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Continuation of Periodic Solutions for a Class of Lienard Equations under Two Parametric Nonautonomous Perturbations

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Abstract. We consider the two parametric family of perturbed Lienard equations

$$\ddot{x} + f(x)\dot{x} + g(x, \dot{x}, t, \varepsilon) = 0. \qquad (*)$$

Here ε is a parameter and $f(x)$, $g(x, \dot{x}, t, \varepsilon)$ are polynomials with respect to x, y and $C^r, r > 1$ with respect to t. Equation $(*)$ is an effect of nonlinear forcing on the Lienard equation. Our aim is to show the persistence of periodic solutions of Lienard equation (if there is any) under perturbations. Therefore first we find some condition under which the Lienard equation has at least one periodic orbit, and then we investigate the persistence of the periodic orbit under perturbation (equation $(*)$). The techniques that we use are techniques of Chicone and Melnikov

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1. Introduction

The dynamic behavior of Lienard equation has been widely investigated [6, 14, 4, 20], due to their application in many fields such as physics, mechanics and engineering technique. Lienard system is one of the interested systems that are substantial. Lienard equation is widely studied by many mathematicians, for example we give some recent studies. Guo, Ge, and Lu in [13] study periodic solutions for generalized Lienard systems by means of Mawhins continuation theorem, Zhou et al. in [22] by using topological degree theory and some analysis skills study the existence and uniqueness of periodic orbit of a class of generalized Lienard systems, Zou, Chen, and Zhang in [23] study the local bifurcations of critical periods for cubic Lienard equations with cubic damping also Aghajani and Moradifam in [3] study the existence of homoclinic orbits of generalized Lienard equations. They extend the results presented by Hara and Yoneyama (see [15]) and present sufficient and necessary conditions for existence of homoclinic orbit. In such application, it is important to know the existence of periodic solution of Lienard equation and more important is the study of persistence of periodic solution under perturbations. Smale in [21] consider an easier and special class of polynomial Lienard systems

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -x, \end{cases} \tag{1}$$

where $F(x) = a_1 x + a_2 x^2 + \ldots + a_m x^m$. For this system the existence of uniform bound for periodic solution remains unproved. But when the degree m of this system is odd, Ilyushenko and Panov in [16] obtain a uniform bound for the number of limit cycles in a subclass of systems such that F is monic and its coefficients satisfy some estimations. For the Lienard system (1) Lins, de Melo and Pugh [17] conjectured that it has at most k limit cycles if $F(x)$ is a polynomial of degree $m = 2k+1$ or $m = 2k+2$. This conjecture is supported mainly by the following three facts. First, the Lienard system of the form

$$\begin{cases} \dot{x} = y - \varepsilon F(x), \\ \dot{y} = -x, \end{cases}$$

with ε sufficiently small has at most k limit cycles bifurcating from the periodic orbit of the linear oscillator

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x \end{cases}$$

and there are examples with exactly k, see [18]. Second, it is known that system (1) has a center at the origin if and only if $a_i = 0$ for all i odd, and that these a_i with i odd are the Liapunov constant of system (1). Consequently at most k small limit cycles can bifurcate by Hopf from these centers, when we perturbed them inside the class of all Lienard systems of degree $m = 2k+1$ or $m = 2k+2$, see Zuppa [24] and also Blows and Lloyd [5]. Third, Lopez and Lopez-Ruiz [18] have studied the Lienard system (1) in what they call the strongly nonlinear regime. In this regime they show that the conjecture is true when m is odd. More recently it was proved by Chen et al. in [7] that the conjecture holds restricted to Lienard system (1) with the function $F(x)$ odd. Therefore Sec. 2 is preliminary which gives those subjects necessary for next sections and in Sec. 3 we study the nonautonomous perturbation on Lienard equation.

2. Preliminaries

Consider the Lienard equation $\ddot{x}+f(x)\dot{x}+g(x)=0$, we can rewrite this equation in the form

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \tag{2}$$

where $F(x) = \int_0^x f(u)du$. In this paper we consider $F(x)$ as a polynomial of degree $2m+1$ and $g(x) = x$. So we have

$$\begin{cases} \dot{x} = y - (a_1 x + a_2 x^2 + ... + a_{2m+1} x^{2m+1}), \\ \dot{y} = -g(x). \end{cases} \tag{3}$$

We choose $\alpha > 0$ such that $a_i = \alpha\mu_i$ for $i = 1, \ldots, 2m+1$ and by using theorem below (see [19]) we find conditions which imply the existence of periodic solutions.

Theorem 2.1. *For $\alpha \neq 0$ sufficiently small system (2) with $g(x) = x$ and $F(x) = \alpha(\mu_1 x + \mu_2 x^2 + ... + \mu_{2m+1} x^{2m+1})$ has at most m limit cycles; also for $\alpha \neq 0$ sufficiently small, this system has exactly m limit cycles which are asymptotic to circles of radius r_j, $j = 1, 2, ..., m$, centered at the origin as $\alpha \to 0$ if and only if the m degree equation*

$$\frac{a_1}{2} + \frac{3a_3}{8}\rho + \ldots + \binom{m+1}{2m+2}\frac{a_{2m+1}}{2^{2m+2}}\rho^m = 0$$

has m positive roots $\rho = r_j^2$.

Now we investigate the persistence of periodic solutions of system (3) under perturbation. For the perturbations which are independent with respect to t, in [1] we present some results. For effect of nonautonomous perturbation on the Lienard equation we use the techniques in the Chicone papers [8, 9, 10, 11]. But since we are going to consider the two parametric perturbations we use the extended method of the Chicone paper (see [3]). Therefore in the following section the Chicon method and its extension are going to be given. First consider the system

$$\dot{x} = f(x) + \varepsilon g(x, t, \varepsilon), \qquad \varepsilon \in R, \quad x \in R^2 \tag{4}$$

in a neighborhood of U of a periodic orbit γ of period T for (4), where f, g are smooth and g is periodic with respect to t. The period of the function g is $\tau = \frac{m}{n}$, where m, n are prime. Perko in [19] uses the subharmonic Melnikov function

$$M = \int_0^{2n\pi} f(\gamma(t)) \wedge g(\gamma(t), t, 0)dt$$

and gives the following

Theorem 2.2. *If the subharmonic Melnikov function along a subharmonic periodic orbit $\gamma_r(t)$, of period $\frac{2n\pi}{m}$ has a simple zero in $[0, 2n\pi]$ then for all sufficiently small ε system* (4) *has a subharmonic periodic orbit of period $2n\pi$ in a ε-neighborhood of $\gamma_r(t)$ (see* [19]*).*

But which we use, is an extension of the above theorem that we explain below: Assuming there are no zeros of f in U then we have a basis $B(\xi) = \{f(\xi), f^{\perp}(\xi)\}$. Let $V \subseteq U$ be the zeros of P_0 that is the set of T-periodic solutions in U. Then we use the result about the Diliberto Theorem [12], see also [9]. This theorem illustrates the important result about the geometric meaning of the two functions α and β, see the following

Theorem 2.3. *For $\xi \in V$ the map P^1 evaluated at ξ has the form*

$$P^1 = (N(\xi), M(\xi))$$

with respect to the basis $B(\xi)$, where

$$N(\xi) = \int_0^{mT} \|f\|^{-2} \left\{ f.g - \frac{\alpha(t)}{\beta(t)} f^{\perp}.g \right\} \varphi_t(\xi) dt,$$

$$M(\xi) = \int_0^{mT} \left(\frac{1}{\beta(t)} f^{\perp}.g \right) \|f\|^{-2} \varphi_t(\xi) dt,$$

$$\alpha(t) = \int_0^{t} \{\|f\|^{-2}(2\kappa\|f\| - \text{curl} f)\beta\}(s, \xi) ds,$$

$$\beta(t) = \exp \int_0^{t} \text{div} f(s, \xi) ds.$$

Let us suppose that we have the vector field $V = (V_1, V_2)$, and then we define

$$\kappa = \|V\|^{-3}(V_1 \dot{V_2} - V_2 \dot{V_1}),$$

where the functions α and β corresponding respectively to the first order variation in time and in displacement transverse to the orbit $\gamma(t)$ of ξ for orbit of the unperturbed system (3). In case of $\alpha \neq 0$ or $\beta \neq 0$, $\gamma(t)$ is called normally nondegenerate, for more information and proof of Theorem 2.3, see [10]. Here $\gamma(t)$ is an isolated periodic orbit.

3. Two Parameters Method

Now we give an extended method of the Chicone. First we consider two parameters perturbations, therefore in this case we put $P^1 = (P^1_1, P^1_2)$ then

$P^1 : V \to R^2$. Hence P^1 is the map which takes V to 2×2 matrix, where i^{th} column of the matrix P^1 is P^1_i. Let us write $\varepsilon = ps$, where $p > 0$ and s belongs to the unit sphere $S^1 \in R^2$. In this case we have

$$\delta(\xi, \varepsilon) = P(\xi, \varepsilon) - \xi = \varepsilon P^1(\xi) + O(\varepsilon^2) = pP^1(\xi)s + O(p^2), \tag{5}$$

where P^1 is the derivative of $P_\varepsilon(\xi)$ with respect to $\varepsilon = (\varepsilon_1, \varepsilon_2)$ at $\varepsilon = (0, 0)$. From this formula, it is clear that if s_0 does not belong to the $\mathrm{Ker}P^1(\xi_0)$, then there is no solution for $\delta(\xi, \varepsilon) = 0$, for more information see [2]. Now we are able to state the following result

Lemma 3.1. *A necessary condition for a solution branch to emanate from $\xi_0 \in V$ as ε moves from the origin in R^2 in the direction of $s_0 \in S^1$ is that $s_0 \in \mathrm{Ker}P^1(\xi_0)$.*

This condition is necessary but it is not sufficient. However, the condition is sufficient if $s_0 \in \mathrm{Ker}P^1(\xi_0)$ in a way that which is nondegenerate with respect to the family of $\mathrm{Ker}P^1(\xi)$ as ξ varies near ξ_0. So we look at the $\mathrm{Ker}P^1$ which should be at least one-dimensional, see [11].

4. Nonautonomous Perturbation

In this section we study the effect of nonautonomous perturbation on Lienard equation with the form

$$\begin{cases} \dot{x} = y - \alpha(\mu_1 x + \mu_2 x^2 + ... + \mu_{2m+1} x^{2m+1}) + \varepsilon_1 \sum\limits_{i+j \le l} a_{i,j} x^i y^j, \\ \dot{y} = -x + \varepsilon_2 g(x, y, t), \end{cases}$$

where g is a periodic function with respect to t with period $\frac{2n\pi}{m}$. This kind of perturbations can consider as effect of forced and damping on Lienard equation. As before we let $a_i = \alpha\mu_i$. Suppose we have $\varepsilon_2 = \varepsilon_1 f(\varepsilon_1)$, hence we are on a curve in parameter space $(\varepsilon_1, \varepsilon_2)$. The above system can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(x, y) + \alpha P(x) + \varepsilon_1 G(x, y, t, \varepsilon_1), \tag{6}$$

where

$$F(x, y) = \begin{pmatrix} y \\ -x \end{pmatrix}, \quad G(x, y, t, \varepsilon_1) = \begin{pmatrix} \sum\limits_{i+j \le l} a_{i,j} x^i y^j \\ f(\varepsilon_1) g(x, y, t) \end{pmatrix}, \quad P(x) = \begin{pmatrix} \sum\limits_{i=1}^{2k+1} \mu_i x^i \end{pmatrix}.$$

Since g is periodic with respect to t, we choose

$$g(x, y, t) = \left[a \cos\left(\frac{n}{m} t\right) + b \sin\left(\frac{n}{m} t\right) \right] xy.$$

For equation (6), $\operatorname{div}(F) = 0$ and $\gamma_r(t) = (r\cos(t), r\sin(t))$ is a periodic solution for unperturbed equation $(\varepsilon_1, \alpha) = (0, 0)$.

Here we want to consider the perturbed system (6), from geometric point of view. For this purpose, we use the extended method of Chicone, see [8, 9, 10, 3]. Therefore now we solve the above problem by *bifurcation matrix* explained in Preliminaries.

We must compute $(P^1_\alpha, P^1_{\varepsilon_1})$ with

$$P^1_{\varepsilon_1} = (N_{\varepsilon_1}, M_{\varepsilon_1}), \qquad P^1_\alpha = (N_\alpha, M_\alpha),$$

where

$$\begin{aligned}
N_{\varepsilon_1} = \int_0^{2n\pi} \frac{1}{r^2}\Big(&\sum_{i+j\le l} a_{ij} r^{1+i+j}(\cos(t))^i(\sin(t))^{j+1}\\
&- f(0)(a\cos(\frac{n}{m}t) + b\sin(\frac{n}{m}t))r^3\sin(t)\cos^2(t)\\
&- \frac{4t}{r^2}(\sum_{i+j\le l} a_{ij} r^{1+i+j}\cos^{i+1}(t)\sin^j(t)\\
&+ f(0)(a\cos(\frac{n}{m}t) + b\sin(\frac{n}{m}t))r^3\cos(t)\sin^2(t))\Big)dt,\\
M_{\varepsilon_1} = \int_0^{2n\pi} \frac{1}{r^2}\Big(&\sum_{i+j\le l} a_{ij} r^{1+i+j}\cos^{i+1}(t)\sin^j(t)\\
&+ f(0)(a\cos(\frac{n}{m}t) + b\sin(\frac{n}{m}t)r^3\cos(t)\sin^2(t))\Big)dt,\\
N_\alpha = \int_0^{2n\pi} \frac{1}{r^2}(r\sin(t)&\Big(\sum_{i=1}^{2k+1}\mu_i r^i\cos^i(t)) - \frac{4t}{r}\cos(t)(\sum_{i=1}^{2k+1}\mu_i r^i\cos^i(t)\Big)dt,\\
M_\alpha = \int_0^{2n\pi} \frac{1}{r}\cos(t)(&\sum_{i=1}^{2k+1}\mu_i r^i\cos^i(t))dt.
\end{aligned}$$

We put

$$P(r) = \begin{pmatrix} N_{\varepsilon_1} & N_\alpha \\ M_{\varepsilon_1} & M_\alpha \end{pmatrix}.$$

We look for $\operatorname{Ker}P$ which at least should be one-dimensional. By computing the above integrals for $n = m = k = l = 1$, we have

$$P(r) = \begin{pmatrix} \frac{-\pi}{4r}f(0)(-br^2 + 4a\pi - 3b) & \frac{\pi}{4}(4a_1 + 3a_3r^2) \\ \frac{-1}{4}f(0)ra\pi & \frac{-\pi^2}{r^2}(4a_1 + 3a_3r^2) \end{pmatrix}.$$

Theorem 4.1. *The matrix $P(r)$ has at least one-dimensional kernel if:*
Case A: $(4a_1 + 3a_3r^2) = 0$,

or
Case B: $\frac{-\pi}{r^3}(-br^2 + 4a\pi - 3b) - \frac{1}{4}ra = 0.$

Proof. Suppose (x, y) is a vector such that $P(r).(x, y) = 0$ so we must have

$$\left(\frac{-\pi}{r^3}(-br^2 + 4a\pi - 3b) - \frac{1}{4}ra\right).x = 0, \quad (4a_1 + 3a_3r^2).y = 0. \tag{7}$$

Hence if we want to have at least one-dimensional kernel, we should have $(x, y) \neq 0$ such that conditions (7) is satisfied, so we must have case A or case B. ■

So by choosing suitable coefficient, the 1:1 resonance curve persists and for change of parameter we have a family of close curves (Hopf bifurcation).

Remark 4.2. In all of the above computations we do for a special case, i.e. for special value of $n = m = k = l = 1$. In fact we can find some result by changing their values.

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