Vietnam Journal of Mathematics **38**:3(2010) 373-380

Vietnam Journal of MATHEMATICS

The Structured Controllability Radii of Higher Order Descriptor Systems

Nguyen Khoa Son[1] **and Do Duc Thuan**[2]

[1] *Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam*

[2] *Department of Applied Mathematics and Informatics, Hanoi University of Technology, 1 Dai Co Viet Street, Hanoi, Vietnam*

Received July 12, 2010

Abstract. In this paper we formulate some formulas for computing the controllability radii for higher order descriptor systems under the assumption that the system matrices are subjected to structured perturbations. The proofs are based on the multi-valued generalization of the classical Eckart-Young theorem which identifies the distance to singularity of a non-singular square matrix.

2000 Mathematics Subject Classification: 06B99, 34D99, 47A10, 47A99, 65P99.

Key words: Multi-valued linear operators, distance to non-surjectivity, higher order descriptor systems, controllability radius, structured perturbations.

1. Introduction

Let $W \in \mathbb{C}^{n\times m}$ with $n \le m$ be a full row rank matrix (or equivalently $W\mathbb{C}^m = \mathbb{C}^n$), then it is well-known (see, e.g. [19]) that the distance from W to the set Σ of all rank-deficient matrices is

$$\mathrm{dist}(W,\Sigma) = \inf\{\|\Delta\| : W+\Delta \in \Sigma\} = \frac{1}{\max_{v\in\mathbb{B}^1_{\mathbb{C}^n}} \min\{\|x\| : Wx = v\}},$$

and $\mathrm{dist}(W,\Sigma)$ is called the *distance to non-surjectivity.* In the case of square matrix W (i.e. when $m = n$), the above result is reduced to the classical Eckart-Young theorem ([7]) which reads

$$\mathrm{dist}(W, \Sigma) = \frac{1}{\|W^{-1}\|}.$$

It is clear that $W \notin \Sigma$ if and only if the linear system $Wx = b, x \in \mathbb{C}^m$ is feasible for all $b \in \mathbb{C}^n$ or W is well-posed. So $\mathrm{dist}(W, \Sigma)$ is sometimes called the distance of W to infeasibility or the distance of W to ill-posedness. The problem of calculation of the above mentioned "distance" is of great importance in mathematical control and optimization theory and atracts thereby a good deal of attention from researchers over several last decades, see [5, 6, 13, 14, 17, 18, 19, 20]. It worth noticing that in most of papers only the case of "unstructured distance" (or "component-wise distance") was studied, where the perturbation matrices Δ are assumed to be arbitrary. In many cases, however, the perturbations are restricted to some specific structure and ignoring such structure may lead to substantial underestimation of the sensible distance to ill-posedness.

In this paper we formulate a generalization of the above classical result of Eckart and Young to the case where the matrix $W \in \mathbb{C}^{n\times m}$ is subjected to the structured perturbations of the form

$$W \rightsquigarrow \tilde{W} = W + M\Delta N,$$

where M, N are given matrices defining the structure of perturbations. The above perturbed model allows us, by choosing the appropriate structuring matrices M and N, to describe the case where only one row or column of W or even only one entry of W is perturbed. Furthermore, the mentioned result will be applied to calculate the controllability radius of a linear controllable system, a problem of great interest in control theory. We note that this problem has been studied so far only for ordinary linear systems of the form $x^{(1)} := \dot{x} = Ax + Bu$ (see, e.g. [8, 9, 10, 11, 12, 16, 21]) and for descriptor systems of first order $Ex^{(1)} = Ax + Bu$ (see, e.g. [1, 2, 23]). The case of higher order descriptor systems is considered for the first time in this paper. Namely, we shall give some formulas for calculating the controllability radii of higher order descriptor systems of the form

$$Ex^{(k)}(t) = A_{k-1}x^{(k-1)}(t) + A_{k-2}x^{(k-2)}(t) + \ldots + A_1x^{(1)}(t) + A_0x(t) + Bu(t), \quad (1)$$

where $E, A_i \in \mathbb{K}^{n\times n}, B \in \mathbb{K}^{n\times m}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $x^{(i)}$ denotes the i^{th} derivative. The key technique is to make use of some well-known facts from the theory of multi-valued linear operators in representing equations and evaluating the norms of matrices involved in the calculation, see [3] or Section Preliminaries in [21].

2. Main Results

Assume that the matrix $W \in \mathbb{K}^{n\times m}$ is *surjective*, i.e. $W\mathbb{K}^m = \mathbb{K}^n$, and is subjected to affine perturbations of the form:

$$W \rightsquigarrow \widetilde{W} = W + M\Delta N. \quad (2)$$

Here $M \in \mathbb{K}^{n\times l}, N \in \mathbb{K}^{q\times m}$ are given matrices defining the structure of perturbations and $\varDelta \in \mathbb{K}^{l\times q}$ is the disturbance matrix.

Definition 2.1. Let $W \in \mathbb{K}^{n\times m}$ be surjective. Given a norm $\|\cdot\|$ on $\mathbb{K}^{l\times q}$, the structured distance of W to non-surjectivity with respect to affine perturbations of the form (2) is defined by

$$\operatorname{dist}(W; M, N) = \inf\left\{\|\varDelta\| : \varDelta \in \mathbb{K}^{l\times q} \text{ s.t. } \widetilde{W} = W + M\varDelta N \text{ non-surjective}\right\}.$$

If $W + M\varDelta N$ is surjective for all $\varDelta \in \mathbb{K}^{l\times q}$ then we set $\operatorname{dist}(W; M, N) = +\infty$.

Define the multi-valued operator $NW^{-1}M : \mathbb{K}^l \rightrightarrows \mathbb{K}^q$ by setting

$$(NW^{-1}M)(u) = N(W^{-1}(Mu)), \quad \forall u \in \mathbb{K}^l,$$

where $W^{-1} : \mathbb{K}^n \rightrightarrows \mathbb{K}^m$ is the (multi-valued) inverse operator of W which is defined by $W^{-1}(y) = \{x \in \mathbb{K}^m : Wx = y\}, \forall y \in \mathbb{K}^n$.

Theorem 2.2. *Assume that the surjective matrix W is subjected to structured perturbations of the form (2). Then the structured distance of W to non-surjectivity is given by the formula*

$$\operatorname{dist}(W; M, N) = \frac{1}{\|NW^{-1}M\|}.$$

The proof of Theorem 2.2 is given in [22]. We mention additionally that for a multi-valued linear operator $\mathcal{F} : \mathbb{K}^n \rightrightarrows \mathbb{K}^m$, the norm of $\mathcal{F}$ is defined by:

$$\|\mathcal{F}\| = \sup\{\inf_{y\in\mathcal{F}(x)} \|y\| : x \in \operatorname{dom}\mathcal{F}, \|x\| = 1\}.$$

Now, let us consider higher order descriptor system (1). We define the following notion of controllability which extends the concepts of $\mathcal{R}$-controllability and $\mathcal{C}$-controllability in [1, 4], $\mathcal{R}2$-controllability and strong $\mathcal{C}2$-controllability in [15].

Definition 2.3. A set $\mathcal{R} \subset \prod_{i=1}^{k} \mathbb{K}^n$ is called $\mathcal{R}^k$-reachable from $(x_0, x_0^{(1)}, \ldots, x_0^{(k-1)})$ if for every $(x_1, x_1^{(1)}, \ldots, x_1^{(k-1)}) \in \mathcal{R}$ there exist $t_1 \geq t_0 > 0$ and a sufficiently smooth control function that the classical solution of (1) satisfies $x^{(i)}(t_0) = x_0^{(i)}, x^{(i)}(t_1) = x_1^{(i)}, i = 0, \ldots k-1$. Then we say that the extended state $(x_1, x_1^{(1)}, \ldots, x_1^{(k-1)})$ is reached from the extended state $(x_0, x_0^{(1)}, \ldots, x_0^{(k-1)})$ in time $t_1 - t_0 \geq 0$.

System (1) is called $\mathcal{R}^k$-controllable if any extended state $(x_1, x_1^{(1)}, \ldots, x_1^{(k-1)}) \in \mathcal{R}$ can be reached from any admissible extended state $(x_0, x_0^{(1)}, \ldots, x_0^{(k-1)})$ in finite time.

System (1) is called $\mathcal{SC}^k$-controllable if any extended state $(x_1, x_1^{(1)}, \ldots, x_1^{(k-1)}) \in \prod_{i=1}^{k} \mathbb{K}^n$ can be reached from any admissible extended state $(x_0, x_0^{(1)}, \ldots, x_0^{(k-1)})$ in finite time.

For establishing the characterizations of $\mathcal{R}^k$-controllability and $\mathcal{SC}^k$-controllability, we define the matrix polynomials

$$P(s) = A_0 + sA_1 + \ldots + s^{k-1}A_{k-1} - s^k E,$$

and

$$Q(s) = s^k A_0 + s^{k-1}A_1 + \ldots + sA_{k-1} - E.$$

Theorem 2.4. *System* (1) *is $\mathcal{R}^k$-controllable if and only if*

$$\operatorname{rank}[P(s), B] = n, \forall s \in \mathbb{C} \Longleftrightarrow \begin{cases} \operatorname{rank}[P(s), B] = n, \forall s \in \mathbb{C} : |s| \le 1, \\ \operatorname{rank}[Q(s), B] = n, \forall s \in \mathbb{C} : |s| \le 1, s \neq 0. \end{cases}$$

Theorem 2.5. *System* (1) *is $\mathcal{SC}^k$-controllable if and only if*

$$\mathcal{R}^k\text{-controllable and } \operatorname{rank}[E, B] = n \Longleftrightarrow \begin{cases} \operatorname{rank}[P(s), B] = n, \forall s \in \mathbb{C} : |s| \le 1, \\ \operatorname{rank}[Q(s), B] = n, \forall s \in \mathbb{C} : |s| \le 1. \end{cases} \tag{3}$$

The proofs of the above results are similar to those given in [15] and therefore omitted.

Assume that system (1) is subjected to structured perturbations of the form

$$\widetilde{E}x^{(k)}(t) = \widetilde{A}_{k-1}x^{(k-1)}(t) + \widetilde{A}_{k-2}x^{(k-2)}(t) + \ldots + \widetilde{A}_1 x'(t) + \widetilde{A}_0 x(t) + \widetilde{B}u(t),$$

with

$$[E, A_{k-1}, \ldots, A_0, B] \rightsquigarrow [\widetilde{E}, \widetilde{A}_{k-1}, \ldots, \widetilde{A}_0, \widetilde{B}] = [E, A_{k-1}, \ldots, A_0, B] + M\Delta N. \tag{4}$$

Here $M \in \mathbb{K}^{n\times l}, N \in \mathbb{K}^{q\times(n+kn+m)}$ are given structuring matrices and $\Delta \in \mathbb{K}^{l\times q}$ is unknown disturbance. We denote $\underline{A} = [A_{k-1}, \ldots, A_0]$.

Definition 2.6. Let system (1) be $\mathcal{R}^k$-controllable. Given a norm $\|\cdot\|$ on $\mathbb{K}^{l\times q}$, the $\mathcal{R}^k$-controllability radius of system (1) with respect to affine perturbations of the form (4) is defined by

$$r^{\mathcal{R}}_{\mathbb{K}}(E, \underline{A}, B; M, N) = \inf\{\|\Delta\| : \Delta \in \mathbb{K}^{l\times q} \text{ s.t. } [E, \underline{A}, B] + M\Delta N \text{ not } \mathcal{R}^k\text{-controllable}\}. \tag{5}$$

If $[E, \underline{A}, B] + M\Delta N$ is $\mathcal{R}^k$-controllable for all $\Delta \in \mathbb{K}^{l\times q}$ then we set

$$r^{\mathcal{R}}_{\mathbb{K}}(E, \underline{A}, B; M, N) = +\infty.$$

Definition 2.7. Let system (1) be $\mathcal{SC}^k$-controllable. Given a norm $\|\cdot\|$ on $\mathbb{K}^{l\times q}$, the $\mathcal{SC}^k$-controllability radius of system (1) with respect to affine perturbations of the form (4) is defined by

$$r_{\mathbb{K}}^{\mathcal{SC}}(E,\underline{A},B;M,N)=\inf\big\{\|\Delta\|:\Delta\in\mathbb{K}^{l\times q}\ \text{s.t. } [E,\underline{A},B]+M\Delta N \text{ not } \mathcal{SC}^k\text{-controllable}\big\}.$$

If $[E,\underline{A},B]+M\Delta N$ is $\mathcal{SC}^k$-controllable for all $\Delta\in\mathbb{K}^{l\times q}$ then we set

$$r_{\mathbb{K}}^{\mathcal{SC}}(E,\underline{A},B;M,N)=+\infty.$$

Now, we will use Theorem 2.2 to establish computable formulas for the radii $r_{\mathbb{K}}^{\mathcal{R}}$ and $r_{\mathbb{K}}^{\mathcal{SC}}$. To this end, we define the matrix functions

$$W_1(s)=[P(s),B],\qquad W_2(s)=[Q(s),B],$$

$$H_1(s)=\begin{bmatrix}-s^kI_n & 0\\ s^{k-1}I_n & 0\\ \vdots & \vdots\\ I_n & 0\\ 0 & I_m\end{bmatrix},\qquad H_2(s)=\begin{bmatrix}-I_n & 0\\ sI_n & 0\\ \vdots & \vdots\\ s^kI_n & 0\\ 0 & I_m\end{bmatrix},$$

$$N_1(s)=NH_1(s),\qquad N_2(s)=NH_2(s).$$

Let $W_1(s)^{-1},W_2(s)^{-1}:\mathbb{K}^n\rightrightarrows\mathbb{K}^{n+m}$ be the (multi-valued) inverse operators of $W_1(s),W_2(s)$, respectively, then we define the multi-valued linear operators

$$N_1(s)W_1(s)^{-1}M,\ \ N_2(s)W_2(s)^{-1}M:\mathbb{K}^l\rightrightarrows\mathbb{K}^q$$

by setting

$$(N_1(s)W_1(s)^{-1}M)(u)=N_1(s)(W_1(s)^{-1}(Mu)),$$
$$(N_2(s)W_2(s)^{-1}M)(u)=N_2(s)(W_2(s)^{-1}(Mu)),$$

for all $u\in\mathbb{K}^l$.

Theorem 2.8. *Assume that system* (1) *is $\mathcal{R}^k$-controllable and subjected to structured perturbations of the form* (4). *If $\mathbb{K}=\mathbb{C}$ then the $\mathcal{R}^k$-controllability radius of system* (1) *is given by the formula*

$$r_{\mathbb{C}}^{\mathcal{R}}(E,\underline{A},B;M,N)=\frac{1}{\sup_{s\in\mathbb{C}}\|N_1(s)W_1(s)^{-1}M\|}$$
$$=\min\left\{\frac{1}{\sup_{|s|\le 1}\|N_1(s)W_1(s)^{-1}M\|};\frac{1}{\sup_{0<|s|\le 1}\|N_2(s)W_2(s)^{-1}M\|}\right\}.$$

Theorem 2.9. *Assume that system* (1) *is $\mathcal{SC}^k$-controllable and subjected to structured perturbations of the form* (4). *If $\mathbb{K}=\mathbb{C}$ then the $\mathcal{SC}^k$-controllability radius of system* (1) *is given by the formula*

$$r_{\mathbb{C}}^{\mathcal{SC}}(E,\underline{A},B;M,N)=\min\left\{\frac{1}{\sup_{s\in\mathbb{C}}\|N_1(s)W_1(s)^{-1}M\|};\frac{1}{\|N_2(0)W_2(0)^{-1}M\|}\right\}$$
$$=\min\left\{\frac{1}{\sup_{|s|\le 1}\|N_1(s)W_1(s)^{-1}M\|};\frac{1}{\sup_{|s|\le 1}\|N_2(s)W_2(s)^{-1}M\|}\right\}.$$

The proofs of above Theorems 2.8 and 2.9 are based on Theorem 2.2 and will be published elsewhere, in a separate paper.

We illustrate the above results by an example.

Example 2.10. Consider the linear control system (E,A,B) described by

$$E\dot{x}(t)=Ax(t)+Bu(t),$$

where $E=\begin{bmatrix}0&0\\0&1\end{bmatrix}$, $A=\begin{bmatrix}0&2\\1&-1\end{bmatrix}$, $B=\begin{bmatrix}1\\0\end{bmatrix}$. We see that

$$W_1(s)=[A-sE,B]=\begin{bmatrix}0&2&1\\1&-1-s&0\end{bmatrix},\qquad W_2(s)=[sA-E,B]=\begin{bmatrix}0&2s&1\\s&-s-1&0\end{bmatrix}.$$

It follows that $\operatorname{rank}W_1(s)=\operatorname{rank}W_2(s)=2$ for all $s\in\mathbb{C}:|s|\le 1$. Therefore, by (3), this system is $\mathcal{SC}^1$-controllable. Assume that, the control matrix $[E,A,B]$ is subjected to structured perturbation of the form

$$\begin{bmatrix}0&0&0&2&1\\0&1&1&-1&0\end{bmatrix}\rightsquigarrow\begin{bmatrix}\delta_1&\delta_1&\delta_2&2+\delta_2&1+\delta_2\\\delta_1&1+\delta_1&1+\delta_2&-1+\delta_2&\delta_2\end{bmatrix},$$

where $\delta_i\in\mathbb{C}, i=1,2$ are disturbance parameters. The above perturbed model can be represented in the form

$$[E,A,B]\rightsquigarrow[\widetilde{E},\widetilde{A},\widetilde{B}]=[E,A,B]+M\Delta N$$

with $M=\begin{bmatrix}1\\1\end{bmatrix}$, $N=\begin{bmatrix}1&1&0&0&0\\0&0&1&1&1\end{bmatrix}$. By definition, we get $N_1(s)=\begin{bmatrix}-s&-s&0\\1&1&1\end{bmatrix}$ and $N_2(s)=\begin{bmatrix}-1&-1&0\\s&s&1\end{bmatrix}$. Therefore, we have

$$N_1(s)W_1(s)^{-1}M(v)=N_1(s)W_1(s)^{-1}\begin{pmatrix}v\\v\end{pmatrix}$$
$$=\left\{N_1(s)\begin{pmatrix}p\\q\\r\end{pmatrix}:2q+r=p-(1+s)q=v\right\}$$
$$=\left\{\begin{pmatrix}-sv-(2s+s^2)q\\2v+sq\end{pmatrix}:q\in\mathbb{C}\right\}.$$

Thus, for each $v\in\mathbb{C}$, the problem of computing $d(0,E[A-\lambda I,B]^{-1}D(v))$ is reduced to the calculation of the distance from the origin to the straight line in $\mathbb{C}^2$ whose equation can be rewritten in the form

$$x_1 + (2+s)x_2 = (4+s)v$$

with $x_1 = -sv-(2s+s^2)q, x_2 = 2v+sq$. Note that if $s = 0$ then this line reduced to the point $\begin{pmatrix} 0 \\ 2v \end{pmatrix}$. Assume $s \neq 0$ and let $\mathbb{C}^2$ be endowed with the vector norms $\|\cdot\|_\infty$, then we can deduce

$$|4+s||v| \leq |x_1| + |2+s||x_2| \leq (1+|2+s|)\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_\infty.$$

This implies

$$\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_\infty \geq \frac{|4+s||v|}{1+|2+s|}.$$

which yields the equality if $x_1 = \dfrac{(4+s)v}{1+|2+s|}$ and $x_2 = e^{i\varphi}x_1$, where φ is chosen such that $(2+s)e^{i\varphi} = |2+s|$. Therefore, we get

$$\|N_1(s)W_1(s)^{-1}M\| = \begin{cases} \dfrac{|4+s|}{1+|2+s|} & \text{if } s \neq 0, \\ 2 & \text{if } s = 0. \end{cases}$$

By applying Theorem 2.8, we obtain $r_{\mathbb{C}}^{\mathcal{R}}(E, A, B; M, N) = \dfrac{1}{2}$. Moreover, by a simple calculation we have $\|N_2(0)W_2(0)^{-1}M\| = 2$. Thus, by Theorem 2.9, we obtain

$$r_{\mathbb{C}}^{\mathcal{R}}(E, A, B; M, N) = r_{\mathbb{C}}^{\mathcal{SC}}(E, A, B; M, N) = \frac{1}{2}.$$

Acknowledgments. This work was supported financially by NAFOSTED (Vietnam National Foundation for Science and Technology Development) under the research Project "Qualitative properties of nonlinear control systems under perturbations and their applications".

References

1. A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols, Feedback design for regularizing descriptor systems, *Linear Algebra Appl.* **299** (1999), 119–151.
2. R. Byers, The descriptor controllability radius, in: *U. Helmke, R. Mennicken, J. Saurer (Eds.), Proceedings of the Conference on the Mathematical Theory of Networks and Systems*, MTNS'93, Akademie Verlag, Berlin, 1994, pp. 85-88.
3. R. Cross, *Multi-valued Linear Operators*, Marcel Dekker, New York, 1998.
4. L. Dai, *Singular Control Systems*, Springer, Berlin-Heidelberg, 1989.
5. A. L. Donchev, A. S. Lewis, and R. T. Rockafellar, The radius of metric regularity, *Trans. Amer. Math. Soc.* **355** (2003), 493–517.
6. A. L. Dontchev and R. T. Rockafellar, *Implicit Functions and Solution Mappings*, Monographs in Mathematics, Springer-Verlag, Berlin, 2009.
7. C. Eckart, G. Young, The approximation of one matrix by another of lower rank, *Psychometrika* **1** (1936), 211–218.

8. R. Eising, Between controllable and uncontrollable, *Systems & Control Letters*, **5** (1984), 263–264.
9. P. Gahinet and A. J. Laub, Algebraic Riccati equations and the distance to the nearest uncontrollable pair, *SIAM J. Control Optim.* **4**(1992) 765–786.
10. M. Gu, New methods for estimating the distance to uncontrollability, *SIAM J. Matrix Anal. Appl.* **21** (2000), 989–1003.
11. M. Gu, E. Mengi et al., Fast methods for estimating the distance to uncontrollability, *SIAM J. Matrix Anal. Appl.* **28** (2006), 447–502.
12. M. Karow, D. Kressner, On the structured distance to uncontrollability, *Systems & Control Letters*, **58** (2009), 128–132.
13. A. Lewis, Ill-conditioned convex processes and linear inequalities, *Math. Oper. Res.* **24** (1999), 829–834.
14. A. Lewis, Ill-conditioned inclusions, *Set-Valued Anal.* **9** (2001), 375–381.
15. P. Losse and V. Mehrmann, Controllability and Observability of Second Order Descriptor Sytems, *SIAM J. Control Optim.* **47** (2008), 1351–1379.
16. E. Mengi, On the estimation of the distance to uncontrollability for higher order systems, *SIAM J. Matrix Anal. Appl.* **30** (2006), 154–172.
17. J. Pena, A characterization of the distance to infeasibility under block-structured perturbations, *Linear Algebra Appl.* **370** (2003), 193–216.
18. J. Pena, On the block-structured distance to non-surjectivity of sublinear mappings, *Math. Program.*- Ser.A **103** (2005), 561–573.
19. J. Renegar, Some perturbation theory for linear programming, *Math. Program.* **65** (1994), 73–91.
20. J. Renegar, Linear programming, complexity theory and elementary functional analysis, *Math. Program.* **70** (1995), 279–351.
21. N. K. Son and D. D. Thuan, Structured distance to uncontrollability under multi-perturbations: an approach using multi-valued linear operators, *Systems & Control Letters* **59**(2010), 476–483.
22. N. K. Son and D. D. Thuan, The structured distance to non-surjectivity and its application to calculating the controllability radius of descriptor systems, *J. Math. Anal. Appl.* (submitted).
23. Z. Yun and Y. Chengwu, Formulae for the distance between controllable and uncontrollable linear systems, *Systems & Control Letters* **21** (1993), 173–180.