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Random Fixed Points of Probabilistic Contractions and Applications to Random Equations*

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Abstract. In this paper, we present some necessary and sufficient conditions for the existence of random fixed points of probabilistic contractions and give some applications of these results to random equations.

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1. Introduction and Preliminaries

The theory of random fixed points is an important topic of the stochastic analysis and has been investigated by various authors (see e.g [1, 3, 4, 8, 9]), in recent years. In these researches, random operators are considered in each sample path (that is $T(\omega, .)$ for each fixed $\omega \in \Omega$) and any assumption about random operators is imposed on each sample path. In this paper, we approach random fixed point problems by a viewpoint of the probability, random operators are considered globally. In Sec. 2, we give some necessary and sufficient conditions for the

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existence of random fixed point of a probabilistic contraction. Sec. 3 presents some applications of random fixed point theorems to random equations.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a separable Banach space. We denote by $\mathcal{B}(X)$ the Borel σ-algebra of X, by $\mathcal{F} \times \mathcal{B}(X)$ the σ-algebra on $\Omega \times X$, by $L_0^X(\Omega)$ the set of X-valued random variables and by $L_p^X(\Omega)$ the set of X-valued random variables ξ such that $E\|\xi\|^p < +\infty$, where $p > 0$. Convergence in probability of a sequence of random variables $\{x_n\}$ to random variable x is denoted by $x_n \xrightarrow{\mathbb{P}} x$.

Definition 1.1. (a) A mapping $T : \Omega \times X \to X$ is said to be a random operator on X if for each $x \in X$, the mapping $T(., x)$ is an X-valued random variable, where $T(., x)$ denotes the mapping $\omega \mapsto T(\omega, x)$.
(b) The random operator $T : \Omega \times X \to X$ is said to be measurable if the mapping $T : \Omega \times X \to X$ is $\mathcal{F} \times \mathcal{B}(X)$-measurable.

Definition 1.2. Let T be a measurable random operator on X.
(a) T is said to be stochastically continuous if $T(\omega, x_n) \xrightarrow{\mathbb{P}} T(\omega, x)$ as $x_n \xrightarrow{\mathbb{P}} x$, where $x, x_n \in L_0^X(\Omega)$ $(n = 1, 2, ...)$.
(b) T is called a probabilistic q-contraction where $q \in (0; 1)$ if for any $x, y \in L_0^X(\Omega)$ we have

$$\mathbb{P}(\|T(\omega, x) - T(\omega, y)\| > q.t) \leq \mathbb{P}(\|x - y\| > t)$$

for any $t > 0$.

Definition 1.3. Let T be a random operator on X. A random variable $\xi(\omega) \in L_0^X(\Omega)$ is said to be a random fixed point of T if $T(\omega, \xi(\omega)) = \xi(\omega)$ a.s.

2. Random Fixed Points of Probabilistic Contractions

Proposition 2.1. *Let T be a probabilistic q-contraction on X. If T has a random fixed point ξ then it has a unique random fixed point and $T^n(\omega, x) \xrightarrow{\mathbb{P}} \xi$ for any $x \in L_0^X(\Omega)$, where $T^0(w, x) = x$, $T^n(w, x) = T(w, T^{n-1}(w, x))$ for any $n \geq 1$.*

Proof. For each $x \in L_0^X(\Omega)$, let $x_n = T^n(\omega, x)$ $(n = 0, 1, 2, ...)$. For any $t > 0$ we have

$$\begin{aligned}\mathbb{P}(\|x_n - \xi\| > t) &= \mathbb{P}(\|T^n(\omega, x) - T^n(\omega, \xi)\| > t) \\ &\leq \mathbb{P}(\|T^{n-1}(\omega, x) - T^{n-1}(\omega, \xi)\| > t/q) \\ &\leq ... \leq \mathbb{P}(\|x - \xi\| > t/q^n).\end{aligned}$$

Let $n \to \infty$ we have $\lim\limits_{n\to\infty} \mathbb{P}(\|x_n - \xi\| > t) \leq \lim\limits_{n\to\infty} \mathbb{P}(\|x - \xi\| > t/q^n) = 0$. Thus, $x_n \xrightarrow{\mathbb{P}} \xi$.

We suppose that T has two different random fixed points denoted by ξ_1 and ξ_2. By the above assertion, we have $x_n \xrightarrow{\mathbb{P}} \xi_1$ and $x_n \xrightarrow{\mathbb{P}} \xi_2$ which is impossible. Thus, T has a unique random fixed point. ■

Recall a subset M of $L_0^X(\Omega)$ is said to be probabilistic bounded if

$$\lim_{t\to\infty} \sup_{u\in M} \mathbb{P}(\|u\| > t) = 0.$$

For each $x_0 \in L_0^X(\Omega)$, let $O_{(T,x_0)} = \{T^n(\omega, x_0) : n = 0, 1, 2, ...\}$.

Theorem 2.2. *Let T be a probabilistic q-contraction on X. Then T has a unique random fixed point if and only if there exists a random variable $x_0 \in L_0^X(\Omega)$ such that $O_{(T,x_0)}$ is probabilistic bounded. Moreover, $\{T^n(\omega, x)\}$ converges in probability to a random fixed point of T for any $x \in L_0^X(\Omega)$.*

Proof. We now suppose that T has a random fixed point denoted by ξ. Let $x_0 = \xi$ then $O_{(T,x_0)} = \{\xi\}$. Thus $O_{(T,x_0)}$ is probabilistic bounded.

Conversely, suppose that there exists $x_0 \in L_0^X(\Omega)$ such that $O_{(T,x_0)}$ is probabilistic bounded. Let $x_n = T^n(\omega, x_0)$ $(n = 0, 1, 2, ...)$. We now show that $\{x_n\}$ converges in probability. Indeed, for any $n, m \in \mathbb{N}$ and $t > 0$ we have

$$\begin{aligned}
\mathbb{P}(\|x_{n+m} - x_n\| > t) &= \mathbb{P}(\|T^{n+m}(\omega, x_0) - T^n(\omega, x_0)\| > t) \\
&\le \mathbb{P}(\|T^{n+m-1}(\omega, x_0) - T^{n-1}(\omega, x_0)\| > t/q) \\
&\le ... \le \mathbb{P}(\|T^m(\omega, x_0) - x_0\| > t/q^n) \\
&= \mathbb{P}(\|x_m - x_0\| > t/q^n) \\
&\le 2 \sup_{u\in O_{(T,x_0)}} \mathbb{P}(\|u\| > t/_{(2q^n)}).
\end{aligned}$$

Thus, let $n \to \infty$ we have $\lim_{n\to\infty} \mathbb{P}(\|x_{n+m} - x_n\| > t) = 0$ for any $m \in \mathbb{N}$, $t > 0$. This shows that $\{x_n\}$ is a Cauchy sequence in probability. Therefore, $\{x_n\}$ converges in probability to a random variable ξ. We will point out that ξ is a random fixed point of T. Indeed, for any $t > 0$ we have

$$\begin{aligned}
\mathbb{P}(\|T(\omega, \xi) - \xi\| > t) &\le \mathbb{P}(\|T(\omega, \xi) - T(\omega, x_n)\| > t/2) + \mathbb{P}(\|T(\omega, x_n) - \xi\| > t/2) \\
&\le \mathbb{P}(\|x_n - \xi\| > t/_{(2q)}) + \mathbb{P}(\|x_{n+1} - \xi\| > t/2)
\end{aligned}$$

for any $n \in \mathbb{N}$. Let $n \to \infty$ we have $\mathbb{P}(\|T(\omega, \xi) - \xi\| > t) = 0$, i.e. ξ is a random fixed point of T. The rest of the proof follows from Proposition 2.1. ■

Theorem 2.3. *Let T be a stochastically continuous random operator on X such that T^k is a probabilistic q-contraction for some $k \in \mathbb{N}$. Then T has a unique random fixed point if and only if there exists a random variable $x_0 \in L_0^X(\Omega)$ such that $O_{(T,x_0)}$ is probabilistic bounded. Moreover, $\{T^n(\omega, x)\}$ converges in probability to a random fixed point of T for any $x \in L_0^X(\Omega)$.*

Proof. Firstly, suppose that there exists $x_0 \in L_0^X(\Omega)$ such that $O_{(T,x_0)}$ is probabilistic bounded. By Theorem 2.2, T^k has a unique random fixed point denoted by ξ and the sequence $\{x_n\}$ defined by $x_n = T^{kn}(\omega, x_0)$ $(n = 1, 2, ...)$ converges in probability to ξ. By the continuity of T, $T(\omega, x_n)$ converges in probability to $T(\omega, \xi)$. We now show that ξ is also a random fixed point of T. Indeed, for any $t > 0$ we have

$$\begin{aligned}
&\mathbb{P}(\|T(\omega,\xi) - \xi\| > t) \\
&\leq \mathbb{P}(\|T(\omega,\xi) - T(\omega,x_n)\| > t/3) + \mathbb{P}(\|T(\omega,x_n) - x_n\| > t/3) \\
&\quad + \mathbb{P}(\|x_n - \xi\| > t/3) \\
&= \mathbb{P}(\|T(\omega,\xi) - T(\omega,x_n)\| > t/3) + \mathbb{P}(\|T(\omega,T^{kn}(\omega,x_0)) - T^{kn}(\omega,x_0)\| > t/3) \\
&\quad + \mathbb{P}(\|x_n - \xi\| > t/3) \\
&= \mathbb{P}(\|T(\omega,\xi) - T(\omega,x_n)\| > t/3) + \mathbb{P}(\|T^{kn}(\omega,T(\omega,x_0)) - T^{kn}(\omega,x_0)\| > t/3) \\
&\quad + \mathbb{P}(\|x_n - \xi\| > t/3) \\
&\leq \mathbb{P}(\|T(\omega,\xi) - T(\omega,x_n)\| > t/3) + \mathbb{P}(\|T(\omega,x_0) - x_0\| > t/_{(3q^n)}) \\
&\quad + \mathbb{P}(\|x_n - \xi\| > t/3).
\end{aligned}$$

Let $n \to \infty$, we have $\mathbb{P}(\|T(\omega,\xi) - \xi\| > t) = 0$, i.e. ξ is a random fixed point of T. To prove the uniqueness of ξ, we merely note that if T has more than one random fixed point then so does T^k, which is impossible, by Proposition 2.1, T^k is a probabilistic q-contraction.

Conversely, suppose that T has a random fixed point denoted by ξ. Let $x_0 = \xi$ then $O_{(T,x_0)} = \{\xi\}$. Thus $O_{(T,x_0)}$ is probabilistic bounded.

To finish the proof, we show that $T^n(\omega, x) \xrightarrow{\mathbb{P}} \xi$ for each fixed $x \in L_0^X(\Omega)$. For $n > k$ we have $n = mk + \ell$ where $m, \ell \in \mathbb{N}$ and $0 < \ell < k$. For $t > 0$,

$$\begin{aligned}
\mathbb{P}(\|T^n(\omega,x) - \xi\| > t) &= \mathbb{P}(\|T^n(\omega,x) - T^{mk}(\omega,\xi)\| > t) \\
&= \mathbb{P}(\|T^{mk}(\omega,T^\ell(\omega,x)) - T^{mk}(\omega,\xi)\| > t) \\
&\leq \mathbb{P}(\|T^\ell(\omega,x) - \xi\| > t/q^m) \\
&\leq \max_{0<\ell<k} \mathbb{P}(\|T^\ell(\omega,x) - \xi\| > t/q^m).
\end{aligned}$$

Let $n \to \infty$ we have $m \to \infty$. Thus, $\lim_{n\to\infty} \mathbb{P}(\|T^n(\omega,x) - \xi\| > t) = 0$, and we are done. ■

Lemma 2.4. *Let $\alpha > 0$ and $f(x)$ be a function defined in $(q;1)$ by $f(x) = \left(\frac{x}{x-q}\right)^\alpha . \frac{1}{1-x^\alpha}$. Then*

$$\min_{(q;1)} f(x) = \frac{1}{(1 - q^{\frac{\alpha}{1+\alpha}})^{1+\alpha}} \qquad as \;\; x = q^{\frac{1}{1+\alpha}}.$$

Proof. We have $\lim_{x\to 1} f(x) = \lim_{x\to q} f(x) = +\infty$ and $f'(x) = 0$ if and only if $x = q^{\frac{1}{1+\alpha}}$. By the continuity of $f(x)$, the proof is finished. ■

The following theorem gives another necessary and sufficient condition for the existence of random fixed point of a probabilistic q-contraction and estimates tail probability of the difference of each element of iteration sequence $\{x_n = T^n(\omega, x_0)\}$ and the random fixed point ξ of T.

Theorem 2.5. *Let T be a probabilistic q-contraction on X. Then T has a unique random fixed point if and only if there exist a random variable $x_0 \in L_0^X(\Omega)$ and $\alpha > 0$ such that*

$$\sup_{t>0} t^\alpha \mathbb{P}(\|T(\omega, x_0) - x_0\| > t) < +\infty. \tag{1}$$

Moreover, if (1) holds then $\{x_n = T^n(\omega, x_0)\}$ converges in probability to the random fixed point ξ of T and

$$\mathbb{P}(\|x_n - \xi\| > t) \leq \frac{M}{(1 - q^{\frac{\alpha}{1+\alpha}})^{1+\alpha}} . \frac{(q^\alpha)^n}{t^\alpha},$$

where $M = \sup_{t>0} t^\alpha \mathbb{P}(\|T(\omega, x_0) - x_0\| > t)$.

Proof. Firstly, suppose that $\sup_{t>0} t^\alpha \mathbb{P}(\|T(\omega, x_0) - x_0\| > t) < +\infty$ for some $x_0 \in L_0^X(\Omega)$ and $\alpha > 0$. We will show that $\{x_n\}$ is a Cauchy sequence in probability. For any $n \in \mathbb{N}$ and $t > 0$, by induction

$$\begin{aligned}\mathbb{P}(\|x_{n+1} - x_n\| > t) &= \mathbb{P}(\|T^{n+1}(\omega, x_0) - T^n(\omega, x_0)\| > t) \\ &\leq \mathbb{P}(\|T^n(\omega, x_0) - T^{n-1}(\omega, x_0)\| > t/q) \\ &\leq ... \leq \mathbb{P}(\|x_1 - x_0\| > t/q^n).\end{aligned}$$

For any $t > 0$, by (1) we have

$$\mathbb{P}(\|x_1 - x_0\| > t) = \mathbb{P}(\|T(\omega, x_0) - x_0\| > t) \leq \frac{M}{t^\alpha}.$$

Thus,

$$\mathbb{P}(\|x_{n+1} - x_n\| > t) \leq \frac{M(q^n)^\alpha}{t^\alpha}.$$

Let $r = \dfrac{p}{q}$, where $q < p < 1$. Then $r > 1$ and $(r - 1)\left(\frac{1}{r} + \frac{1}{r^2} + ... + \frac{1}{r^m}\right) = 1 - \frac{1}{r^m} < 1 \quad \forall m \in \mathbb{N}$.
Thus, for any $t > 0$ and $m, n \in \mathbb{N}$ we have

$$\begin{aligned}&\mathbb{P}(\|x_{n+m} - x_n\| > t) \\ &\leq \mathbb{P}(\|x_{n+m} - x_n\| > (1 - 1/r^m)t) \\ &\leq \mathbb{P}(\|x_{n+m} - x_{n+m-1}\| > t(r-1)/_{r^m}) + ...+\end{aligned}$$

$$
\begin{aligned}
&+ \mathbb{P}(\|x_{n+2} - x_{n+1}\| > t(r-1)/_{r^2}) + \mathbb{P}(\|x_{n+1} - x_n\| > t(r-1)/_r) \\
&\leq \frac{M}{[(r-1)t]^\alpha}\big[(r^m)^\alpha (q^{n+m-1})^\alpha + ... + (r^2)^\alpha (q^{n+1})^\alpha + r^\alpha (q^n)^\alpha\big] \\
&= \frac{M}{[(r-1)t]^\alpha}(q^n)^\alpha r^\alpha . \big[(qr)^{\alpha(m-1)} + ... + (qr)^\alpha + 1\big] \\
&= \frac{M}{[(r-1)t]^\alpha}(q^n)^\alpha r^\alpha . \frac{1-(qr)^{m\alpha}}{1-(qr)^\alpha} \\
&< \frac{Mr^\alpha}{[(r-1)t]^\alpha[1-(qr)^\alpha]}(q^\alpha)^n
\end{aligned}
\tag{2}
$$

which tends to 0 as $n \to \infty$. It follows that $\{x_n\}$ is a Cauchy sequence in probability. Hence, $\{x_n\}$ converges in probability to a random variable ξ. By the similar arguments as in the proof of Theorem 2.2, we imply that ξ is a uniqe random fixed point of T.

Conversely, if T has a random fixed point ξ then (1) holds with $x_0 = \xi$ for any $\alpha > 0$.

By (2), let $m \to \infty$, we have

$$\mathbb{P}(\|x_n - \xi\| > t) \leq \frac{Mr^\alpha}{[(r-1)t]^\alpha[1-(qr)^\alpha]}(q^\alpha)^n = \frac{M}{1-p^\alpha}\Big(\frac{p}{p-q}\Big)^\alpha . \frac{(q^\alpha)^n}{t^\alpha}.$$

for any $p \in (q;1)$. Let $p = q^{\frac{1}{1+\alpha}}$. By Lemma 2.4, we have

$$\mathbb{P}(\|x_n - \xi\| > t) \leq \frac{M}{(1-q^{\frac{\alpha}{1+\alpha}})^{1+\alpha}} . \frac{(q^\alpha)^n}{t^\alpha}.$$

Thus, the proof is complete. ■

Corollary 2.6. *Let T be a probabilistic $q-$contraction on X. Then T has a unique random fixed point if and only if there exists a random variable $x_0 \in L_0^X(\Omega)$ such that $E\|T(\omega, x_0) - x_0\|^\alpha < +\infty$ for some $\alpha > 0$.*

Proof. By Chebyshev inequality we have

$$\mathbb{P}(\|T(\omega, x_0) - x_0\| > t) \leq \frac{E\|T(\omega, x_0) - x_0\|^\alpha}{t^\alpha}.$$

Hence,

$$\sup_{t>0} t^\alpha \mathbb{P}(\|T(\omega, x_0) - x_0\| > t) \leq E\|T(\omega, x_0) - x_0\|^\alpha.$$

The proof is finished by applying Theorem 2.5. ■

Corollary 2.7. *Let T be a stochastically continuous random operator on X such that T^k is a probabilistic q-contraction for some $k \in \mathbb{N}$. Then T has a unique random fixed point if and only if there exists $x_0 \in L_0^X(\Omega)$ such that*

$$\sup_{t>0} t^\alpha \mathbb{P}(\|T^k(\omega, x_0) - x_0\| > t) < +\infty \tag{3}$$

for some $\alpha > 0$.

Proof. Firstly, if T has a random fixed point denoted by ξ then (3) holds with $x_0 = \xi$.

Conversely, suppose that (3) is satisfied. By Theorem 2.5, T^k has a unique random fixed point. The rest of the proof is finished by using the same arguments as in the proof of Theorem 2.3. ■

3. Applications to Random Equations

Definition 3.1. Let T be a random operator on X and $\eta(\omega)$ an X-valued random variable. A random variable $\xi(\omega) \in L_0^X(\Omega)$ is said to be a solution of the random equation $T(\omega, x) = \eta(\omega)$ if $T(\omega, \xi(\omega)) = \eta(\omega)$ a.s.

Theorem 3.2. *Let T be a probabilistic lipchitzian operator on X in the sense that there exists a positive real-valued random variable $L(\omega)$ such that for any $t > 0$*

$$\mathbb{P}(\|T(\omega, x) - T(\omega, y)\| > L(\omega)t) \leq \mathbb{P}(\|x - y\| > t)$$

for any $x, y \in L_0^X(\Omega)$, and $k(\omega)$ a real-valued random variable such that $|k(\omega)| > \alpha > 0$ and $\dfrac{L(\omega)}{|k(\omega)|} < q < 1$ a.s. Then the random equation

$$T(\omega, x) - k(\omega)x = \eta(\omega) \tag{4}$$

has a unique solution for any $\eta \in L_p^X(\Omega)$ if and only if there exists $x_0 \in L_0^X(\Omega)$ such that $E\|T(\omega, x_0) - k(\omega)x_0\|^p < +\infty$.

Moreover, for any $x_0 \in L_0^X(\Omega)$, the sequence of random variables $\{x_n\}$ defined by $x_{n+1} = \dfrac{1}{k}\big[T(\omega, x_n) - \eta\big]$ $(n = 0, 1, 2, ...)$ converges in probability to the solution of (4).

Proof. Equation (4) is rewritten in the form $\dfrac{T(\omega, x) - \eta(\omega)}{k(\omega)} = x$. Let G be a random operator defined by $G(\omega, x) = \dfrac{T(\omega, x) - \eta(\omega)}{k(\omega)}$. Then the random equation (4) has a unique solution if and only if the random operator G has a unique random fixed point. For any $x, y \in L_0^X(\Omega)$ and $t > 0$ we have

$$\begin{aligned}\mathbb{P}(\|G(\omega, x) - G(\omega, y)\| > qt) &= \mathbb{P}(\|T(\omega, x) - T(\omega, y)\| > q|k(\omega)|t)\\ &\leq \mathbb{P}(\|T(\omega, x) - T(\omega, y)\| > L(\omega)t)\\ &\leq \mathbb{P}(\|x - y\| > t).\end{aligned}$$

Thus, G is a probabilistic $q-$contraction.

For any $t > 0$ and $x \in L_0^X(\Omega)$, by Chebyshev inequality and C_p inequality, we have

$$\begin{aligned}\mathbb{P}(\|T(\omega,x)-kx-\eta\|>t) &\le \frac{E\|T(\omega,x)-kx-\eta\|^p}{t^p}\\ &\le C_p.\frac{E\|T(\omega,x)-kx\|^p+E\|\eta\|^p}{t^p},\end{aligned}$$

where

$$C_p=\begin{cases}1 & \text{if } \ 0<p<1,\\ 2^{p-1} & \text{if } \ p\ge 1.\end{cases}$$

So, we have

$$\begin{aligned}\sup_{t>0} t^p\mathbb{P}(\|G(\omega,x)-x\|>t) &= \sup_{t>0} t^p\mathbb{P}(\|T(\omega,x)-kx-\eta\|>|k|t)\\ &\le \sup_{t>0} t^p\mathbb{P}(\|T(\omega,x)-kx-\eta\|>\alpha.t)\\ &= \frac{1}{\alpha^p}\sup_{t>0} t^p\mathbb{P}(\|T(\omega,x)-kx-\eta\|>t)\\ &\le \frac{C_p}{\alpha^p}.\big(E\|T(\omega,x)-kx\|^p+E\|\eta\|^p\big). \qquad (5)\end{aligned}$$

Firstly, suppose that there exists $x_0 \in L_0^X(\Omega)$ such that $E\|T(\omega,x_0)-k(\omega)x_0\|^p<+\infty$. For any $\eta\in L_p^X(\Omega)$, by (5) we have $\sup_{t>0} t^p\mathbb{P}(\|G(\omega,x_0)-x_0\|>t)<+\infty$. By Theorem 2.5, the random operator G has a unique random fixed point. Thus, the random equation (4) has a unique solution.

Conversely, for any $\eta\in L_p^X(\Omega)$, suppose that the random equation (4) has a solution denoted by ξ. Let $x_0=\xi$. Then $E\|T(\omega,x_0)-k(\omega)x_0\|^p=E\|\eta\|^p<+\infty$.

To finish the proof we show that $\{x_n\}$ converges in probability to ξ. Since $x_{n+1}=\frac{1}{k}\big[T(\omega,x_n)-\eta\big]=G(\omega,x_n)$, the convergence of $\{x_n\}$ to ξ follows from Proposition 2.1. Thus the proof is complete. ■

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