR$ is δ-small in $\alpha(P)$, by Lemma 2.1 (1). Hence M is principally δ-supplemented.

(2) Let $M \overset{f}{\to} N$ be an epimorphism and nR a cyclic submodule of N. Let $m \in f^{-1}(nR)$ and $P \overset{g}{\to} M/(mR)$ be a projective δ-cover. Define $M/(mR) \overset{h}{\to} N/nR$ by $h(m' + mR) = f(m') + nR$, where $m' + mR \in M/(mR)$. Then $\mathrm{Ker}(g)$ is contained in $\mathrm{Ker}(hg)$. By projectivity of P, there is a map α from P to N such that $hg = \pi\alpha$.



It is routine to check that $(nR) \cap \alpha(P) = \alpha(\mathrm{Ker}(g))$. By Lemma 2.1 (2), $\alpha(\mathrm{Ker}(g))$ is δ-small in N since $\mathrm{Ker}(g)$ is δ-small. Let $x \in \mathrm{Ker}(\pi\alpha)$. Then $hg(x) = (\pi\alpha)(x) = 0$ or $\alpha(x) \in (nR) \cap \alpha(P)$. So $\mathrm{Ker}(\pi\alpha)$ is δ-small. Hence P is a projective δ-cover for $N/(nR)$. ■

Theorem 4.4. *Let P be a projective module with $\mathrm{Rad}_\delta(P)$ δ-small in P. Then the following are equivalent.*

(1) *P is principally δ-lifting.*

(2) *$P/\mathrm{Rad}_\delta(P)$ is principally semisimple and, for any cyclic submodule $\overline{x}R$ of $P/\mathrm{Rad}_\delta(P)$ that is a direct summand of $P/\mathrm{Rad}_\delta(P)$, there exists a cyclic direct summand A of P such that $\overline{x}R = \overline{A}$.*

Proof. (1)⇒ (2) Since P is a principally δ-lifting module, $P/\mathrm{Rad}_\delta(P)$ is principally semisimple by Lemma 3.15. Let $\overline{x}R$ be any cyclic submodule of $P/\mathrm{Rad}_\delta(P)$. By Theorem 3.6, there exists a direct summand A of P and a δ-

small submodule B such that $xR = A \oplus B$. Since B is contained in $\mathrm{Rad}_\delta(R)$, $xR + \mathrm{Rad}_\delta(R) = A + \mathrm{Rad}_\delta(R)$. Hence $\overline{x}R = \overline{A}$.

(2)$\Rightarrow$ (1) Let xR be any cyclic submodule of P. Then we have $P/\ \mathrm{Rad}_\delta(P) = [(xR+\mathrm{Rad}_\delta(P))/\mathrm{Rad}_\delta(P)] \oplus [U/\ \mathrm{Rad}_\delta(P)]$ for some $U \leq P$. By (2), there exists a direct summand A of P such that $P = A \oplus B$ and $U = B+\ \mathrm{Rad}_\delta(P)$. Then $P = A \oplus B = A + U+\ \mathrm{Rad}_\delta(P)$. Since $\mathrm{Rad}_\delta(P)$ is δ-small in P, there exists a projective and semisimple submodule Y of P such that $P = A \oplus B = (A+U) \oplus Y$. Since P is projective, $A + B$ is also projective and so by Lemma 3.13, we have $A + B = V \oplus B$ for some $V \leq A$. Hence $P = V \oplus B \oplus Y$. On the other hand $(xR) \cap (B \oplus Y) = (xR) \cap B \leq (xR) \cap U \leq \mathrm{Rad}_\delta(R)$. Since $\mathrm{Rad}_\delta(R)$ is δ-small in P, it is δ-small in $B \oplus Y$ by Lemma 2.1 (3). Thus P is principally δ-lifting. ■

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