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Weighted Estimates for Solutions of the $\overline{\partial}$-Equation in q-Pseudoconvex Domains

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Abstract. In this paper we give weighted estimates of Hörmander type for solutions of the $\overline{\partial}$-equation of $\overline{\partial}$-closed $(0, r)$-forms in q-pseudoconvex domains of $\mathbb{C}^n$. At the same time, we also establish the norm formula $|.|_\Theta$ of $(0, r)$-forms according to a positive definite Hermitian $(1, 1)$-form Θ.

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1. Introduction

Studying solutions of the $\overline{\partial}$-equation on pseudoconvex domains with weighted L^2-estimates of Hörmander type is one of important problems of complex analysis of several variables. The original Hörmander Theorem (see [11, Lemma 4.4.1]) said that if Ω is a pseudoconvex domain in $\mathbb{C}^n$ and φ is a weighted function in $C^2(\Omega)$ such that

$$c\sum_{j=1}^{n}|w_j|^2 \leqslant \sum_{j,k=1}^{n}\partial^2\varphi(z)/\partial z_j\partial\overline{z}_k w_j\overline{w}_k, \quad z\in\Omega, \quad w\in\mathbb{C}^n,$$

where c is a positive continuous function in Ω. Assume that $g \in L^2_{(p,q+1)}(\Omega)$ with $\overline{\partial}g = 0$. Then there exists $u \in L^2_{(p,q)}(\Omega)$ with $\overline{\partial}u = g$ and we have the following estimate

$$\int |u|^2 e^{-\varphi} dV \leqslant 2 \int |g|^2 e^{-\varphi}/c dV,$$

where dV denotes the Lebesgue measure in $\mathbb{C}^n$.

Later, a number of authors, namely, Donnelly and Fefferman (see [9]), Berndtsson (see [3, 5]) or Blocki (see [7]) extended the above result of Hörmander for $(0,1)$-forms on pseudoconvex domains with estimates through Kähler metric $i\partial\overline{\partial}\varphi$. We recall the following theorem which is essentially contained in [9].

Theorem 1.1. [9] *Let ϕ and ψ be plurisubharmonic functions of class C^2 on a bounded pseudoconvex domain Ω and let ϕ satisfy the condition*

$$i\partial\phi \wedge \overline{\partial}\phi \leqslant mi\partial\overline{\partial}\phi,$$

where m is a constant. Assume that g is a $\overline{\partial}$-closed $(0,1)$-form on Ω. Then there exists $u \in L^2(\Omega, \psi)$ such that $\overline{\partial}u = g$ and the estimate

$$\int |u|^2 e^{-\psi} dV \leqslant Cm \int |g|^2_{i\partial\overline{\partial}\phi} e^{-\psi} dV$$

holds, where C is an absolute constant.

Later on, the result of Donnelly and Fefferman has been proved by another method by Berndtsson with the constant $C = \frac{4}{\delta(1-\delta)^2}$, $0 < \delta < 1$ (see [3, Theorem 3.1]). Notice that all the above results have been proved under the hypothesis that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and for $\overline{\partial}$-closed (0,1)-forms. In 1991, in the paper "$\overline{\partial}$-problem on weakly q-convex domains" on Math. Ann., L.-H. Ho proved the existence of solutions of the $\overline{\partial}$-problem for $\overline{\partial}$-closed (p,r)-forms on weakly q-convex domains (see [10]) without weighted L^2-estimates of Hörmander type. By modifying techniques of Hörmander [11], Berndtsson [2] and Blocki [6, 7] in this paper we will study solutions of the $\overline{\partial}$-equation on q-pseudoconvex domains for $\overline{\partial}$-closed $(0,r)$-forms with weighted L^2-estimates of Hörmander type. Notice that the class of q-pseudoconvex domains is larger than the class of weakly q-convex domains introduced by L.-H. Ho in [10]. Now we outline the main contents and the organization of the paper.

Throughout this paper let Ω be a q-pseudoconvex domain in $\mathbb{C}^n$ and let $q \leqslant r \leqslant n$. We will study the equation

$$\overline{\partial}u = g, \tag{1}$$

where g is a $\overline{\partial}$-closed $(0,r)$-form in Ω.

Assume that there are a weight function $\varphi \in \mathcal{C}^2(\Omega)$ and a nonnegative function $H \in L^1(\Omega, \mathrm{loc})$ satisfying

$$\sum_{|J|=r}{}' \sum_{|L|=r}{}' \alpha_J \overline{\alpha}_L \det(\varphi_{J,\overline{L}}) \leqslant H \sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \varphi_{j\overline{k}} \alpha_{jK} \overline{\alpha}_{kK} \tag{2}$$

for all $(0,r)$-forms $\alpha = \sum_{|J|=r}{}' \alpha_J d\overline{z}_J$, where $(\varphi_{J,\overline{L}}) = (\varphi_{j,\overline{l}})_{j\in J, l\in L}$. Note that if $r = 1$ and we take $H = 1$ then condition (2) is obvious. The first result of the paper is the following.

Theorem 1.2. *Let Ω be a q-pseudoconvex domain in $\mathbb{C}^n$ and let φ be a strictly $\mathcal{C}^2$ plurisubharmonic function in Ω satisfying condition* (2) *and $-e^{-\varphi}$ a q-subharmonic function. Assume that $\delta \in (0,1)$ and ψ is a q-subharmonic function in Ω. Then for any $\overline{\partial}$-closed $(0,r)$-form g in Ω, there is a solution, u, to equation* (1) *such that*

$$\int_\Omega |u|^2 e^{-\psi+\delta\varphi} dV \leqslant \frac{1}{\delta(1-\sqrt{\delta})^2} \int_\Omega H|g|^2_{i\partial\overline{\partial}\varphi} e^{-\psi+\delta\varphi} dV.$$

Here $|.|_{i\partial\overline{\partial}\varphi}$ denotes the norm in the Kähler metric with Kähler form $i\partial\overline{\partial}\varphi$.

Next we obtain the following result which is a slight extension of a result in [4] (see [4, Theorem 4]) for the case Ω is a q-pseudoconvex domain and φ is a plurisubharmonic function on Ω. Namely we prove the following.

Theorem 1.3. *Let Ω be a q-pseudoconvex domain in $\mathbb{C}^n$ and let φ be a plurisubharmonic function on Ω. Assume that $\Theta = i \sum_{j,k=1}^n \Theta_{j,\overline{k}} dz_j \wedge d\overline{z}_k$ is a positive definite hermitian $(1,1)$-form with $\Theta_{j,\overline{k}}$ continuous on Ω and ω is a positive C^2-function satisfying*

$$i\partial\overline{\partial}\omega \leqslant \omega(i\partial\overline{\partial}\varphi - \Theta) \tag{3}$$

in the sense of currents. Then there exists a solution, u, of (1) *such that*

$$\int_\Omega |u|^2 e^{-\varphi}\omega dV \leqslant \frac{1}{r^2} \sum_{|K|=r-1}{}' \sum_{j,k=1}^n \int_\Omega \Theta^{j,\overline{k}} g_{jK}\overline{g}_{kK} e^{-\varphi}\omega dV.$$

The paper is organized as follows. In Sec. 2 we recall the notions of q-subharmonic functions and q-pseudoconvex domains used in the paper and list some of their basic properties. For details of results concerning with q-subharmonic functions and q-pseudoconvex domains we refer the reader to the papers of Ahn and Dieu [1] and [8]. Sec. 3 is devoted to establish the norm formula of $(0,r)$-forms in the Kähler metric induced by a positive definite hermitian $(1,1)$-form Θ. Moreover, we prove some auxiliary results which will be used for proofs of the main results of this paper in Sec. 4. together with some corollaries from these theorems.

2. q-pseudoconvex domains in $\mathbb{C}^n$

In this section we recall the notions of q-subharmonic functions introduced and investigated by L.-H. Ho [10] and q-pseudoconvex domains in $\mathbb{C}^n$ introduced by

Ahn and Dieu [1] recently, where $1 \leqslant q \leqslant n$. First we assume that the reader is familiar with plurisubharmonic functions. For details concerning with these functions we refer the reader to the monograph of Klimek [12]. Now we come back the definition of q-subharmonic functions. Note that in the following definition of q-subharmonic functions we do not assume that they are in $C^2(\Omega)$ as in [10]. It seems that this is a slight extension of the definition of q-subharmonic functions introduced by L.- H. Ho.

Definition 2.1. Let Ω be an open set in $\mathbb{C}^n$. The function φ defined in Ω with values in $[-\infty; +\infty)$ is called q-subharmonic if it is upper semicontinuous and

$$\int\limits_{\Omega} \varphi \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} (\alpha_{jK} \overline{\alpha}_{kK}) dV \geqslant 0 \tag{4}$$

for every $\alpha = \sum\limits_{|J|=q}{}' \alpha_J d\overline{z}_J \in \mathcal{D}_{(0,q)}(\Omega)$. Here $'$ means that the summation is over increasing indices and $\alpha_{jK} = \varepsilon^J_{jK} \alpha_J$, where

$$\varepsilon^J_{jK} = \begin{cases} \text{the sign of the permutation taking } \{j\} \cup K \text{ to } J, & \text{if } \{j\} \cup K = J, \\ 0, & \text{if } \{j\} \cup K \neq J. \end{cases}$$

The function φ is called strictly q-subharmonic if it is q-subharmonic and satisfies (4) with strictly inequality for all $\alpha \neq 0$. If $q = 1$ then 1-subharmonic exactly is plurisubharmonic.

We will denote the set of all such functions by q- $SH(\Omega)$. Note that in the case $\varphi \in \mathcal{C}^2(\Omega)$ condition (4) is equivalent to

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \alpha_{jK} \overline{\alpha}_{kK} \geqslant 0$$

for every $(0, q)$-form $\alpha = \sum\limits_{|J|=q}{}' \alpha_J d\overline{z}_J$. That is the definition of q-subharmonic functions introduced by L.-H. Ho [10].

We list the basic properties of q-subharmonic functions which the reader can find from Proposition 1.2 in [1].

Proposition 2.2. *Let Ω be an open set of $\mathbb{C}^n$ and $1 \leqslant q \leqslant n$. Then the following hold:*

(a) *If ψ is q-subharmonic in Ω, then ψ is subharmonic in Ω.*

(b) *If ψ is q-subharmonic, then ψ is also r-subharmonic for all $q \leqslant r \leqslant n$.*

(c) *If ψ is q-subharmonic in Ω, then $\psi * \varrho_\varepsilon$ is smooth q-subharmonic in Ω_ε, where $\Omega_\varepsilon = \{z \in \Omega : d(z, \partial\Omega) > \varepsilon\}$. Moreover, $\psi * \varrho_\varepsilon \searrow \psi$ when $\varepsilon \longrightarrow 0$, where $\varrho_\varepsilon = \varrho(z/\varepsilon)/|\varepsilon|^{2n}$, ϱ is a nonnegative smooth function in $\mathbb{C}^n$ vanishing outside the unit ball and satisfying $\int\limits_{\mathbb{C}^n} \varrho dV = 1$.*

(d) *If χ is a convex increasing function and ψ is q-subharmonic in Ω, then $\chi \circ \psi$ is q-subharmonic in Ω.*

Now the following comes from [1].

Definition 2.3. An open set $\Omega \subset \mathbb{C}^n$ is called q-pseudoconvex if it admits a continuous q-subharmonic exhaustion function on Ω. Here a function φ is a q-subharmonic exhaustion function on Ω if it is q-subharmonic and for all $c \in \mathbb{R}$ the set $\Omega_c = \{\varphi < c\} \Subset \Omega$.

We have some following remarks on q-pseudoconvex domains.

Remark 2.4. (a) If Ω is q-pseudoconvex in $\mathbb{C}^n$ then Ω is also r-pseudoconvex for all $q \leqslant r \leqslant n$.

(b) Assume that Ω is a q-pseudoconvex domain in $\mathbb{C}^n$. By using arguments as in [11, Theorem 2.6.11] we can find an exhaustion function $s \in \mathcal{C}^\infty(\Omega)$ which is strictly q-subharmonic on Ω.

3. Norm Formula $|.|_\Theta$ for $(0, r)$-forms and Some Auxiliary Results

Let $\Theta = i \sum\limits_{j,k=1}^{n} \Theta_{j,\overline{k}} dz_j \wedge d\overline{z}_k$ be a positive definite hermitian $(1,1)$-form. In this section we will establish the norm formula $|.|_\Theta$ for $(0,r)$-forms. First note that if $\beta(z) = \sum\limits_{j=1}^{n} \beta_j(z) dz_j$ is a $(1,0)$-form then

$$|\beta|^2_\Theta(z) = \sum_{j,k=1}^{n} \Theta^{j,\overline{k}}(z)\beta_j(z)\overline{\beta}_k(z),$$

where $(\Theta^{j,\overline{k}})$ is the inverse matrix of the matrix $(\Theta_{j,\overline{k}})$. Moreover, assume that $f = \sum\limits_{|J|=r}{}' f_J \overline{\omega}^J$, $\omega^J = \omega^{j_1} \wedge \cdots \wedge \omega^{j_r}$, $\omega^j = \sum\limits_{h=1}^{n} c_{hj} dz_h$ are $(1,0)$-forms satisfying

$$\langle \omega^j, \omega^k \rangle_\Theta = \sum_{h,l=1}^{n} \Theta^{h,\overline{l}} c_{hj} \overline{c}_{lk} = \delta_{jk},$$

where $\delta_{j,k}$ is the Kronecker symbol. Then

$$|f|^2_\Theta = \langle f, f \rangle_\Theta = \sum_{|J|=r}{}' |f_J|^2.$$

(See [11, p. 119]).

Let $\lambda_1(z), \lambda_2(z), \ldots, \lambda_n(z)$ be n eigenvalues of the matrix $\left(\Theta_{j,\overline{k}}\right)$. Then $\dfrac{1}{\lambda_1(z)}, \dfrac{1}{\lambda_2(z)}, \ldots, \dfrac{1}{\lambda_n(z)}$ are also n eigenvalues of the inverse matrix $\left(\Theta^{j,\overline{k}}\right)$. Let C be the matrix of unitary change of coordinates such that $\overline{C}^t\left(\Theta_{j,\overline{k}}\right)C$ is the diagonal matrix. We set

$$\omega^j(z) = \sqrt{\lambda_j(z)}\sum_{h=1}^{n} c_{hj}(z)dz_h.$$

It is clear that $\{\omega^j\}$ is an orthogonal basis for the Kähler metric induced by Θ. We have

$$d\overline{z}_j = \sum_{h=1}^{n} \frac{c_{jh}}{\sqrt{\lambda_h}}\overline{\omega}^h.$$

So for all $|J| = r$ it follows that

$$\begin{aligned}
d\overline{z}_J &= d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_r} \\
&= \sum_{h_1,\ldots,h_r=1}^{n} \left(\prod_{k=1}^{r} \frac{c_{j_k h_k}}{\sqrt{\lambda_{h_k}}}\right)\overline{\omega}^{h_1} \wedge \ldots \wedge \overline{\omega}^{h_r} \\
&= \sum_{h_1,\ldots,h_r=1}^{n} \frac{1}{\sqrt{\lambda_H}}\left(\prod_{k=1}^{r} c_{j_k h_k}\right)\overline{\omega}^{h_1} \wedge \ldots \wedge \overline{\omega}^{h_r},
\end{aligned}$$

where $\lambda_H = \prod\limits_{k=1}^{r} \lambda_{h_k}$. Thus, we have

$$\begin{aligned}
f &= {\sum_{|J|=r}}' f_J d\overline{z}_J \\
&= {\sum_{|J|=r}}' f_J \sum_{h_1,\ldots,h_r=1}^{n} \left(\prod_{k=1}^{r} \frac{c_{j_k h_k}}{\sqrt{\lambda_{h_k}}}\right)\overline{\omega}^{h_1} \wedge \ldots \wedge \overline{\omega}^{h_r} \\
&= \sum_{h_1,\ldots,h_r=1}^{n} {\sum_{|J|=r}}' f_J \left(\prod_{k=1}^{r} \frac{c_{j_k h_k}}{\sqrt{\lambda_{h_k}}}\right)\overline{\omega}^{h_1} \wedge \ldots \wedge \overline{\omega}^{h_r} \\
&= {\sum_{|H|=r}}' \left[\sum_{h_1,\ldots,h_r\in H} \varepsilon_H^{h_1,\ldots,h_r} {\sum_{|J|=r}}' f_J \left(\prod_{k=1}^{r} \frac{c_{j_k h_k}}{\sqrt{\lambda_{h_k}}}\right)\right]\overline{\omega}^H,
\end{aligned}$$

where $\varepsilon_H^{h_1,\ldots,h_r}$ is the sign of the permutation taking $\{h_1, \ldots, h_r\}$ to H. Hence

$$|f|_\Theta^2 = {\sum_{|H|=r}}' \frac{1}{\lambda_H}\left|\sum_{h_1,\ldots,h_r\in H} \varepsilon_H^{h_1,\ldots,h_r} {\sum_{|J|=r}}' f_J \left(\prod_{k=1}^{r} c_{j_k h_k}\right)\right|^2$$

$$= \sum_{|H|=r}{}' \frac{1}{\lambda_H} \sum_{|J|=r}{}' \sum_{|L|=r}{}' f_J \overline{f}_L \left(\sum_{h_1,\dots,h_r \in H} \varepsilon_H^{h_1,\dots,h_r} \prod_{k=1}^{r} c_{j_k h_k} \right) \times$$
$$\times \left(\sum_{h_1',\dots,h_r' \in H} \varepsilon_H^{h_1',\dots,h_r'} \prod_{k=1}^{r} \overline{c}_{l_k h_k'} \right)$$
$$= \sum_{|H|=r}{}' \frac{1}{\lambda_H} \sum_{|J|=r}{}' \sum_{|L|=r}{}' f_J \overline{f}_L \left(\sum_{h_1,\dots,h_r \in H} \sum_{h_1',\dots,h_r' \in H} \varepsilon_{h_1,\dots,h_r}^{h_1',\dots,h_r'} \prod_{k=1}^{r} c_{j_k h_k} \overline{c}_{l_k h_k'} \right)$$
$$= \sum_{|J|=r}{}' \sum_{|L|=r}{}' f_J \overline{f}_L \left(\sum_{h_1,\dots,h_r=1}^{n} \sum_{h_1',\dots,h_r'=1}^{n} \times \right.$$
$$\left. \times\, \varepsilon_{h_1,\dots,h_r}^{h_1',\dots,h_r'} \frac{1}{\sqrt{\lambda_{h_1,\dots,h_r}}} \frac{1}{\sqrt{\lambda_{h_1',\dots,h_r'}}} \prod_{k=1}^{r} c_{j_k h_k} \overline{c}_{l_k h_k'} \right)$$
$$= \sum_{|J|=r}{}' \sum_{|L|=r}{}' f_J \overline{f}_L \left(\sum_{\sigma \in S_r} \operatorname{sgn} \sigma \sum_{h_1,\dots,h_r=1}^{n} \frac{1}{\lambda_{h_1,\dots,h_r}} \prod_{k=1}^{r} c_{j_k h_k} \prod_{k'=1}^{r} \overline{c}_{l_{\overline{k}} h_{\sigma(k')}} \right)$$
$$= \sum_{|J|=r}{}' \sum_{|L|=r}{}' f_J \overline{f}_L \left(\sum_{\sigma \in S_r} \operatorname{sgn} \sigma \sum_{h_1,\dots,h_r=1}^{n} \frac{1}{\lambda_{h_1,\dots,h_r}} \prod_{k=1}^{r} c_{j_k h_k} \overline{c}_{l_{\sigma^{-1}(k)} h_k} \right)$$
$$= \sum_{|J|=r}{}' \sum_{|L|=r}{}' f_J \overline{f}_L \det \left(\Theta^{J,\overline{L}} \right).$$

Therefore, we obtain the following.

Proposition 3.1. *Let* $\Theta = i \sum_{j,k=1}^{n} \Theta_{j,\overline{k}} dz_j \wedge d\overline{z}_k$ *be a positive definite hermitian* $(1,1)$*-form. Then for any* $(0,r)$*-forms* $f = \sum_{|J|=r}{}' f_J d\overline{z}_J$ *the following norm formula holds*

$$|f|_\Theta^2 = \sum_{|J|=r}{}' \sum_{|L|=r}{}' f_J \overline{f}_L \det \left(\Theta^{J,\overline{L}} \right),$$

where

$$\left(\Theta^{J,\overline{L}} \right) = \left(\Theta^{j,\overline{l}} \right)_{j \in J, l \in L}.$$

Next, we establish a general Cauchy-Schwarz inequality in the following form.

Proposition 3.2. *Let* $\Theta = i \sum_{j,k=1}^{n} \Theta_{j,\overline{k}} dz_j \wedge d\overline{z}_k$ *be a positive definite hermitian* $(1,1)$*-form and* α, β *be two* $(0,r)$*-forms. Then*

$$|\alpha.\overline{\beta}|^2 \leqslant \sum_{|J|=r}{}' \sum_{|L|=r}{}' \alpha_J \overline{\alpha}_L \det \left(\Theta_{J,\overline{L}} \right) . |\beta|_\Theta^2,$$

where

$$\left(\Theta_{J,\overline{L}}\right) = \left(\Theta_{j,\overline{l}}\right)_{j\in J, l\in L}, \qquad \alpha = \sum_{|J|=r} \alpha_J d\overline{z}_J, \qquad \beta = \sum_{|L|=r} \beta_L d\overline{z}_L.$$

Proof. Let C be the matrix of unitary change of coordinates such that $\overline{C}^t \left(\Theta_{j,\overline{k}}\right) C$ is the diagonal matrix. Calculating as in proof of Proposition 3.1, we have

$$|\beta|^2_\Theta = \sum_{|H|=r}{}' \frac{1}{\lambda_H} \left| \sum_{h_1,\dots,h_r\in H} \varepsilon_H^{h_1,\dots,h_r} \sum_{|J|=r}{}' \beta_J \left(\prod_{k=1}^r c_{j_k h_k}\right)\right|^2$$

and

$$\sum_{|J|=r}{}' \sum_{|L|=r}{}' \alpha_J \overline{\alpha}_L \det\left(\Theta_{J,\overline{L}}\right)$$
$$= \sum_{|H|=r}{}' \lambda_H \left| \sum_{h_1,\dots,h_r\in H} \varepsilon_H^{h_1,\dots,h_r} \sum_{|J|=r}{}' \alpha_J \left(\prod_{k=1}^r c_{j_k h_k}\right)\right|^2.$$

Hence

$$\sum_{|J|=r}{}' \sum_{|L|=r}{}' \alpha_J \overline{\alpha}_L \det\left(\Theta_{J,\overline{L}}\right) . |\beta|^2_\Theta$$
$$\geqslant \left| \sum_{|H|=r}{}' \sum_{|J|=r}{}' \sum_{|L|=r}{}' \alpha_J \overline{\beta}_L \left(\sum_{h_1,\dots,h_r\in H} \sum_{h'_1,\dots,h'_r\in H} \varepsilon_{h_1,\dots,h_r}^{h'_1,\dots,h'_r} \prod_{k=1}^r c_{j_k h_k} \overline{c}_{l_k h'_k}\right)\right|^2$$
$$= \Big| \sum_{|J|=r}{}' \alpha_J \overline{\beta}_J \Big|^2 = \left|\alpha.\overline{\beta}\right|^2$$

and the desired conclusion follows. ■

Now we study solutions of the $\overline{\partial}$-problem on q-pseudoconvex domains with weighted L^2-estimates of Hörmander type. Techniques which we use here come from [6, 7, 11].

Let Ω be a q-pseudoconvex domain in $\mathbb{C}^n$ and let φ be a $\mathcal{C}^2$ q-subharmonic function in Ω such that there is a nonnegative function $h \in L^1(\Omega, \mathrm{loc})$ satisfying

$$\Big| \sum_{|J|=r}{}' g_J \overline{\alpha}_J \Big|^2 \leqslant h \sum_{|K|=r-1}{}' \sum_{j,k=1}^n \varphi_{j\overline{k}} \alpha_{jK} \overline{\alpha}_{kK} \tag{5}$$

for all $(0,r)$-forms $\alpha = \sum_{|J|=r}{}' \alpha_J d\overline{z}_J$.

The following result is a form of Theorem A5.1 in [6] for q-pseudoconvex domains and $(0,r)$-forms.

Proposition 3.3. *Let Ω be a q-pseudoconvex domain in $\mathbb{C}^n$ and φ a $\mathcal{C}^2$ q-subharmonic function in Ω satisfying condition* (5)*. Assume that g is a $\overline{\partial}$-closed $(0,r)$-form on Ω. Then there is a $(0,r-1)$-form u to* (1) *satisfying the estimate*

$$\int\limits_{\Omega} |u|^2 e^{-\varphi} dV \leqslant \int\limits_{\Omega} h e^{-\varphi} dV. \tag{6}$$

Proof. If the right-hand side of (6) is infinite then the theorem is clear. Hence we assume that it is finite and even equal to 1.

Since Ω is a q-pseudoconvex domain in $\mathbb{C}^n$ so there exists a smooth strictly q-subharmornic function s in Ω such that $K_a = \{z \in \Omega : s(z) < a\} \Subset \Omega$. It is clear that s is strictly r-subharmornic with $q \leqslant r \leqslant n$. We fix $a > 0$ and choose $\eta_v \in \mathcal{D}(\Omega),\ v = 1, 2, \dots$ such that $0 \leqslant \eta_v \leqslant 1$ and $K_{a+1} \subset \{\eta_v = 1\} \uparrow \Omega$ as $v \uparrow \infty$. Let $\psi \in \mathcal{C}^\infty(\Omega)$ vanish in K_a and satisfy $|\partial\eta_v|^2 \leqslant e^{\psi}$ for every $v = 1, 2, \dots$ Let $\chi \in \mathcal{C}^\infty(\Omega)$ be a convex increasing function such that $\chi = 0$ on $(-\infty, a)$, $\chi \circ s \geqslant 2\psi$ and

$$\chi' \circ s \sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 s}{\partial z_j \partial \overline{z}_k} \alpha_{jK} \overline{\alpha}_{kK} \geqslant (1+a)|\partial\psi|^2 |\alpha|^2$$

for all $(0,r)$-forms $\alpha = \sum\limits_{|J|=r} \alpha_J d\overline{z}_J$. From here by repeating the proof of Theorem A5.1 in [6] we finish the proof of Proposition 3.3. ∎

We also will discuss a generalization of Theorem 3.2 in [7] for q-pseudoconvex domains and $(0,r)$-forms.

Proposition 3.4. *Let Ω and φ be as in Proposition 3.3. Let $\delta \in (0,1)$ and assume that $-e^{-\varphi/\delta}$ is a q-subharmonic function in Ω. Assume that g is a $\overline{\partial}$-closed $(0,r)$-form on Ω and $\psi \in PSH(\Omega)$. Then there is a $(0,r-1)$-form u to* (1) *satisfying*

$$\int\limits_{\Omega} |u|^2 e^{\varphi-\psi} dV \leqslant \frac{1}{(1-\sqrt{\delta})^2} \int\limits_{\Omega} h e^{\varphi-\psi} dV.$$

Proof. First we assume that φ and ψ are $\mathcal{C}^2$-smooth up to the boundary. Note that since $-e^{-\varphi/\delta}$ is a q-subharmonic function so $-e^{-\varphi/\delta}$ is also r-subharmonic, $q \leqslant r \leqslant n$. Hence, we have

$$\sum_{|K|=r-1}{}' \Big|\sum_{j=1}^{n} \varphi_j \alpha_{jK}\Big|^2 \leqslant \delta \sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \varphi_{j,\overline{k}} \alpha_{jK} \overline{\alpha}_{kK}.$$

Now by using techniques in the proof of Theorem 3.2 in [7] we obtain the proof of Proposition 3.4 for the case φ and ψ are $\mathcal{C}^2$-smooth up to the boundary.

For the general case, we carry out the standard exhaustion procedure as in [11] (see [11, Theorem 4.4.2]). Since Ω is a q-pseudoconvex domain then there exists a strictly q-subharmonic and smooth exhaustion function s. The sublevel sets $K_a = \{s < a\}$ of Ω are smoothly, bounded, q-pseudoconvex for almost every a. We fix such a. Then $\psi_\varepsilon = \psi * \varrho_\varepsilon \in \mathcal{C}^\infty(K_a)$, for all ε small enough. By the beginning of this proof we can find u_ε such that $\overline{\partial} u_\varepsilon = g$ in K_a and

$$\int\limits_{K_a} |u_\varepsilon|^2 e^{\varphi-\psi_\varepsilon} dV \leqslant \frac{1}{(1-\sqrt{\delta})^2} \int\limits_{K_a} h e^{\varphi-\psi_\varepsilon} dV \leqslant \frac{1}{(1-\sqrt{\delta})^2} \int\limits_{\Omega} h e^{\varphi-\psi} dV.$$

Since ψ_ε decreases with ε this shows that the L^2 norm of u_ε over K_a is bounded for every fixed a. We can choose a sequence $\varepsilon_j \to 0$ such that u_{ε_j} converges weakly in K_a for every a to a limit u in $L^2_{(0,r-1)}(\Omega, \text{loc})$ and the desired conclusion follows. ■

4. Weighted L^2-Estimates for the $\overline{\partial}$-equation on q-pseudoconvex domains

In this section we give the proof of Theorems 1.2 and 1.3 and some corollaries from them.

Proof of Theorem 1.2. Set $\widetilde{\varphi} = \delta\varphi$. Applying Proposition 3.2, we get

$$\begin{aligned}
|g.\overline{\alpha}|^2 = |\alpha.\overline{g}|^2 &\leqslant \sum_{|J|=r}{}' \sum_{|L|=r}{}' \alpha_J \overline{\alpha}_L \det\left(\varphi_{J,\overline{L}}\right).|g|^2_{i\partial\overline{\partial}\varphi} \\
&\leqslant H|g|^2_{i\partial\overline{\partial}\varphi} \sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \varphi_{j\overline{k}} \alpha_{jK} \overline{\alpha_{kK}} \\
&\leqslant \frac{1}{\delta} H|g|^2_{i\partial\overline{\partial}\varphi} \sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \widetilde{\varphi}_{j\overline{k}} \alpha_{jK} \overline{\alpha_{kK}}.
\end{aligned}$$

It is easy to see that $H|g|^2_{i\partial\overline{\partial}\varphi}$ is in $L^1(\Omega, \text{loc})$ then Proposition 3.4 implies the existence of a solution, u, to (1) satisfying

$$\begin{aligned}
\int\limits_{\Omega} |u|^2 e^{\widetilde{\varphi}-\psi} dV &\leqslant \frac{1}{(1-\sqrt{\delta})^2} \int\limits_{\Omega} \frac{1}{\delta} H|g|^2_{i\partial\overline{\partial}\varphi} e^{\widetilde{\varphi}-\psi} dV \\
&= \frac{1}{\delta(1-\sqrt{\delta})^2} \int\limits_{\Omega} H|g|^2_{i\partial\overline{\partial}\varphi} e^{\widetilde{\varphi}-\psi} dV.
\end{aligned}$$

Therefore the proof is complete. ■

Corollary 4.1. *Let Ω be a q-pseudoconvex domain in $\mathbb{C}^n$ and let φ be a strictly $\mathcal{C}^2$-plurisubharmonic function in Ω satisfying condition* (2)*. Then for any $\overline{\partial}$-closed $(0,r)$-form g in Ω, there is a solution u to equation* (1) *such that*

$$\int\limits_{\Omega} |u|^2 e^{-\varphi} dV \leqslant \int\limits_{\Omega} H|g|^2_{i\partial\overline{\partial}\varphi} e^{-\varphi} dV.$$

Proof. For any $(0, r)$-form α, using Proposition 3.2, we get

$$|g.\overline{\alpha}|^2 \leqslant \sum_{|J|=r}{}' \sum_{|L|=r}{}' \alpha_J \overline{\alpha}_L \det\left(\varphi_{J,\overline{L}}\right).|g|^2_{i\partial\overline{\partial}\varphi}.$$

Combining this with (2), we arrive at

$$|g.\overline{\alpha}|^2 \leqslant H|g|^2_{i\partial\overline{\partial}\varphi} \sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \varphi_{j\overline{k}} \alpha_{jK} \overline{\alpha_{kK}}.$$

The desired conclusion follows from Proposition 3.3 and the corollary is completely proved. ■

The next result is due to Ahn and Dieu (see [1, Theorem 1.5]).

Corollary 4.2. *Let Ω be a q-pseudoconvex domain in $\mathbb{C}^n$ and let ψ be a q-subharmonic function in Ω. Let $\varphi \in \mathcal{C}^2(\Omega)$ be a strictly plurisubharmonic function and $-e^{-\varphi}$ be q- subharmonic. Assume that $\delta \in (0, 1)$. Then for every $\overline{\partial}$-closed $(0, r)$-form g there is a solution u of equation* (1) *such that*

$$\int\limits_{\Omega} |u|^2 e^{-\psi+\delta\varphi} dV \leqslant \frac{1}{\delta(1-\sqrt{\delta})^2}.\frac{1}{r^2} \sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \int\limits_{\Omega} \varphi^{j,\overline{k}} g_{jK} \overline{g}_{kK} e^{-\psi+\delta\varphi} dV.$$

Proof. We set $\widetilde{\varphi} = \delta\varphi$. Since

$$\begin{aligned} &\Big|\sum_{|J|=r}{}' g_J.\overline{\alpha}_J\Big|^2 \\ &= \frac{1}{r^2}\Big|\sum_{|K|=r-1}{}' \sum_{j=1}^{n} g_{jK}.\overline{\alpha}_{jK}\Big|^2 \\ &\leqslant \frac{1}{r^2}.\frac{1}{\delta}\left(\sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \varphi^{j\overline{k}} g_{jK}\overline{g_{kK}}\right)\left(\sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \widetilde{\varphi}_{j\overline{k}} \alpha_{jK}\overline{\alpha_{kK}}\right). \end{aligned}$$

Applying Proposition 3.4 the desired conclusion follows. ■

The following is a slight extension of Theorem 4 in [4].

Proof of Theorem 1.3. First we assume that φ is smooth. Put $\psi = -\ln\omega$. Then $\omega = e^{-\psi}$ and

$$i\partial\overline{\partial}\omega = e^{-\psi}(i\partial\psi \wedge \overline{\partial}\psi - i\partial\overline{\partial}\psi).$$

Hence (3) is equivalent to

$$i\partial\psi\wedge\overline{\partial}\psi+\Theta\leqslant i\partial\overline{\partial}(\varphi+\psi).$$

It follows that $\varphi+\psi$ is a strictly plurisubharmonic function in Ω and

$$\Theta\leqslant i\partial\overline{\partial}(\varphi+\psi).$$

Thus, we get

$$\begin{aligned}
&\Big|\sum_{|J|=r}{}' g_J\overline{\alpha}_J\Big|^2\\
&=\frac{1}{r^2}\Big|\sum_{|K|=r-1}{}'\sum_{j=1}^{n} g_{jK}.\overline{\alpha}_{jK}\Big|^2\\
&\leqslant\frac{1}{r^2}\left(\sum_{|K|=r-1}{}'\sum_{j,k=1}^{n}\Theta^{j\overline{k}}g_{jK}\overline{g_{kK}}\right)\left(\sum_{|K|=r-1}{}'\sum_{j,k=1}^{n}\Theta_{j\overline{k}}\alpha_{jK}\overline{\alpha_{kK}}\right)\\
&\leqslant\frac{1}{r^2}\left(\sum_{|K|=r-1}{}'\sum_{j,k=1}^{n}\Theta^{j\overline{k}}g_{jK}\overline{g_{kK}}\right)\left(\sum_{|K|=r-1}{}'\sum_{j,k=1}^{n}(\varphi+\psi)_{j\overline{k}}\alpha_{jK}\overline{\alpha_{kK}}\right).
\end{aligned}$$

Applying Proposition 3.3 we obtain the proof of the theorem in the case φ is smooth.

Now we prove the general case. Since Ω is a q-pseudoconvex domain, there exists a strictly q-subharmonic and smooth exhaustion function s. The sublevel sets $K_a=\{s<a\}$ of Ω are smoothly, bounded, q-pseudoconvex for almost every a. We fix such a. Put $\Theta_\varepsilon=\Theta*\varrho_\varepsilon$. We prove Θ_ε is a positive definite hermitian $(1,1)$-form on K_a when ε is small enough. Indeed, using the arguments as in [11] there exists $\chi\in \mathrm{C}^\infty(\Omega),\ \chi>0$ such that

$$\Theta\geqslant\chi i\partial\overline{\partial}|w|^2$$

on Ω. Then $\Theta\geqslant\chi_0 i\partial\overline{\partial}|w|^2$ on K_a, where χ_0 is a constant. We have

$$(\Theta-\Theta_\varepsilon)\leqslant \mathrm{C}\chi_1 i\partial\overline{\partial}|w|^2,$$

where χ_1,C are some constants. Hence $\Theta_\varepsilon\geqslant\Theta-\mathrm{C}\chi_1 i\partial\overline{\partial}|w|^2$. If we choose χ_1 small enough then it follows that

$$\Theta_\varepsilon\geqslant(\chi_0-\mathrm{C}\chi_1)i\partial\overline{\partial}|w|^2>0$$

on K_a. The desired conclusion follows. As above, we have

$$\Theta\leqslant i\partial\overline{\partial}(\varphi+\psi)$$

in the sense of currents. Thus $\Theta_\varepsilon\leqslant i\partial\overline{\partial}(\varphi+\psi)_\varepsilon$ on K_a when ε is small enough. By the result of the beginning of the proof it follows that there exists a solution

$u_{a,\varepsilon}$ of equation (1) satisfying

$$\int\limits_{K_a} |u_{a,\varepsilon}|^2 e^{-(\varphi+\psi)_\varepsilon} dV \leqslant \frac{1}{r^2} \sum_{|K|=r}{}' \sum_{j,k=1}^{n} \int\limits_{K_a} \Theta_\varepsilon^{j,\overline{k}} g_{j,K} \overline{g}_{k,K} e^{-(\varphi+\psi)_\varepsilon} dV$$
$$\leqslant \frac{1}{r^2} \sum_{|K|=r}{}' \sum_{j,k=1}^{n} \int\limits_{K_a} \Theta_\varepsilon^{j,\overline{k}} g_{j,K} \overline{g}_{k,K} e^{-(\varphi+\psi)} dV.$$

By applying arguments as in the proof of Theorem 4.4.2 in [11] we finish the proof of Theorem 1.3. ■

The following result is an extension of Lemma 4.4.1 in [11] for q-pseudoconvex domains and $(0,r)$-forms.

Corollary 4.3. *Assume that φ is a q-subharmonic function in Ω, where Ω is a q-pseudoconvex domain in $\mathbb{C}^n$, such that*

$$\int\limits_{\Omega} h|\alpha|^2 dV \leqslant \int\limits_{\Omega} \varphi \sum_{|K|=r-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} (\alpha_{jK} \overline{\alpha}_{kK}) dV \tag{7}$$

for every $(0,r)$-form $\alpha = \sum_{|J|=r}{}' \alpha_J d\overline{z}_J \in \mathcal{D}_{(0,r)}(\Omega)$, where h is a positive continuous function. Then for every $\overline{\partial}$-closed $(0,r)$-form g, there exists a solution, u, to equation (1) *such that*

$$\int\limits_{\Omega} |u|^2 e^{-\varphi} dV \leqslant \int\limits_{\Omega} \frac{|g|^2}{h} e^{-\varphi} dV. \tag{8}$$

Proof. We may assume that the right-hand side of (8) is finite and equal to 1. We first consider the case when φ is a smooth function. Repeating the proof of Theorem A5.1 in [6] the proof of the corollary follows.

For the general case we assume that φ is arbitrary q-subharmonic. Because Ω is q-pseudoconvex then there exists a strictly q-subharmonic and smooth exhaustion function s. The sublevel sets $K_a = \{s < a\} \Subset \Omega$ are smoothly, bounded, q-pseudoconvex for almost every a. Since h is continuous on $\overline{K}_a$ for every fixed a then h is uniformly continuous on K_a. Hence we have

$$\lim_{\varepsilon \to 0} \int\limits_{K_a} \frac{|g|^2}{h * \varrho_\varepsilon} e^{-\varphi} dV = \int\limits_{K_a} \frac{|g|^2}{h} e^{-\varphi} dV \leqslant 1. \tag{9}$$

Thus, for each $i = 1, 2, \ldots$ take $\varepsilon_i > 0$ small sufficiently such that $K_i + B(0, \varepsilon_i) \Subset \Omega$, $\varphi_{\varepsilon_i} := \varphi * \varrho_{\varepsilon_i} \in \mathcal{C}^\infty(\overline{\Omega}_{1/i})$ and

$$\int\limits_{K_i} \frac{|g|^2}{h * \varrho_{\varepsilon_i}} e^{-\varphi} dV < 1 + \frac{1}{i}.$$

We can choose ε_i such that the sequence $\{\varepsilon_i\} \downarrow 0$ as $i \uparrow \infty$. For every $w \in B(0, \varepsilon_i)$ and $\alpha \in \mathcal{D}_{(0,r)}(K_i)$ we have $\alpha(. + w) \in \mathcal{D}_{(0,r)}(\Omega)$. By the hypothesis (7) we get

$$\int_\Omega h(z)|\alpha(z+w)|^2 dV(z) \leqslant \int_\Omega \varphi(z) \sum_{|K|=r-1}{}' \sum_{j,k=1}^n \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \Big[\alpha_{jK}(z+w)\overline{\alpha}_{kK}(z+w)\Big] dV(z).$$

After a change of variables we can write

$$\int_{K_i} h(z-w)|\alpha(z)|^2 dV(z) \leqslant \int_{K_i} \varphi(z-w) \sum_{|K|=r-1}{}' \sum_{j,k=1}^n \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \Big[\alpha_{jK}(z)\overline{\alpha}_{kK}(z)\Big] dV(z)$$

for all $w \in B(0, \varepsilon_i)$ and $\alpha \in \mathcal{D}_{(0,r)}(K_i)$.

By multiplying by $\varrho_{\varepsilon_i}(w)$ and integrating with respect to $dV(w)$ we have

$$\int_{K_i} h * \varrho_{\varepsilon_i} |\alpha|^2 dV \leqslant \int_{K_i} \varphi * \varrho_{\varepsilon_i} \sum_{|K|=r-1}{}' \sum_{j,k=1}^n \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \left(\alpha_{jK}\overline{\alpha}_{kK}\right) dV.$$

Since K_i is also a q-pseudoconvex domain then using the results of the above part we can find $u_{\varepsilon_j} \in L^2_{(0,r-1)}(K_i, \mathrm{loc})$ such that $\overline{\partial} u_{\varepsilon_i} = g$ in K_i and

$$\int_{K_i} |u_{\varepsilon_i}|^2 e^{-\varphi_{\varepsilon_i}} dV \leqslant \int_{K_i} \frac{|g|^2}{h * \varrho_{\varepsilon_i}} e^{-\varphi_{\varepsilon_i}} dV \leqslant \int_{K_i} \frac{|g|^2}{h * \varrho_{\varepsilon_i}} e^{-\varphi} dV \leqslant 1 + \frac{1}{i}.$$

Now using arguments as at the end of the proof of Proposition 3.4 the desired conclusion follows. ■

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