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Attractors for Quasilinear Parabolic Equations Involving Weighted p-Laplacian Operators

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Abstract. In this paper, using the asymptotic *a priori* estimate method, we prove the existence of global attractors for a class of quasilinear degenerate parabolic equations involving weighted p-Laplacian operators in bounded domains. The obtained results, in particular, improve some recent known results about the p-Laplacian equations.

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1. Introduction

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a system having some dissipativity properties is to analyse the existence and structure of its global attractor. The existence of the global attractor has been derived for a large class of partial differential equations (see e.g. [4, 9, 16] and references therein). However, to the best of our knowledge, little seems to be known about the asymptotic behavior of solutions of degenerate equations.

In this paper we consider the following problem

$$\begin{aligned} u_t - \operatorname{div}(\sigma(x)|\nabla u|^{p-2}\nabla u) + f(u) = g(x), \quad x \in \Omega, t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{1}$$

where $p \geq 2$, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with its boundary $\partial\Omega$, $u_0 \in L^2(\Omega)$ given, the coefficient σ, the nonlinearity f and the external force g satisfy some conditions specified later.

Denote

$$L_{p,\sigma}u := -\operatorname{div}(\sigma|\nabla u|^{p-2}\nabla u);$$

p' is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$;

$p_\gamma^* := \frac{pN}{N-p+\gamma}$, for $\gamma \in \mathbb{R}^+$;

$\mathcal{I}[p,q] := \{tp + (1-t)q : 0 \leqslant t \leqslant 1\}$.

We assume the following conditions:

(H1) The function $\sigma : \Omega \to \mathbb{R}$ satisfies the following assumptions:

($\mathcal{H}_\alpha$) $\sigma \in L^1_{\mathrm{loc}}(\Omega)$ and for some $\alpha \in (0,p)$, $\liminf_{x \to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$;

(H2) $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying:

$$C_1|u|^q - C_0 \leq f(u)u \leq C_2|u|^q + C_0, \quad q \geq 2, \tag{2}$$

$$f'(u) \geq -C_3 \quad \text{for all} \quad u \in \mathbb{R}, \tag{3}$$

where C_0, C_1, C_2, C_3 are positive constants;

(H3) $g \in L^s(\Omega)$ with $s \in \mathbb{R}$ satisfies

$$s \geq \min\left\{2, \frac{N(q+p-2)}{p(N+q-1) - N - \alpha(q-1)}\right\}; \tag{4}$$

(H4) $[1, p_\alpha^*) \cap \mathcal{I}[p', q'] \neq \emptyset$, where α is given in assumption ($\mathcal{H}_\alpha$).

The degeneracy of problem (1) is considered in the sense that the measurable, nonnegative diffusion coefficient $\sigma(x)$ is allowed to have at most a finite number of (essential) zeroes at some points. The physical motivation of the assumption ($\mathcal{H}_\alpha$) is related to the modelling of reaction diffusion processes in composite materials, occupying a bounded domain Ω, in which at some points they behave as *perfect insulator*. Following [10, p. 79], when at some points the medium is perfectly insulating, it is natural to assume that $\sigma(x)$ vanishes at these points. Note that in various diffusion processes, the equation involves diffusion $\sigma(x) \sim |x|^\alpha, \alpha \in (0,p)$.

In the case $\sigma(x)$ satisfying condition $(\mathcal{H}_\alpha)$, problem (1) contains some important classes of parabolic equations, such as the semilinear heat equations (when $\sigma = 1, p = 2$), semilinear degenerate parabolic equations (when $p = 2$), the p-Laplacian equations (when $\sigma = 1, p \neq 2$), etc. It is noticed that the existence and long-time behavior of solutions to problem (1) when $p = 2$, the semilinear case, have been studied in [11, 12] and improved recently in [1, 3]. The existence of a (generalized) global attractor in $L^2(\Omega)$ for the multi-valued semiflow generated by problem (1) in the case without uniqueness of solutions was proved recently in [2].

The existence of global attractors for the p-Laplacian equations has been studied extensively by many authors in the last years (see [4, 5, 7, 9, 16] and references therein). In [4], Babin and Vishik established the existence of an $(L^2(\Omega), (W_0^{1,p}(\Omega) \cap L^q(\Omega))_w)$-global attractor; in Temam [16] only the special case $f = ku$ was discussed; in [5], Carvalho, Cholewa and Dlotko considered the existence of global attractors for problems with monotone operators, and as such an application, they got the existence of an $(L^2(\Omega), L^2(\Omega))$-global attractor for the p-Laplacian equation, see also Cholewa and Dlotko [9].

Recently, Carvalho and Gentile in [7], combining with their comparison results developed in [6], obtained that the corresponding semigroup has an $(L^2(\Omega), W_0^{1,p}(\Omega))$-global attractor. However they need some additional assumptions, i.e. either assume that $p > \frac{N}{2}$ and $f = f_1 + f_2$, where f_1 satisfies $(f_1, u) \geqslant 0$ and f_2 is a global $(L^2(\Omega), L^2(\Omega))$-Lipschitz mapping, or assume that f satisfies some growth condition such that it can be dominated by the p-Laplacian operator.

In order to study problem (1) we introduce the energy space $\mathcal{D}_0^{1,p}(\Omega, \sigma)$ defined as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} := \left(\int_\Omega \sigma(x) |\nabla u|^p dx \right)^{1/p},$$

and prove some compact embedding results (see Sec. 2.1 for details). In this paper, under assumptions (H1)-(H4), we prove the global existence of a weak solution and the existence of an $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega, \sigma) \cap L^q(\Omega))-$global attractor for the semigroup generated by problem (1). Thus, in some sense, we improve the above results about the p-Laplacian equations in bounded domains.

Let us explain the methods used in the paper. First, using the compactness and monotonicity methods [14, Chapters 1-2] we prove the existence and uniqueness of a global weak solution to problem (1). Then we study the existence of global attractors in some function spaces for the semigroup associated to problem (1). Thanks to *a priori* estimates of the solutions in $\mathcal{D}_0^{1,p}(\Omega, \sigma) \cap L^q(\Omega)$ and the compactness of the embedding $\mathcal{D}_0^{1,p}(\Omega, \sigma) \hookrightarrow L^2(\Omega)$, we immediately obtain the existence of an $(L^2(\Omega), L^2(\Omega))$-global attractor. However, some significant difficulties arise when we want to prove the existence of an $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega, \sigma) \cap L^q(\Omega))$-global attractor. Indeed, to this end we need to verify that the semigroup generated by problem (1) is asymptotically compact in

$\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$. Under conditions (H1)-(H4), the solutions are at most in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$, so there is no compact embedding results for these cases. Thus, it seems to be difficult that we can directly verify the asymptotic compactness in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$. In order to overcome the difficulty since lacking of a compact embedding, we exploit the asymptotic *a priori* estimate method initiated in [13] to testify the asymptotic compactness of the corressponding semigroup. Such an approach has been used recently for some kinds of partial differential equations (see e.g. [15, 17, 18]).

The rest of the paper is organized as follows. In Sec. 2, we recall some compactness results and prove some results on bi-spaces global attractors, which are frequently used later. Sec. 3 is devoted to a proof of the global existence of a weak solution to problem (1) by using compactness and monotonicity methods. In Sec. 4, we prove the existence of an $(L^2(\Omega), L^q(\Omega))$-global attractor and an $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega))$-global attractor for the semigroup corresponding to problem (1).

Notations. Throughout the paper, we write $\Omega(u \geqslant M) = \{x \in \Omega : u(x) \geqslant M\}$ and $\Omega(u \leqslant M) = \{x \in \Omega : u(x) \leqslant M\}$, and we use C to denote various constants whose values may change with each appearance.

2. Preliminary Results

2.1. Function Spaces and Operator

In order to study problem (1), we introduce the weighted Sobolev space $\mathcal{D}_0^{1,p}(\Omega,\sigma)$, defined as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} := \left(\int_\Omega \sigma(x)|\nabla u|^p dx \right)^{\frac{1}{p}},$$

and denote by $\mathcal{D}^{-1,p'}(\Omega,\sigma)$ the dual space of $\mathcal{D}_0^{1,p}(\Omega,\sigma)$.

We recall some compactness results in [2], which are generalizations of the results in the case $p = 2$ of Caldiroli and Musina [8].

Proposition 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N, N \geq 2$, and σ satisfies $(\mathcal{H}_\alpha)$. Then the following embeddings hold:*

(i) $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow W_0^{1,\beta}(\Omega)$ *continuously if* $1 \leq \beta < \dfrac{pN}{N+\alpha}$;

(ii) $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow L^r(\Omega)$ *compactly if* $1 \leq r < p_\alpha^*$, *where* $p_\alpha^* = \dfrac{pN}{N-p+\alpha}$.

Putting

$$L_{p,\sigma}u = -\mathrm{div}(\sigma|\nabla u|^{p-2}\nabla u), \quad u \in \mathcal{D}_0^{1,p}(\Omega,\sigma).$$

The following proposition, whose proof is straightforward, gives some important properties of the operator $L_{p,\sigma}$.

Proposition 2.2. *The operator $L_{p,\sigma}$ maps $\mathcal{D}_0^{1,p}(\Omega,\sigma)$ into its dual $\mathcal{D}^{-1,p'}(\Omega,\sigma)$. Moreover,*

(i) $L_{p,\sigma}$ *is hemicontinuous, i.e., for all $u, v, w \in \mathcal{D}_0^{1,p}(\Omega,\sigma)$, the map $\lambda \mapsto \langle L_{p,\sigma}(u+\lambda v), w\rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$;*

(ii) $L_{p,\sigma}$ *is strongly monotone when $p \geq 2$, i.e.,*

$$\langle L_{p,\sigma}u - L_{p,\sigma}v, u-v\rangle \geq \delta \|u-v\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} \ \text{ for all } \ u, v \in \mathcal{D}_0^{1,p}(\Omega,\sigma).$$

2.2. Global Attractors for Semigroups

We recall some basic concepts about bi-spaces global attractors (see [4] for more details).

Definition 2.3. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space X and Z be a metric space.

(i) A set $\mathcal{A} \subset X \cap Z$, which is invariant, closed in X, compact in Z and attracts bounded subsets of X in the topology of Z, is called an (X, Z)-global attractor of the semigroup $S(t)$;

(ii) A bounded subset B_0 of Z, which satisfies that for any bounded subset $B \subset X$, there is a time $T = T(B)$ such that $S(t)B \subset B_0$ for any $t \geq T$, is called an (X, Z)-bounded absorbing set;

(iii) $\{S(t)\}_{t\geq 0}$ is called (X, Z)-asymptotically compact, if for any bounded (in X) sequence $\{x_n\}_{n=1}^\infty \subset X$ and $t_n \geq 0, t_n \to \infty$, $\{S(t_n)x_n\}_{n=1}^\infty$ has a convergent subsequence with respect to the topology of Z.

Definition 2.4. Let X be a Banach space and $B \subset X$ be any bounded subset. The semigroup $\{S(t)\}_{t\geq 0}$ on X is called norm-to-weak continuous on B if for any $\{x_n\}_{n=1}^\infty \subset B$, $x_n \to x$ and $t_n \geq 0, t_n \to t$, we have $S(t_n)x_n \rightharpoonup S(t)x$ in X.

Definition 2.5. Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a semigroup on X. For any bounded subset B of X, we define the stationary set $S(B)$ of B by

$$S(B) = \{x \in B | \ S(t)x \in B \quad \text{for any} \quad t \geq 0\}.$$

The following result [18, Corollary 3.6] is very useful for verifying that a semigroup is norm-to-weak continuous.

Proposition 2.6. *Let X, Y be two Banach spaces and X^*, Y^* be their dual spaces, respectively, such that*

$$X \hookrightarrow Y \quad \text{and} \quad Y^* \hookrightarrow X^*,$$

where the injection $i : X \to Y$ is continuous and its adjoint $i^* : Y^* \to X^*$ is densely injective. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on X and Y, respectively, and assume furthermore that $\{S(t)\}_{t\geq 0}$ is continuous or weakly continuous on Y.

Then for any bounded subset B of X, $\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous on $S(B)$.

We now prove the following important result.

Theorem 2.7. *Let X be a Banach space and Z be a metric space. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on X such that:*

(i) *$\{S(t)\}_{t\geq 0}$ has an (X,Z)-bounded absorbing set B_0;*

(ii) *$\{S(t)\}_{t\geq 0}$ is (X,Z)-asymptotically compact;*

(iii) *$\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous on $S(B_0)$.*

Then $\{S(t)\}_{t\geq 0}$ has an (X,Z)-global attractor.

Proof. Set
$$\mathcal{A} = \omega(B_0) := \cap_{s\geq 0}\overline{\cup_{t\geq s}S(t)B_0}^{Z}.$$
It is clear that $\mathcal{A}$ is nonempty, compact in Z. We now prove that $\mathcal{A}$ attracts every bounded subset of X in the topology of Z. Indeed, if not, there exist a bounded set B of X, a $\delta > 0$, a sequence $\{t_n\}$ with $t_n \to \infty$, a sequence $x_n \in B$ such that $\text{dist}(S(t_n)x_n, \mathcal{A}) \geq \delta$. By ii), there is a subsequence $S(t_{n_j})x_{n_j} \to \beta \in Z$. As B_0 is an (X,Z)-bounded absorbing set, $S(t)B \subset B_0$ when $t \geq t_0(B)$. Therefore, $\beta = \lim\limits_{j\to\infty} S(t_{n_j})x_{n_j} = \lim\limits_{j\to\infty} S(t_{n_j} - t_0(B))S(t_0(B))x_{n_j}$. Setting $\beta_j = S(t_0(B))x_{n_j}$, we have $\beta_j \in B_0$, thus $\beta \in \mathcal{A}$. This is a contradiction.

In what follows, we will verify the invariance of $\mathcal{A}$, i.e., $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$. By the equivalent characterziration of ω-limit set, for any $x_0 \in \mathcal{A}$ we know that there exist $\{x_n\} \subset B_0$ and t_n with $t_n \to \infty$ such that $S(t_n)x_n \to x_0$ in Z. Using the norm-to-weak continuity of $\{S(t)\}_{t\geq 0}$ on $S(B_0)$, we have
$$S(t + t_n)x_n = S(t)S(t_n)x_n \rightharpoonup S(t)x_0 \quad \text{in}\, Z.$$
On the other hand, $\{S(t+t_n)x_n\}$ is (X,Z)-asymptotically compact, then it has a subsequence which is convergent in Z. Without loss of generality, we assume that $S(t + t_n)x_n \to x$ in Z. Hence, by the uniqueness of limits, we have $S(t)x_0 = x$, and from the definition of x we know that $x \in \mathcal{A}$. Therefore, combining with the arbitrariness of x_0 and t, we get
$$S(t)\mathcal{A} \subset \mathcal{A}, \quad \text{for all} \quad t \geq 0. \tag{5}$$
Now, we prove the converse conclusion. Since $t_n \to \infty$ as well as $t_n - t$, we can assume that $t_n - t \geq 0$ for each $n \in \mathbb{N}$. Therefore, $\{S(t_n - t)x_n\}_{n=1}^{\infty}$ has a convergent subsequence in Z, without loss of generality, we assume that $S(t_n - t)x_n \to y_0 \in \mathcal{A}$ in Z, and then by use of the norm-to-weak continuity again, we have
$$x_0 \leftarrow S(t_n)x_n = S(t)S(t_n - t)x_n \rightharpoonup S(t)y_0 \quad \text{in}\, Z.$$

Hence, notice the uniqueness of limits again, we have $x_0 = S(t)y_0$, and from the definition of y_0 we know that $y_0 \in \mathcal{A}$, which implies

$$\mathcal{A} \subset S(t)\mathcal{A} \quad \text{for all} \quad t \geq 0. \tag{6}$$

From (5) and (6), we get the invariance of $\mathcal{A}$. ■

3. Existence of Global Solutions

Denote

$$\begin{aligned} \Omega_T &= \Omega \times (0,T), \\ V &= L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma)) \cap L^q(\Omega_T), \\ V^* &= L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma)) + L^{q'}(\Omega_T). \end{aligned}$$

In what follows, we assume that $u_0 \in L^2(\Omega)$ is given.

Definition 3.1. A function u is called a weak solution of (1) on $(0,T)$ if and only if

$$\begin{aligned} &u \in V, \quad \frac{\partial u}{\partial t} \in V^*, \\ &u|_{t=0} = u_0 \ \text{ a.e. in } \ \Omega, \end{aligned}$$

and

$$\int\limits_{\Omega_T} \left(\frac{\partial u}{\partial t}\eta + \sigma(x)|\nabla u|^{p-2}\nabla u \nabla \eta + f(u)\eta - g\eta \right) dxdt = 0, \tag{7}$$

for all test functions $\eta \in V$.

It is known (see [2]) that if $u \in V$ and $\dfrac{du}{dt} \in V^*$, then $u \in C([0,T];L^2(\Omega))$. This makes the initial condition in problem (1) meaningful.

Lemma 3.2. *Let $\{u_n\}$ be a bounded sequence in $L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))$ such that $\{u_n'\}$ is bounded in V^*. If* (H1) *and* (H4) *hold, then $\{u_n\}$ converges almost everywhere in Ω_T up to a subsequence.*

Proof. By hypothesis (H4) and Proposition 2.1, one can take a number $r \in [1, p_\alpha^*) \cap \mathcal{I}[p',q']$ such that

$$\mathcal{D}_0^{1,p}(\Omega,\sigma) \subset\subset L^r(\Omega). \tag{8}$$

Since $r' \in \mathcal{I}[p,q]$, we have

$$L^p(\Omega) \cap L^q(\Omega) \subset L^{r'}(\Omega),$$

and therefore

$$L^r(\Omega) \subset L^{p'}(\Omega) + L^{q'}(\Omega). \tag{9}$$

Using Proposition 2.1 again, we see that

$$\mathcal{D}_0^{1,p}(\Omega,\sigma) \subset L^p(\Omega).$$

This and (9) imply that

$$L^r(\Omega) \subset W^* := \mathcal{D}^{-1,p'}(\Omega,\sigma) + L^{q'}(\Omega).$$

Now with (8), we have an evolution triple

$$\mathcal{D}_0^{1,p}(\Omega,\sigma) \subset\subset L^r(\Omega) \subset W^*. \tag{10}$$

The boundedness of $\{u_n'\}$ in V^* ensures that $\{u_n'\}$ is also bounded in $L^s(0,T;W^*)$, where $s = \min\{p', q'\}$. Thanks to the Aubin-Lions Lemma in [14, p. 58], $\{u_n\}$ is precompact in $L^p(0,T;L^r(\Omega))$ and therefore in $L^t(0,T;L^t(\Omega))$, $t = \min(p,r)$, so it has an a.e. convergent subsequence. ∎

The following lemma is a direct consequence of Young's inequality and the embedding $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow L^{p_\alpha^*}(\Omega)$, where $p_\alpha^* = \frac{pN}{N-p+\alpha}$, given by Proposition 2.1.

Lemma 3.3. *If assumption* (H3) *holds, then for every* $u \in \mathcal{D}_0^{1,p}(\Omega,\sigma)$*, we have*

$$\left|\int_\Omega gudx\right| \le \varepsilon\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} + C(\varepsilon)\|g\|^s_{L^s(\Omega)},\ \forall\epsilon > 0.$$

Theorem 3.4. *Under assumptions* (H1) - (H4)*, for each* $u_0 \in L^2(\Omega)$ *and* $T > 0$ *given, problem* (1) *has a unique weak solution on* $(0,T)$*. Moreover, the mapping* $u_0 \mapsto u(t)$ *is* $(L^2(\Omega), L^2(\Omega))$*-continuous.*

Proof. (i) Existence. Consider the approximating solution $u_n(t)$ of the form

$$u_n(t) = \sum_{k=1}^{n} u_{nk}(t)e_k,$$

where $\{e_j\}_{j=1}^\infty$ is a basis of $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$, which is orthogonal in $L^2(\Omega)$. We get u_n from solving the problem

$$\begin{cases} \langle \dfrac{du_n}{dt}, e_k\rangle + \langle L_{p,\sigma}u_n, e_k\rangle + \langle f(u_n), e_k\rangle = \langle g, e_k\rangle, \\ (u_n(0), e_k) = (u_0, e_k),\ k = 1, \ldots, n. \end{cases}$$

Since $f \in C^1(\mathbb{R})$, using the Peano theorem, we get the local existence of u_n. We now establish some *a priori* estimates for u_n. Since

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \int_\Omega \sigma(x)|\nabla u_n|^p dx + \int_\Omega f(u_n)u_n dx = \int_\Omega g u_n dx, \tag{11}$$

using (2) and Lemma 3.3 we obtain

$$\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + C(\int_\Omega \sigma(x)|\nabla u_n|^p dx + \int_\Omega |u_n|^q dx) \le C(\| g \|_{L^s(\Omega)}, |\Omega|).$$

Then

$$\begin{aligned}&\|u_n(t)\|^2_{L^2(\Omega)} + C\int_{\Omega_t} \sigma(x)|\nabla u_n|^p dx + C\int_{\Omega_t} |u_n|^q dx \\ &\le \|u_n(0)\|^2_{L^2(\Omega)} + tC(\|g\|_{L^s(\Omega)}, |\Omega|). \end{aligned} \tag{12}$$

It follows from here that

- $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$;
- $\{u_n\}$ is bounded in $L^p(0,T;\mathcal{D}^{1,p}_0(\Omega,\sigma))$;
- $\{u_n\}$ is bounded in $L^q(\Omega_T)$.

We rewrite equation (1) in V^* as

$$u_n' = g(x) - L_{p,\sigma}u_n - f(u_n) \tag{13}$$

and perform the following estimate deduced from Hölder's inequality

$$\begin{aligned}|\int_0^T \langle L_{p,\sigma}u_n, v\rangle dt| &= |\int_0^T\int_\Omega \sigma(x)|\nabla u_n|^{p-2}\nabla u_n \nabla v dx dt| \\ &\le \int_0^T\int_\Omega \big(\sigma(x)^{\frac{p-1}{p}}|\nabla u_n|^{p-1})(\sigma(x)^{\frac{1}{p}}|\nabla v|\big) dx dt \\ &\le \|u_n\|^{\frac{p}{p'}}_{L^p(0,T;\mathcal{D}^{1,p}_0(\Omega,\sigma))}\|v\|_{L^p(0,T;\mathcal{D}^{1,p}_0(\Omega,\sigma))},\end{aligned}$$

for any $v \in L^p(0,T;\mathcal{D}^{1,p}_0(\Omega,\sigma))$. Using the boundedness of $\{u_n\}$ in $L^p(0,T;\mathcal{D}^{1,p}_0(\Omega,\sigma))$, we infer that $\{L_{p,\sigma}u_n\}$ is bounded in $L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma))$. From (2), we have

$$|f(u)| \le C(|u|^{p-1} + 1).$$

This and the boundedness of $\{u_n\}$ in $L^q(\Omega_T)$ imply that $\{f(u_n)\}$ is bounded in $L^{q'}(\Omega_T)$. Therefore $\{u_n'\}$ is bounded in V^*. From above estimates, we can assume that

- $u_n' \rightharpoonup u'$ in V^*;
- $L_{p,\sigma}u_n \rightharpoonup \psi$ in $L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma))$;

- $f(u_n) \rightharpoonup \chi$ in $L^{q'}(\Omega_T)$.

Using Lemma 3.2, we see that $u_n \to u$ a.e. in Ω_T, so $f(u_n) \to f(u)$ a.e. in Ω_T. Thus, $\chi = f(u)$ thanks to Lemma 1.3 in [14]. Now taking (13) into account, we obtain the following equation in V^*,

$$u' = g(x) - \psi - f(u). \tag{14}$$

We now show that $\psi = L_{p,\sigma} u$. We have

$$X_n := \int_0^T \langle L_{p,\sigma} u_n - L_{p,\sigma} v, u_n - v \rangle \geq 0,$$

for every $v \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))$. Notice that

$$\begin{aligned}
\int_0^T \langle L_{p,\sigma} u_n, u_n \rangle dt &= \int_0^T \int_\Omega \sigma(x) |\nabla u_n|^p dx \\
&= \int_0^T \int_\Omega (g(x) u_n - f(u_n) u_n - u_n' u_n) dx dt \\
&= \int_0^T \int_\Omega (g(x) u_n - f(u_n) u_n) dx + \frac{1}{2} \|u_n(0)\|^2_{L^2(\Omega)} \\
&\quad - \frac{1}{2} \|u_n(T)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{15}$$

Therefore

$$\begin{aligned}
X_n = &\int_0^T \int_\Omega (g(x) u_n - f(u_n) u_n) dx + \frac{1}{2} \|u_n(0)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_n(T)\|^2_{L^2(\Omega)} \\
&- \int_0^T \langle L_{p,\sigma} u_n, v \rangle dt - \int_0^T \langle L_{p,\sigma} v, u_n - v \rangle dt.
\end{aligned}$$

It follows from the formulation of $u_n(0)$ that $u_n(0) \to u_0$ in $L^2(\Omega)$. Moreover, by the lower semi-continuity of $\|.\|_{L^2(\Omega)}$ we obtain

$$\|u(T)\|_{L^2(\Omega)} \leq \lim_{n\to\infty} \inf \|u_n(T)\|_{L^2(\Omega)}. \tag{16}$$

Meanwhile, by the Lebesgue dominated theorem, one can check that

$$\int_0^T \int_\Omega (g(x)u - f(u)u)dxdt = \lim_{n\to\infty} \int_0^T \int_\Omega (g(x)u_n - f(u_n)u_n)dxdt.$$

This fact and (15), (16) imply that

$$\lim_{n\to\infty} \sup X_n \le \int_0^T \int_\Omega (g(x)u - f(u)u)dxdt + \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} - \int_0^T \langle \psi, v\rangle dt - \int_0^T \langle L_{p,\sigma} v, u - v\rangle dt. \tag{17}$$

In view of (14), we have

$$\int_0^T \int_\Omega (g(x)u - f(u)u)dxdt + \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} = \int_0^T \langle \psi, u\rangle.$$

This and (17) deduce

$$\int_0^T \langle \psi - L_{p,\sigma} v, u - v\rangle dt \ge 0. \tag{18}$$

Put $v = u - \lambda w,\ w \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma)),\ \lambda > 0$. Since (18) we have

$$\lambda \int_0^T \langle \psi - L_{p,\sigma}(u - \lambda w), w\rangle dt \ge 0.$$

Then

$$\int_0^T \langle \psi - L_{p,\sigma}(u - \lambda w), w\rangle dt \ge 0.$$

Taking the limit $\lambda \to 0$ and noticing that $L_{p,\sigma}$ is hemicontinuous, we obtain

$$\int_0^T \langle \psi - L_{p,\sigma} u, w\rangle dt \ge 0,$$

for all $w \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))$. Thus $\psi = L_{p,\sigma} u$.

We now prove $u(0) = u_0$. Choosing some test function $\varphi \in C^1([0,T];\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega))$ with $\varphi(T) = 0$ and integrating by parts in t in the approximate equations, we have

$$\int_0^T -\langle u_n, \varphi' \rangle dt + \int_0^T \langle L_{p,\sigma} u_n, \varphi \rangle dt + \int_{\Omega_T} (f(u_n)\varphi - g(x)\varphi) dx dt = (u_n(0), \varphi(0)).$$

Taking limits as $n \to \infty$, we obtain

$$\int_0^T -\langle u, \varphi' \rangle dt + \int_0^T \langle L_{p,\sigma} u, \varphi \rangle dt + \int_{\Omega_T} (f(u)\varphi - g(x)\varphi) dx dt = (u_0, \varphi(0)), \quad (19)$$

since $u_n(0) \to u_0$. On the other hand, for the "limiting equation", we have

$$\int_0^T -\langle u, \varphi' \rangle dt + \int_0^T \langle L_{p,\sigma} u, \varphi \rangle dt + \int_{\Omega_T} (f(u)\varphi - g(x)\varphi) dx dt = (u(0), \varphi(0)). \quad (20)$$

Comparing (19) with (20) we get $u(0) = u_0$.

(ii) Uniqueness and continuous dependence. Let u, v be two weak solutions of problem (1) with initial datum u_0, v_0 in $L^2(\Omega)$. Then $w := u - v$ satisfies

$$\begin{cases} w_t + (L_{p,\sigma} u - L_{p,\sigma} v) + (f(u) - f(v)) = 0, & x \in \Omega, t > 0, \\ w|_{\partial\Omega} = 0, \ \ w(0) = u_0 - v_0. \end{cases}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \langle L_{p,\sigma} u - L_{p,\sigma} v, u - v \rangle + \int_\Omega (f(u) - f(v))(u - v) dx = 0.$$

Using condition (3) and the monotonicity of $L_{p,\sigma}$, we have

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} \leq 2C_3 \|w\|^2_{L^2(\Omega)}.$$

Applying the Gronwall inequality, we obtain

$$\|w(t)\|_{L^2(\Omega)} \leq \|w(0)\|_{L^2(\Omega)} e^{2C_3 t}.$$

This implies the uniqueness (if $u_0 = v_0$) and the continuous dependence of the solution. ■

4. Existence of Global Attractors

From Theorem 3.4, we can define a semigroup

$$S(t) : L^2(\Omega) \to L^2(\Omega), \ u_0 \mapsto S(t)u_0 := u(t), \quad (21)$$

where $u(t)$ is the unique solution of problem (1) with initial datum u_0.

Proposition 4.1. $\{S(t)\}_{t\geq 0}$ *has an* $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega))$*-bounded absorbing set* B_0*, that is, there is a positive constant* ρ*, such that for any bounded subset* B *in* $L^2(\Omega)$*, there exists a positive constant* T *which depends only on the* L^2*-norm of* B *such that*

$$\int_\Omega \sigma(x)|\nabla u(t)|^p dx + \int_\Omega |u(t)|^q dx \leq \rho,$$

for all $t \geq T$ *and* $u_0 \in B$*, where* $u(t)$ *is the unique weak solution of problem* (1) *with initial datum* u_0*.*

Proof. Multiplying (1) by u and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} + \int_\Omega f(u)u dx = \int_\Omega g u dx,$$

combining with (2), we find that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} + C_1\int_\Omega |u|^q dx \leq \int_\Omega g u dx + C_0|\Omega|. \tag{22}$$

Using Lemma 3.3, we get

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + C(\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} + \int_\Omega |u|^q dx) \leq C(\|g\|_{L^s(\Omega)}, |\Omega|).$$

Noticing that $p \geq 2$, we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + C\int_\Omega |u|^2 dx \leq C(\|g\|_{L^s(\Omega)}, |\Omega|). \tag{23}$$

Applying the Gronwall lemma, we get

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u(0)\|^2_{L^2(\Omega)} e^{-Ct} + C(\|g\|_{L^s(\Omega)}, |\Omega|)(1 - e^{-Ct}). \tag{24}$$

From (24), we see that $\{S(t)\}_{t\geq 0}$ has an $(L^2(\Omega), L^2(\Omega))$-bounded absorbing set, i.e., for any bounded subset B in $L^2(\Omega)$, there exists $T_1 = T_1(B)$ which depends only on the L^2-norm of B such that

$$\|S(t)u_0\|^2_{L^2(\Omega)} \leq \rho_0, \tag{25}$$

for all $t \geq T_1$, $u_0 \in B$, where the constant ρ_0 is independent of u_0. Taking $t \geq T_1$, integrating (22) on $[t, t+1]$ and combining with (25), we have

$$\int_t^{t+1}\Big(\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)}+\|u\|^q_{L^q(\Omega)}-\int_\Omega g(x)udx\Big)ds\le C(\|g\|_{L^s(\Omega)},|\Omega|,\rho_0),\qquad(26)$$

for all $t\ge T_1$. Meanwhile, let $F(s)=\int_0^s f(\tau)d\tau$; then by (2) again, we can deduce that

$$\tilde C_1|u|^q-\tilde C_0\le F(u)\le \tilde C_2|u|^q+\tilde C_0,\qquad(27)$$

and then,

$$\tilde C_1\|u\|^q_{L^q(\Omega)}-\tilde C_0|\Omega|\le\int_\Omega F(u)\le \tilde C_2\|u\|^q_{L^q(\Omega)}+\tilde C_0|\Omega|.\qquad(28)$$

Hence, from (26), we get

$$\int_t^{t+1}\Big(\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)}+\int_\Omega F(u)dx-\int_\Omega g(x)udx\Big)ds\le C(\|g\|_{L^s(\Omega)},|\Omega|,\rho_0,\tilde C_2,\tilde C_0).\qquad(29)$$

On the other hand, multiplying (1) by u_t, we obtain

$$\frac{d}{dt}\Big(\frac1p\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)}+\int_\Omega F(u)dx-\int_\Omega g(x)udx\Big)=-\|u_t\|^2_{L^2(\Omega)}\le 0.\qquad(30)$$

Therefore, from (29) and (30), by virtue of the uniform Gronwall lemma [16, p. 91], we get

$$\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)}+\int_\Omega F(u)dx-\int_\Omega g(x)udx\le C(\|g\|_{L^s(\Omega)},|\Omega|,\rho_0,\tilde C_2,\tilde C_0).\qquad(31)$$

Thanks to (28) and Lemma 3.2, (31) implies that for all $t\ge T_1+1$,

$$\int_\Omega \sigma(x)|\nabla u|^pdx+\int_\Omega|u|^q\le C(\|g\|_{L^s(\Omega)},|\Omega|,\rho_0,\tilde C_2,\tilde C_0).\qquad(32)$$

Now, take $\rho=C(\|g\|_{L^s(\Omega)},|\Omega|,\rho_0,\tilde C_2,\tilde C_0)$ and $T=T_1+1$, we complete the proof. ∎

Proposition 4.2. $\{S(t)\}_{t\ge0}$ *is norm-to-weak continuous on* $S(B_0)$*, where* B_0 *is the* $(L^2(\Omega),\mathcal{D}_0^{1,p}(\Omega,\sigma)\cap L^q(\Omega))$ *-bounded absorbing set obtained in Proposition 4.1.*

Proof. Choosing $Y=L^2(\Omega), X=\mathcal{D}_0^{1,p}(\Omega,\sigma)\cap L^q(\Omega)$, the conclusion follows immediately from Proposition 2.6. ∎

The set B_0 obtained in Proposition 4.1 is also of course an $(L^2(\Omega), L^2(\Omega))$-and $(L^2(\Omega), L^q(\Omega))$ -bounded absorbing set for the semigroup $S(t)$. By Theorem 2.7, to prove the existence of a global attractor, we only need to verify that $S(t)$ is asymptotically compact.

4.1. $(L^2(\Omega), L^2(\Omega))$-global attractor

From Proposition 4.1 and the compactness of the embedding $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow L^2(\Omega)$, we obtain the following result.

Theorem 4.3. *Assume that assumptions* (H1)-(H4) *are satisfied. Then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated to* (1) *has an* $(L^2(\Omega), L^2(\Omega))$*-global attractor* $\mathcal{A}_2$.

4.2. $(L^2(\Omega), L^q(\Omega))$-global attractor

To prove the existence of an $(L^2(\Omega), L^q(\Omega))$-global attractor, we need the following lemma ([18, Corollary 5.7]).

Lemma 4.4. *Let* $\{S(t)\}_{t\geq 0}$ *be a semigroup on* $L^2(\Omega)$ *and have an* $(L^2(\Omega), L^2(\Omega))$*-global attractor. Then* $\{S(t)\}_{t\geq 0}$ *has an* $(L^2(\Omega), L^q(\Omega))$*-global attractor provided that the following conditions hold:*

(i) $\{S(t)\}_{t\geq 0}$ *has an* $(L^2(\Omega), L^q(\Omega))$*-bounded absorbing set;*

(ii) *for any* $\epsilon > 0$ *and any bounded subset* B *of* $L^2(\Omega)$*, there exist positive constants* $M = M(\epsilon)$ *and* $T = T(\epsilon, B)$ *such that*

$$\int_{\Omega(|S(t)u_0|\geq M)} |S(t)u_0|^q < \epsilon, \tag{33}$$

for all $u_0 \in B$ *and* $t \geq T$.

Theorem 4.5. *Assume that asumptions* (H1)-(H4) *are satisfied. Then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated to* (1) *has an* $(L^2(\Omega), L^q(\Omega))$*-global attractor* $\mathcal{A}_q$.

Proof. By Lemma 4.4, it is sufficient to prove that for any $\epsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist two positive constants $T = T(B, \epsilon)$ and $M = M(\epsilon)$ such that

$$\int_{\Omega(|u|\geq M)} |u|^q \leq C\epsilon,$$

for all $t \geq T$ and $u_0 \in B$, where the constant C is independent of ϵ and B.

First, noting that for any fixed $\epsilon > 0$, there exists $\delta > 0$ such that for any $e \subset \Omega$ with $m(e) \leq \delta$, we have

$$\int_e |g|^s < \epsilon. \tag{34}$$

We now multiply (1) by $(u-M)_+^{q-1}$ to get that

$$\begin{aligned}(u-M)_+^{q-1}u_t - \operatorname{div}(\sigma(x)|\nabla u|^{p-2}\nabla u)(u-M)_+^{q-1} + f(u)(u-M)_+^{q-1} \\ = g(x)(u-M)_+^{q-1},\end{aligned} \tag{35}$$

where $(u-M)_+$, $M \geq M_1$, denotes the positive part of $(u-M)$, that is,

$$(u-M)_+ = \begin{cases} u-M, & u \geq M, \\ 0, & u \leq M, \end{cases}$$

and M_1 is a large enough positive constant. Note that, from (2) we have $f(u) \geq \tilde{C}|u|^{q-1}, u \geq M$. Thus

$$\begin{aligned} f(u)(u-M)_+^{q-1} &\geq \tilde{C}|u|^{q-1}(u-M)_+^{q-1} \\ &\geq \frac{\tilde{C}}{2}|u|^{q-1}(u-M)_+^{q-1} + \frac{\tilde{C}}{2}|u|^{q-1}(u-M)_+^{q-1} \\ &\geq \frac{\tilde{C}}{2}|u|^{q-1-\frac{1}{s-1}}(u-M)_+^{(q-1)\frac{s}{s-1}} + \frac{\tilde{C}}{2}|u|^{q-2}(u-M)_+^{q} \\ &\geq \frac{\tilde{C}}{2}M^{q-1-\frac{1}{s-1}}(u-M)_+^{(q-1)\frac{s}{s-1}} + \frac{\tilde{C}}{2}M^{q-2}(u-M)_+^{q}. \end{aligned} \tag{36}$$

From (H3), we have

$$g(u-M)_+^{q-1} \leq \frac{\tilde{C}}{2}M^{q-1-\frac{1}{s-1}}(u-M)_+^{(q-1)\frac{s}{s-1}} + \hat{C}|g|^s, \tag{37}$$

where $\hat{C}$ is a positive constant depending on $\tilde{C}$ and M. From (36), (37) and integrating (35) we get that

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int\limits_{\Omega(u\geq M)}(u-M)_+^q + (q-1)\int\limits_{\Omega(u\geq M)}\sigma(x)|\nabla u|^p(u-M)_+^{q-2} \\ +\frac{\tilde{C}}{2}M^{q-2}\int\limits_{\Omega(u\geq M)}(u-M)_+^q \leq \hat{C}\int\limits_{\Omega(u\geq M)}|g|^s.\end{aligned}$$

And then

$$\frac{d}{dt}\int\limits_{\Omega(u\geq M)}(u-M)_+^q + \frac{\tilde{C}}{2}qM^{q-2}\int\limits_{\Omega(u\geq M)}(u-M)_+^q \leq q\hat{C}\int\limits_{\Omega(u\geq M)}|g|^s.$$

By the Gronwall lemma, we have

$$\int\limits_{\Omega(u\geq M)}(u-M)_+^q \leq C\frac{\epsilon}{2^{q+2}}, \quad M \geq M_1, \tag{38}$$

where C is independent of ϵ and M.

Repeating the same steps above, just taking $(u+M)_-$ instead of $(u-M)_+$, where

$$(u+M)_- = \begin{cases} u+M, & u \le -M, \\ 0, & u \ge M, \end{cases}$$

we deduce that

$$\int_{\Omega(u \ge -M)} |u|^q \le C\frac{\epsilon}{2^{q+2}}, \quad M \ge M_2, \tag{39}$$

where M_2 is a large enough positive constant. Choose $M \ge M_3 = \max\{M_1, M_2\}$, we get that

$$\int_{\Omega(|u| \ge M)} (|u| - M)^q \le C\frac{\epsilon}{2^{q+1}}. \tag{40}$$

From (40), we have

$$\begin{aligned} \int_{\Omega(|u| \ge 2M)} |u|^q &= \int_{\Omega(|u| \ge 2M)} |u - M + M|^q \\ &\le 2^q \int_{\Omega(|u| \ge 2M)} (|u| - M)^q + 2^q \int_{\Omega(|u| \ge 2M)} M^q \\ &\le 2^{q+1} \int_{\Omega(|u| \ge 2M)} (|u| - M)^q < C\epsilon, \end{aligned}$$

where the constant C is independent of ϵ and M. This completes the proof. ■

Remark 4.6. In fact, if we are only concerned with the existence of the $(L^2(\Omega), L^2(\Omega))$-global attractor and the $(L^2(\Omega), L^q(\Omega))$-global attractor for the semigroup $S(t)$, then the assumption (H2) can be replaced by a weaker assumption: $f \in C(\mathbb{R})$ satisfying

$$C_1|u|^q - C_0 \le f(u)u \le C_2|u|^q + C_0, \quad q \ge 2,$$
$$(f(u) - f(v))(u - v) \ge -C|u - v|^2 \text{ for any } u, v \in \mathbb{R},$$

and we only need to assume that $p > 1$, which ensures that the operator $L_{p,\sigma}$ is monotone (but not strongly monotone when $1 < p < 2$). However, we need to use the stronger assumptions, namely $f \in C^1(\mathbb{R})$ satisfying (H2) and $p \ge 2$, in the next section for proving the existence of an $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega, \sigma) \cap L^q(\Omega))$-global attractor.

4.3. $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega))$−global attractor

At first, we will give some *a priori* estimates for u_t endowed with $L^2(\Omega)$-norm.

Lemma 4.7. *Assume that assumptions* (H1)-(H4) *hold. Then for any bounded subset B in $L^2(\Omega)$, there exists a positive constant $T = T(B)$ such that*

$$\|u_t(s)\|^2_{L^2(\Omega)} \le \rho_1 \quad \text{for all} \quad u_0 \in B \quad \text{and} \quad s \ge T,$$

where $u_t(s) = \dfrac{d}{dt}(S(t)u_0)|_{t=s}$ *and ρ_1 is a positive constant independent of B.*

Proof. We give some formal calculations, whose rigorous proof is done by using Galerkin approximations. By differentiating (1) in time and denoting $v = u_t$, we get

$$v_t - \mathrm{div}(\sigma(x)|\nabla u|^{p-2}\nabla v) - (p-2)\mathrm{div}(\sigma(x)|\nabla u|^{p-4}(\nabla u.\nabla v)\nabla u) + f'(u)v = 0. \tag{41}$$

Multiplying the above equality by v, intergrating over Ω and using (3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Omega)} + \int_\Omega \sigma(x)|\nabla u|^{p-2}|\nabla v|^2 + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u.\nabla v)^2 \le C_3\|v\|^2_{L^2(\Omega)}, \tag{42}$$

hence,

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \le C_3\|v\|^2_{L^2(\Omega)}. \tag{43}$$

On the other hand, integrating (30) from t to $t+1$ and using (31), we get

$$\int_t^{t+1} \|u_t\|^2_{L^2(\Omega)} \le C(\rho, \|g\|_{L^s(\Omega)}, |\Omega|), \tag{44}$$

as t large enough. Combining (43) with (44), and using the uniform Gronwall lemma, we have

$$\|u_t\|^2_{L^2(\Omega)} \le C(\rho, \|g\|_{L^s(\Omega)}, |\Omega|),$$

as t large enough. The proof is complete. ∎

We are now in a position to state the main result of the paper.

Theorem 4.8. *Assume that assumptions* (H1)-(H4) *are satisfied. Then the semigroup $\{S(t)\}_{t\ge0}$ associated to* (1) *has an $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega))$-global attractor $\mathcal{A}$.*

Proof. By Theorem 2.7 and Propositions 4.1 - 4.2, we only need to prove that the semigroup $\{S(t)\}_{t\ge0}$ is $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega))$-asymptotically compact.

Let B be a bounded subset of $L^2(\Omega)$, we will show that for any $\{u_{0n}\} \subset B$ and $t_n \to \infty$, $\{u_n(t_n)\}_{n=1}^\infty$ is precompact in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$, where $u_n(t_n) = S(t_n)u_{0n}$. By Theorem 4.5, it is sufficient to verify that for any $\{u_{0n}\} \subset B$ and $t_n \to \infty$, $\{u_n(t_n)\}_{n=1}^{\infty}$ is precompact in $\mathcal{D}_0^{1,p}(\Omega, \sigma)$.

We will prove that $\{u_n(t_n)\}$ is a Cauchy sequence in $\mathcal{D}_0^{1,p}(\Omega, \sigma)$. Thanks to Theorems 4.3 and 4.5, one can assume that $\{u_n(t_n)\}$ is a Cauchy sequence in $L^2(\Omega)$ and in $L^q(\Omega)$. Since $L_{p,\sigma}$ is strongly monotone when $p \geq 2$, we have

$$
\begin{aligned}
&\delta\|u_n(t_n) - u_m(t_m)\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} \\
&\leq \langle L_{p,\sigma} u_n(t_n) - L_{p,\sigma} u_m(t_m), u_n(t_n) - u_m(t_m)\rangle \\
&\quad = \langle -\frac{d}{dt} u_n(t_n) - f(u_n(t_n)) + \frac{d}{dt} u_m(t_m) + f(u_m(t_m)), u_n(t_n) - u_m(t_m)\rangle \\
&\leq \int_\Omega |\frac{d}{dt} u_n(t_n) - \frac{d}{dt} u_m(t_m)||u_n(t_n) - u_m(t_m)|dx \\
&\quad + \int_\Omega |f(u_n(t_n)) - f(u_m(t_m))||u_n(t_n) - u_m(t_m)|dx \\
&\leq \|\frac{d}{dt} u_n(t_n) - \frac{d}{dt} u_m(t_m)\|_{L^2(\Omega)} \|u_n(t_n) - u_m(t_m)\|_{L^2(\Omega)} \\
&\quad + \|f(u_n(t_n)) - f(u_m(t_m))\|_{L^{q'}(\Omega)} \|u_n(t_n) - u_m(t_m))\|_{L^q(\Omega)}.
\end{aligned}
$$

Because $\{f(u_n(t_n))\}$ is bounded in $L^{q'}(\Omega)$ and from Lemma 4.7, it implies the desired result. ■

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