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On Connes Subgroups and Graded Semirings

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Abstract. Semirings are studied either in their own right, in an attempt to broaden techniques coming from semigroup theory or generalization of group theory and ring theory or in connection with their applications. For the generalization of ring theoretic results, in the absence of additive inverses, one has to impose some weaker versions of additive inverses on semirings. In [4] Montgomery and Passman introduced Connes subgroup of a group G and related it to the ideal structures of a G-graded ring and its smash product. This paper generalizes the above ring theoretic relation for additively cancellative yoked semirings and computes the Connes subgroup Γ_R of a G-graded semiring R in terms of support (R), when R_1 is prime.

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1. Introduction

In the absence of additive inverses in semirings, the conditions to find the validity of the results of ring theory become complicated, thus one needs a weaker version of additive inverses, i.e. cancellation of the elements ($a + b = a + c$ implies that $b = c$ in a semiring R). Another weak version of the condition of having additive inverses, i.e. R being yoked (for $a,\ b \in R$, there exists an element r of R such that $a + r = b$ or $b + r = a$) is also required for some results. The ring theoretic results of [6] are studied for such semirings in [8]. If R is an additively cancellative semiring, then R is isomorphic to a subsemiring of the ring of differences R^{Δ} such that every element of R^{Δ} is the difference between two elements in the image of R [3]. It is evident from ([3, Proposition 9.42]) that there are plenty of such semirings. In $R^{\triangle} = \{a - b \mid a, b \in R\}$, we have $a - b = c - d$ if and only if there exist $r, r^{'} \in R$ such that $a + r = c + r^{'}$ and $b + r = d + r^{'}$. The set R^{Δ} becomes a ring under componentwise addition and multiplication given by $(a - b)(c - d) = (ac + bd) - (ad + bc)$. The zero element of R^{Δ} is $a - a$, denoted by 0 and multiplicative identity is 1. Clearly $R^{\triangle}$ contains R by way of embedding $a \mapsto a - 0$ (simply written as a). Thus the ring of differences $R^{\triangle}$ is an important tool to analyze the validity of ring theoretic results for a semiring. For basic definitions of semirings one can refer [3].

Throughout this paper, K be an additively cancellative commutative semiring and R an additively cancellative K-semialgebra graded by a finite group G. If R is a semiring graded by G, then for any graded ideal A of R, $A^{\Delta} = \{a - b | a, b \in A\}$ is a graded ideal of R^{Δ} (c.f. [5]). Also for any graded ideal I of R^{Δ}, $I \cap R$ is a graded ideal of R. Furthermore when R is a G-graded semiring, then there exists an extension semiring (known as smash product), with same 1, which comes from the study of semi-Hopf algebras. This smash product is denoted by $R \# K[G]^{\star}$, where R is a $K-$semialgebra. This semiring is a free left R-semimodule with basis $\{p_x \mid x \in G\}$ such that $\sum\limits_{x \in G} p_x = 1$ is decomposition of $1 \in R$ into orthogonal idempotents(c.f. [5]). We started this article with the aim to find out the validity of results proved by Montgomery and Passman [4] regarding the connection between the Connes subgroup of a group G (which is a purely analogue of the Connes spectrum introduced by Connes [2] in the context of action of locally compact groups on Von Neumann algebras) and the ideal structure of a G-graded ring R, its smash product $R \# K[G]^{\star}$.

Since R and K are additively cancellative, so their rings of differences $R^{\triangle}$ and $K^{\triangle}$ exist. Moreover, if a semiring R is graded by G, then $R^{\triangle}$ becomes a ring graded by G. That is, $R^{\triangle} = \sum\limits_{g \in G} \oplus \left(R^{\triangle}\right)_g$, where

$$(R^{\triangle})_g = \{p_g(a - a') \mid a,\ a' \in R\} = \{a_g - a'_g \ \mid \ a_g,\ a'_g \in R_g\}.$$

Therefore for $x \in R^{\triangle}$, $x = a - b$ ($a = \sum\limits_{g \in G} a_g,\ b = \sum\limits_{g \in G} b_g \in R$), we have the unique representation $a - b = \sum\limits_{g \in G} (a - b)_g$, where $(a - b)_g = a_g - b_g$. Thus we also have the smash product $R^{\triangle} \# K^{\triangle}[G]^{\star}$ which is isomorphic to $(R \# K[G]^{*})^{\triangle}$ (c.f. [5]) and hence $R \# K[G]^{*}$ embeds in $R^{\triangle} \# K^{\triangle}[G]^{*}$, whereas R embeds in $R^{\triangle}$. These embeddings become useful as the results of Montgomery and Passman are valid for the ring $R^{\triangle}$ and the smash product $R^{\triangle} \# K^{\triangle}[G]^{*}$, thereby providing us with an incisive technique for analyzing these results for R and $R \# K[G]^{*}$.

2. Basic Results of Semiring R, Its Ring of Differences R^{Δ} and Their Smash Products

The results proved in this section play an important role for semirings in the absence of additive inverses. First we state a lemma from [5] which is felt to be inseparable part of this paper.

Lemma 2.1. *Let R be a semiring graded by a finite group G and A any subset of R. Then for $g \in G$,*

(i) *Each R_g is subtractive;*

(ii)

(a) *Each A_g is subtractive, if A is a subtractive subset of R;*

(b) *A_G is subtractive, if A is a subtractive submonoid of $(R, +)$;*

(iii) $(R_g)^{\Delta} = (R^{\Delta})_g$;

(iv) $R_g = (R^{\Delta})_g \cap R$;

(v) $(A_g)^{\Delta} \subseteq (A^{\Delta})_g$. *Equality holds if R is yoked and A is a subtractive submonoid of R.*

(vi) $(A_G)^{\Delta} \subseteq (A^{\Delta})_G$. *Equality holds if R is yoked and A is a subtractive submonoid of R.*

(vii) *Let I be an ideal of R^{Δ}. Then*

(a) $(I \cap R)_g = I_g \cap R$, $g \in G$;

(b) $(I \cap R)_G = I_G \cap R$.

(viii) *If R is a yoked semiring, then R_1 is a yoked subsemiring of R.*

Using some basic results from [7], we relate the primeness (semiprimeness) of R and R^{Δ}.

Lemma 2.2. *Let R be a semiring and R^{Δ} its ring of differences.*

(i) *If R^{Δ} is prime (semiprime), then R is prime (semiprime).*

(ii) *If R is yoked and prime (semiprime), then R^{Δ} is prime (semiprime).*

(iii) (i) *and* (ii) *replacing prime by graded prime.*

Proof. We prove the result for primeness and the result for semiprimeness follows in the same way.

(i) Let A, B be two ideals of R such that $AB = 0$. By Lemma 3.2 (ii) of [7], we have $A^{\triangle}B^{\triangle} \subseteq (AB)^{\triangle} = 0$. Since $R^{\triangle}$ is prime, so either $A^{\triangle} = 0$ or $B^{\triangle} = 0$ which implies that $A^{\triangle} \cap R = 0$ or $B^{\triangle} \cap R = 0$. Hence $0 = A^{\triangle} \cap R \supseteq A$ or $0 = B^{\triangle} \cap R \supseteq B$ (c.f. [7, Lemma 3.2 (vi)].

(ii) Let I and J be two ideals of R^{Δ}, such that $IJ = 0$. This implies that $IJ \cap R = 0$. By using ([7, Lemma 3.2 (iv)]), we have $(I \cap R)(J \cap R) \subseteq IJ \cap R = 0$.

Also R is prime, which implies that either $I \cap R = 0$ or $J \cap R = 0$. Since R is yoked, so using ([7, Lemma 3.2 (vii)]), we get $I = (I \cap R)^{\triangle} = 0$ or $J = (J \cap R)^{\triangle} = 0$. Hence R^{Δ} is prime. ■

Using the above lemma, we prove

Lemma 2.3. *Let R be a yoked semiring, graded by a group G. Then*

(i) *$R\#K[G]^{\star}$ is semiprime if and only if R is graded semiprime.*

(ii) *$R\#K[G]^{\star}$ is G-prime if and only if R is graded prime.*

Proof. (i) Let R be graded semiprime. Since R is yoked, by Lemma 2.2 (ii), $R^{\triangle}$ is graded semiprime. This implies that $R^{\triangle}\#K^{\triangle}[G]^{\star} \cong \left(R\#K[G]^{\star}\right)^{\triangle}$ is semiprime (c.f. [1, Theorem 2.9]). Since R is yoked, so by Lemma 2.2 (i), $R\#K[G]^{\star}$ is semiprime. The converse follows from the fact that if I is a nilpotent graded ideal of R, then it generates a nilpotent ideal of $R\#K[G]^{\star}$.

(ii) Let $I,\ J$ be two G-invariant ideals of $R \# K[G]^{*}$ such that $I\ J = 0$. This implies $(I \cap R)\ (J \cap R) \subseteq (IJ) \cap R = 0$ (c.f. [7, Lemma 3.2 (iv)]). By Lemma 4.3 (ii) of [5], we get $I \cap R$ and $J \cap R$ are graded ideals of R. Now the graded primeness of R gives that either $I \cap R = 0$ or $J \cap R = 0$. This implies that either $0 = I = (I \cap R) \# K[G]^{*}$ or $0 = J = (J \cap R) \# K[G]^{*}$ (c.f. [5, Lemma 4.3 (iii)]). Hence $R \# K[G]^{*}$ is G-prime. The converse follows from the fact that for any graded ideal P of R, $P\#K[G]^{\star}$ is G-invariant. ■

The following result regarding the nonnilpotent ideals of R and R^{Δ} will be used to relate various subgroups of G.

Lemma 2.4. *Let R be a semiring and R^{Δ} its ring of differences.*

(i) *If I is a nonnilpotent ideal of R, then $I^{\triangle}$ is a nonnilpotent ideal of R^{Δ}.*

(ii) *If R is yoked and J a nonnilpotent ideal of $R^{\triangle}$, then $J \cap R$ is nonnilpotent.*

Proof. (i) This is obvious as $I \subseteq I^{\triangle}$.

(ii) Let R be yoked and J a nonnilpotent ideal of $R^{\triangle}$. Suppose $J \cap R$ is nilpotent, i.e., there exists a positive integer n such that $a_1 a_2 \ldots\ldots\ldots a_n = 0$, for $a_1, a_2, \ldots\ldots\ldots, a_n \in J \cap R$. Let $x_1, x_2, \ldots\ldots\ldots, x_n \in J$, where $x_i = a_i - b_i$ for $a_i, b_i \in R$. As R is a yoked semiring this implies that either $a_i - b_i \in R$ or $b_i - a_i \in R$. Thus for each x_i either $x_i \in R$ or $-x_i \in R$. Therefore, either $x_1.x_2 \ldots\ldots\ldots x_n$ or $-x_1.x_2 \ldots\ldots\ldots x_n$ is a product of n elements of $J \cap R$. In both cases $x_1.x_2 \ldots\ldots\ldots x_n = 0$ in $R^{\triangle}$. Hence J is nilpotent which is a contradiction, so we conclude that $J \cap R$ is nonnilpotent. ■

Definition 2.5. The graded semiring R is said to be *strongly graded* if $R_g\ R_h = R_{gh}$ for all $g,\ h \in G$.

The strongly G-graded semirings play an important role in simplification of the Connes subgroup. Thus, we include

Lemma 2.6. *Let R be a yoked semiring graded by a group G.*

(i) *The subsemiring R_1 contains the identity 1 for any G-graded semiring R.*

(ii) *A G-graded semiring R is strongly G-graded if and only if it satisfies $1 \in R_g R_{g^{-1}}$, for all $g \in G$. Equivalently, $R_{g^{-1}}\ R_g = R_1$ for all $x \in G$.*

(iii) *If R is strongly graded, then $R^{\triangle}$ is strongly graded.*

Proof. (i) Since $R^{\triangle}$ is G-graded, we have $1 \in \left(R^{\triangle}\right)_1 = (R_1)^{\triangle}$ (c.f. Lemma 2.1 (iii)). As $1 \in R$, we get $1 \in R \cap (R_1)^{\triangle}$. By Lemma 2.1 (i), R_1 is subtractive and so by Lemma 2.1 (iv), $R \cap (R_1)^{\triangle} = R_1$. Therefore we conclude that $1 \in R_1$.

(ii) If R is strongly G-graded, then $R_1 = R_g R_{g^{-1}}$. So using (i), we have $1 \in R_g R_{g^{-1}}$for all $g \in G$. Conversely, suppose that $1 \in R_g R_{g^{-1}}$for all $g \in G$ holds. Then using (i), $R_{gh} = 1R_{gh} \subseteq R_g R_{g^{-1}} R_{gh} \subseteq R_g R_{g^{-1}\ gh} = R_g R_h \subseteq R_{gh}$, for any $g,\ h \in G$. So $R_g R_h = R_{gh}$ for all $g,\ h \in G$.

(iii) Let R be a strongly graded semiring. Then by (ii), $R_1 = R_g R_{g^{-1}}$ for all $g \in G$. But by Lemma 3.2 (ii) of [7], $(R_g R_{g^{-1}})^{\triangle} = R_g^{\triangle} R_{g^{-1}}^{\triangle}$, so $(R^{\triangle})_1 = \left(R^{\triangle}\right)_g \left(R^{\triangle}\right)_{g^{-1}}$for all $g \in G$. Hence the ring $R^{\triangle}$ is strongly graded. ∎

Definition 2.7. We define the support of R by

$$\mathrm{sp}R = \{g \in G \mid R_g \neq 0\}.$$

The elements of R_g are called *homogeneous of degree g*.

Finally, in this section, we prove a result which will be used to relate the nilpotency of a subsemiring of a graded semiring to its identity component.

Lemma 2.8. *Let $|\mathrm{sp}R| = n$.*

(i) *Suppose for suitable n and d, $\lambda_1, \ldots \lambda_{nd} \in G$, and for all i, $i = 1, \ldots, nd$ $\alpha_i = \prod_{j=1}^{i} \lambda_j \in$ supp R. Furthermore, assume that $\{\alpha_i\}$ takes on at most n distinct values, then there exists $0 \leq j_0 < j_1 < \ldots < j_d$ with $1 = \lambda_{j_0+1}\lambda_{j_0+2} \ldots \lambda_{j_1} = \lambda_{j_1+1} \ldots \lambda j_2 = \ldots = \ldots \lambda_{j_d}$.*

(ii) *If A is a subsemiring of a graded semiring R with $A_1 = 0$, then $A^n = 0$.*

Proof. (i) Consider the products $\alpha_0 = 1, \alpha_i = \prod_{j=1}^{i} \lambda_j$, where $i = 1, \ldots, nd$. We are done unless they are all distinct from 1. Now, since there are only n possible distinct α_i, there must be some $d+1$ $\alpha_i' s$ which are equal (say). $\alpha_{j_0} = \alpha_{j_1} = \ldots = \alpha_{j_d}$ with $j_0 < j_1 < \ldots < j_d$. But then for each $0 \leq k \leq d-1$, we have $\lambda_1 \ldots \lambda_{j_k} = (\lambda_1 \ldots \lambda_{j_k}) \ldots \lambda_{j_k+1}$ implying $1 = \lambda_{j_k+1} \ldots \lambda_{j_{k+1}}$, as desired.

(ii) Since $A = \sum A_\lambda$, we need to show that $A_{\lambda_1} \ldots A_{\lambda_n} = 0$ for any given sequence $\lambda_1 \ldots \lambda_n \in G$. This is certainly true if $\lambda_1, \ldots, \lambda_i \notin$ supp R for some

$i \leq n$. So assume that $\prod_{j=1}^{i} \lambda_j \in$ supp R for each $i \leq n$, then by (i) , $A_{\lambda_i} \ldots A_{\lambda_k} \subset A_1 = 0$, since $1 \in A_1$ (c.f. Lemma 2.6). ■

3. The Connes Subgroup

We denote the set of all graded left ideals of R by $\mathrm{Gr}\, L$ and the set of all graded right ideals by $\mathrm{Gr}\, R$.

Definition 3.1. A subsemiring B of R is said to be *graded hereditary* if $B = AL$ for some $A \in \mathrm{Gr}\, R$, $L \in \mathrm{Gr}\, L$ and B is a nonnilpotent graded subsemiring of R. The set of all such B is denoted by $\mathrm{Gr}\, H$.

Using Lemma 2.8, we get

Lemma 3.2. *Let B be a graded subsemiring of R. Then B is nilpotent if and only if B_1 is nilpotent.*

Proof. B is nilpotent obviously implies that B_1 is nilpotent. Conversely, suppose that B_1 is nilpotent, then there exists a positive integer m such that $B_1^m = 0$. This implies that $(B^m)_1 = (B_1)^m = 0$. Now by using Lemma 2.8 (ii), we get $(B^m)^n = 0$ implying $B^{mn} = 0$, where $n = |\mathrm{sp}R|$. Hence B is nilpotent. ■

Now, we prove some results which are necessary for the simplification of the Connes subgroup.

Lemma 3.3. *Let R be strongly G-graded. Then $L \in \mathrm{Gr}\, L$ if and only if $L = RL_1$, where L_1 is a left ideal of R_1 and similarly $A \in \mathrm{Gr}\, R$ if and only if $A = A_1 R$, where A_1 is a right ideal of R_1. Thus $L_x = R_x L_1$, $A_x = A_1 R_x$ and if $B = AL$, then $B = A_1 R L_1$.*

Proof. Since R is strongly G-graded and $1 \in R_1$, we have $1 = a_g b_{g^{-1}}, a_g \in R_g$ and $b_{g^{-1}} \in R_{g^{-1}}$. Now, for $r_g \in L$, we have $r_g = 1.r_g = a_g b_{g^{-1}}.r_g = a_g b_{g^{-1}}.r_g \in RL_1$ (since $r_g \in R, b_{g^{-1}}.r_g \in L$ and L is a left ideal). Thus $L \subseteq RL_1$ and $RL_1 \subseteq L$ is trivial, hence $L = RL_1$. ■

Definition 3.4. If I is an ideal of R_1 and $x \in G$, we define $I^x = R_{x^{-1}} I R_x \subseteq R_1$.

Since R_x is an (R_1, R_1)-bisemimodule we see that I^x is an ideal of R_1. Furthermore $I^1 = I, (I^x)^y \subseteq I^{xy}$ and $(I^x J)^x \subseteq (IJ)^x$. If R is strongly graded, then this yields a permutation action on the ideals of R_1.

Lemma 3.5. *Let R be strongly graded semiring, and I, J two ideals of R_1. Then $(I^x)^y = I^{xy}, I^1 = I$ and $(IJ)^x = I^x J^x$ for $x, y \in G$.*

Proof. We have

$$(I^x)^y = (R_{x^{-1}}IR_x)^y = R_{y^{-1}}R_{x^{-1}}IR_xR_y = R_{(xy)^{-1}}IR_{(xy)} = I^{xy}$$

and $I^1 = R_1IR_1 = I$, as I is an ideal of R_1. Now

$$(IJ)^x = R_{x^{-1}}IJR_x = R_{x^{-1}}IR_1JR_x = R_{x^{-1}}IR_xR_{x^{-1}}JR_x = I^xJ^x.$$

■

Remark 3.6. Let $a, b \in R$ and $x, y \in G$. Then it follows by multiplication in $R \# K[G]^\star$ that $ap_x.bp_y = ab_{xy^{-1}}p_y$. This implies that

$$p_xb_{xy^{-1}} = \sum_{z\in G} p_xb_{xy^{-1}}p_z = \sum_{z\in G} \left(b_{xy^{-1}}\right)_{xz^{-1}} p_z = b_{xy^{-1}}p_y = p_x.bp_y.$$

So, $p_xR_x = p_xRp_1 = R_xp_1$.

Using the above facts, we prove

Lemma 3.7. *Let R be strongly G-graded. If I, I' are two ideals of $R \# K[G]^\star$, then $Ip_gI' = II'$ for all $g \in G$.*

Proof. If R is strongly graded, then it follows that $IR_g = I$ for $g \in G$. For, if $a_h \in I$ then $a_h = a_h.1 = a_h\left(b_{g^{-1}}a_g\right) = \left(a_hb_{g^{-1}}\right)a_g \in IR_g$ implying $I \subseteq IR_g$ and obviously $IR_g \subseteq I$. Similarly, $R_gI = I$. Now, using $R_gI' = I', p_xR_x = R_xp_1$ and $IR_g \subseteq I$, we get $Ip_gI' = Ip_gR_gI' = IR_gp_1I' = Ip_1I'$. In other words all Ip_gI' are equal and hence $II' = I.1.I' = \sum_{g\in G} Ip_gI' = Ip_xI'$ as required. ■

Now we define the Connes subgroup Γ_R of G.

Definition 3.8. Let R be a G-graded semiring with G finite. We define

$$\Gamma_R = \{x \in G \mid R_{x^{-1}}B_x \text{ is nonnilpotent for all } B \in \operatorname{Gr} H\}$$

and

$$\Gamma_{0R} = \{x \in G \mid B_{x^{-1}}B_x \text{ is nonnilpotent for all } B \in \operatorname{Gr} H\}.$$

Definition 3.9. The group G acts on the smash product $R\#K[G]^\star$ by $(rp_h)^g = rp_{hg}$ and hence it permutes the ideals of the semiring. Thus we can define

$$\Lambda = \left\{x \in G \mid \text{for all nonnilpotent ideals } I \text{ of } R\#K[G]^\star \right.$$
$$\left. \text{we have } I^xI \text{nonnilpotent}\right\}$$

and for each $g \in G$,

$$\Lambda_g = \left\{x \in G \mid \text{for all ideals } I \text{ of } R\#K[G]^\star, \right.$$
$$\left. \text{if } Ip_gI \text{ is nonnilpotent then } I^xI \text{ nonnilpotent}\right\}.$$

Lemma 3.10. *$\Gamma_{0R} \subseteq \Gamma_R$ and Γ_R is a subgroup of G.*

Proof. The inclusion $\Gamma_{0R} \subseteq \Gamma_R$ is obvious. Let x, $y \in \Gamma_R$ and $B \in GrH$, so by definition $R_{x^{-1}}B_x$ is nonnilpotent. Since $R_{x^{-1}}B_x \subseteq R_{x^{-1}}B$, so $R_{x^{-1}}B$ is nonnilpotent. This implies that $C = R_{x^{-1}}B_x \in \mathrm{Gr}\, H$. Now, for $y \in \Gamma_R$, we have $C_y = (R_{x^{-1}}B_x)_y = R_{x^{-1}}B_{xy}$ and since $y \in \Gamma_R$, we know that $R_{y^{-1}}C_y$ is nonnilpotent. But $R_{y^{-1}}C_y = R_{y^{-1}}R_{x^{-1}}B_{xy} \subseteq R_{(yx)^{-1}}B_{xy}$, so $R_{(yx)^{-1}}B_{xy}$ is nonnilpotent and $xy \in \Gamma_R$. Similarly Γ_{0R} is a subgroup of G. ■

Lemma 3.11. *Λ is normal subgroup of G and $\Lambda = \cap_g \Lambda_g$.*

Proof. Let $x, y \in \Lambda$ and I be a nonnilpotent ideal of $R\#K[G]^\star$. For $x \in \Lambda$, we have $J = I^x I$ is nonnilpotent and then $y \in \Lambda$ implies that $J^y J$ is nonnilpotent. But $J^y J = (I^x I)^y \, I^x I \subseteq I^{xy} I$, so $I^{xy} I$ is nonnilpotent and hence $xy \in \Lambda$. Now, let $x \in \Lambda$, $g \in G$. Then $J = I^{g^{-1}}$ is nonnilpotent, so $J^x J$ is nonnilpotent which implies that $(J^x J)^g$ is nonnilpotent. But $(J^x J)^g = \left(I^{g^{-1}x} I^{\,g^{-1}}\right)^g = I^{g^{-1}xg} I$, so $g^{-1}xg \in \Lambda$ and hence Λ is a normal subgroup of G. Finally if $Ip_g I$ is nonnilpotent, then the larger ideal I is nonnilpotent, so it follows that $\Lambda \subseteq \cap_g \Lambda_g$. Conversely, let $x \in \cap_g \Lambda_g$ and I be nonnilpotent. Then $I^2 = \sum_{g \in G} Ip_g I$ and I^2 nonnilpotent implies that $Ip_h I$ is nonnilpotent for some $h \in G$. But $x \in \Lambda_h$, so $I^x I$ is nonnilpotent and $x \in \Lambda$. ■

Lemma 3.12. *Let R be a graded semiring.*

(i) *If I is a left ideal of $R \# K[G]^\star$ and $x \in G$, then there exists $L \in \mathrm{Gr}\, L$ with $Ip_x = Lp_x$. Similarly, if I is a right ideal of $R\#K[G]^\star$, then there exists $A \in \mathrm{Gr}\, R$ with $p_x I = p_x A$.*

(ii) *If L is a graded left ideal of R. Then Lp_x is a left ideal of $R \# K[G]^\star$ for any $x \in G$. Similarly, if A is a right ideal of R, then $p_x A$ a right ideal of $R \# K[G]^\star$. In particular $I = Lp_x A$ is an ideal of $R \# K[G]^\star$ and I is nilpotent if and only if $B = AL$ or equivalently B_1 is nilpotent.*

Proof. (i) Let I be an ideal of $R\#K[G]^\star$ and set $L = \{a \in R \mid ap_x \in Ip_x\}$. Clearly L is a left ideal of R. Now, let $x \in Ip_x$, then $x = rp_x = r_{x^{-1}}p_x \in Lp_x$, where $r \in L$ and hence $Lp_x \supseteq Ip_x$, obviously $Lp_x \subseteq Ip_x$. To prove $L \in GrL$, let $a = \sum a_y \in L$. Then $ap_x \in Ip_x$, so $p_{yx}ap_x \in Ip_x$ as I is an ideal of $R \# K[G]^\star$. Hence $a_y p_x = p_{yx}ap_x \in Ip_x$ implying that $a_y \in L$ for all $g \in G$. The proof for a right ideal of $R \# K[G]^\star$ follows in the same way.

(ii) Let $rp_g \in R \# K[G]^\star, ap_x \in Lp_x$. Then $rp_g ap_x = ra_{gx^{-1}}p_x \in Lp_x$, since $L \in \mathrm{Gr}\, L$. Similarly $p_x A$ is a right ideal of $R\#K[G]^\star$ and thus $I = Lp_x A = Lp_x . p_x A$ is an ideal of $R \# K[G]^\star$. We now show $I^{n+1} = LB_1^n p_x A$, using induction on $n \geq 0$. Obviously the result holds for $n = 0$. Suppose that $I^{n+1} = LB_1^n p_x A$. Then $I^{n+2} = I^{n+1}.I = LB_1^n p_x A \; Lp_x A = LB_1^n \left(p_x A . Lp_x\right) A = LB_1^{n+1} p_x A$ as required. So B_1 nilpotent implies that I is nilpotent. Conversely, suppose $I^{n+1} = 0$. Then $0 = A \; I^{n+1} Lp_x = ALB_1^n p_x ALp_x = BB_1^n \left(p_x Bp_x\right) =$

$BB_1^{n+1}p_x$, so $0 = BB_1^{n+1} \supseteq B_1^{n+2}$ and hence B_1 is nilpotent. Now the result follows from Lemma 3.2. ■

Using the above results, we have the following relations between the different subgroups of G.

Theorem 3.13. *Let R be a semiring graded by a finite group G. Then for all $g \in G$ we have $\Lambda_g = \Gamma_{0R}^g = \Gamma_R^g$. In particular $\Gamma_{0R} = \Gamma_R$ and $\Lambda = \cap_g \Gamma^g$.*

Proof. Since by Lemma 3.10, $\Gamma_{0R}^g \subseteq \Gamma_R^g$, so it suffices to prove that $\Gamma_R^g \subseteq \Lambda_g \subseteq \Gamma_{0R}^g$. First we show that $\Lambda_g \subseteq \Gamma_{0R}^g$. For, let $x \in \Lambda_g$ and $B = AL \in \mathrm{Gr}\, H$ with $A \in \mathrm{Gr}\, R$, $L \in \mathrm{Gr}\, L$. Set $I = Lp_gA$ so that by Lemma 3.12 (ii), I is an ideal of $R \# K[G]^\star$. Since I is an ideal, we get

$$Ip_gI \supseteq (IL)p_g(AI) = Lp_gALp_gALp_gA = Lp_gBp_gBp_gA = LB_1^2p_gA,$$

where $LB_1^2 \in \mathrm{Gr}\, L$. Now, set $C = A.LB_1^2 = BB_1^2$ and so $C_1 = B_1^3$. We have $B \in GrH$, which implies that B_1 is nonnilpotent and hence C is nonnilpotent. Now by using Lemma 3.12 (ii), $LB_1^2p_gA$ is nonnilpotent and hence Ip_gI is nonnilpotent. Now, for $x \in \Lambda_g$, we have $J = I^xI$ is nonnilpotent. But $I^x = Lp_{gx}A$, so $J = Lp_{gx}ALp_gA = Lp_{gx}Bp_gA = LB_{gxg^{-1}}p_gA$. Since J is nonnilpotent, so by Lemma 3.12 (ii), D_1 is nonnilpotent, where $D = ALB_{gxg^{-1}} = BB_{gxg^{-1}}$. Thus $D_1 = B_{(gxg^{-1})^{-1}}B_{gxg^{-1}}$ is nonnilpotent for all $B \in GrH$, implying $gxg^{-1} \in \Gamma_{0R}$ and $x \in g^{-1}\Gamma_{0R}g = \Gamma_{0R}^g$. Hence $\Lambda_g \subseteq \Gamma_{0R}^g$. Now, we have to prove that $\Gamma_R^g \subseteq \Lambda_g$. Let $x \in \Gamma_R^g$ such that $gxg^{-1} \in \Gamma_R$. Let I be an ideal of $R \# K[G]^\star$ with Ip_gI is nonnilpotent. By Lemma 3.12 (i), $Ip_g = Lp_g$ and $p_gI = p_gA$ for some $A \in \mathrm{Gr}\, R$ and $L \in \mathrm{Gr}\, L$.This implies that $Lp_xA = Ip_gI$ is nonnilpotent and by Lemma 3.12 (ii), $B = AL$ is nonnilpotent and hence $B \in \mathrm{Gr}\, H$. Since $I \supseteq p_gI = p_gA$ and $I \supseteq Ip_g = Lp_g$, so we get $I^xI \supseteq (p_gA)^x(Lp_g) = p_{gx}(AL)p_g = B_{gxg^{-1}}p_g$. As I^xI is an ideal of $R\# K[G]^\star$, so we have $I^xI \supseteq R_{(gxg^{-1})^{-1}}B_{gxg^{-1}}p_g$. But $gxg^{-1} \in \Gamma_R$, which implies that $R_{(gxg^{-1})^{-1}}B_{gxg^{-1}} \subseteq R_1$ is nonnilpotent and this subset commutes with p_g so $R_{(gxg^{-1})^{-1}}B_{gxg^{-1}}p_g$ is nonnilpotent. Thus $I^xI \supseteq R_{(gxg^{-1})^{-1}}B_{gxg^{-1}}p_g$ is nonnilpotent, so we get $x \in \Lambda_g$. Hence $\Lambda_g = \Gamma_{0R}^g = \Gamma_R^g$ and the particular case follows by Lemma 3.11. ■

If R is graded semiprime, then the groups Γ_R and Λ becomes simple as observed below.

Lemma 3.14. *If R is yoked and graded semiprime then*

$$\Gamma_R = \{x \in G \mid B_x \neq 0 \text{ for all } B \in \mathrm{Gr}\, H\}$$

and

$$\Lambda = \left\{x \in G \mid \text{for all ideals } I \text{ of } R\#K[G]^\star, I \neq 0 \text{ implies } I^xI \neq 0\right\}.$$

Proof. If $R_{x^{-1}}B_x$ is nonnilpotent, then certainly $B_x \neq 0$. Conversely, if $B_x \neq 0$, then $0 \neq RB_x \in \operatorname{Gr} L$ so by graded semiprimeness of R, $L = RB_x$ is nonnilpotent. Thus by Lemma 3.2, $L_1 = R_{x^{-1}}B_x$ is nonnilpotent. This proves the result about Γ_R. The result for Λ follows from the fact that R is graded semiprime if and only if $R\#K[G]^\star$ is semiprime (c.f. Lemma 2.3). ■

We note that if R is strongly graded, then both Γ_R and Λ coincide.

Lemma 3.15. *If R is strongly G-graded, then*

$$\Lambda = \Gamma_R = \{x \in G \mid \text{ if } J \text{ is nonnilpotent ideal of } R_1 \text{ then } J^x J \text{ is nonnilpotent } \}.$$

Proof. If J is any nonnilpotent ideal of R_1. Then JRJ is nonnilpotent and so $B = JRJ = (JR)(RJ) \in \operatorname{Gr} H$. Conversely, if R is strongly graded and $B = AL \in \operatorname{Gr} H$ with $A \in \operatorname{Gr} R$, $L \in \operatorname{Gr} L$; then by Lemma 3.3, B can be written as $B = A_1RL_1$, where A_1 is a right ideal of R_1 and L_1 a left ideal of R_1. This implies that $B_1 = A_1R_1L_1 = A_1L_1$. Note that UV is nilpotent if and only if VU is. We have $B \in \operatorname{Gr} H$ if and only if $B_1 = A_1L_1$ is nonnilpotent or equivalently $J = L_1A_1$ is nonnilpotent.

We here use the fact $\Gamma_R = \Gamma_{0R}$. So, for a strongly graded semiring R, $x \in \Gamma_R$ if and only if $B_{x^{-1}}B_x = A_1R_{x^{-1}}L_1.A_1R_xL_1 = A_1C.A_1E$ is nonnilpotent, where $C = R_{x^{-1}}L_1$ and $E = R_xL_1$. But $B_{x^{-1}}B_x$ is nonnilpotent if and only if $CA_1.EA_1 = R_{x^{-1}}L_1.A_1R_xL_1.A_1 = R_{x^{-1}}J^xR_xJ = J^xJ$. Since $J = L_1.A_1$ is any nonnilpotent ideal of R_1, so the above characterization follows for Γ_R. Finally let $x \in \Gamma_R$, $g \in G$ and J be a nonnilpotent ideal of R_1. Then $D = J^{g^{-1}}$ is nonnilpotent, so $x \in \Gamma_R$ implies that D^xD is nonnilpotent and hence$(D^xD)^g$ is nonnilpotent. Using Lemma 3.5, $(D^xD)^g = \left(J^{g^{-1}x}J^{g^{-1}}\right)^g = J^{g^{-1}xg}J$, so $g^{-1}xg \in \Gamma_R$, implying that $x \in \Gamma_R^{g^{-1}}$ and hence $\Gamma_R = \Gamma_R^g$ for all $g \in G$. Thus by Lemma 3.11, $\Gamma_R = \cap_{g\in G}\Gamma_R^g = \cap_{g\in G}\Lambda_g$ for all $g \in G$. By Lemma 3.7, we have $I^2 = Ip_gI$ for any ideal I of $R\#K[G]^\star$. Thus it follows that all Λ_g are equal and hence $\Lambda = \Gamma_R$. ■

If R is G-graded, then R^Δ is a G-graded ring. Hence we have two Connes subgroups Γ_R and Γ_{R^Δ} corresponding to R and R^Δ respectively. For a yoked strongly graded semiring, we have

Lemma 3.16. *If R is a yoked and strongly graded semiring and $R^\triangle$ its ring of differences. Then $\Gamma_R \subseteq \Gamma_{R^\triangle}$.*

Proof. Let $x \in \Gamma_R$ and J be a nonnilpotent ideal of $R_1^\triangle$, then by Lemma 2.1 (viii), Lemma 2.4 (ii), $J \cap R_1$ be a nonnilpotent ideal of R_1. By definition of Γ_R, we get $(J \cap R_1)^x (J \cap R_1) \subseteq (J^x \cap R_1)(J \cap R_1) \subseteq J^xJ$ is nonnilpotent. Hence the result is proved. ■

The Connes subgroup Γ_R is useful to relate the primeness of $R\#K[G]^\star$ to the graded primeness of R as follows

Theorem 3.17. *Let R be a yoked and G-graded semiring with G finite. If $R\#K[G]^\star$ is prime, then R is graded prime and $\Gamma_R = G$. The converse follows if R is strongly graded.*

Proof. Obviously, the primeness of $R\#K[G]^\star$ implies its G-primeness and hence the graded primeness of R (c.f. Lemma 2.3). Let J be any nonnilpotent ideal of $R\#K[G]^\star$ and $g \in G$. Then J^g is nonnilpotent. Further, since $R\#K[G]^\star$ is prime, $J^g J$ is nonzero and thus nonnilpotent. This implies $\Lambda = G$. But by Theorem 3.13, $\Lambda \subseteq \Gamma_R$, so $\Gamma_R = G$. Conversely, let R be strongly graded and graded prime. Then its ring of differences $R^\triangle$ is strongly graded and graded prime (c.f. Lemma 2.6 (iii) and Lemma 2.2). Further $G = \Gamma_R \subseteq \Gamma_R^\triangle$, because R is strongly graded. So we have $G = \Gamma_{R^\triangle}$. Now, by using ([4, Corollary 2.8]), we get $R^\triangle \#\, K^\triangle [G]^\star \cong \left(R \,\#\, K[G]^\star\right)^\triangle$ is prime. Hence $R \,\#\, K[G]^\star$ is prime (c.f. Lemma 2.2 (i)). ■

Remark 3.18. Define the support of $R^\triangle$ by $\operatorname{sp}\left(R^\triangle\right) = \left\{x \in G \mid R_x^\triangle \neq 0\right\}$. Since $R \subseteq R^\triangle$, so $\operatorname{sp}(R) \subseteq \operatorname{sp}\left(R^\triangle\right)$ is obvious. Now, let $x \in \operatorname{sp}\left(R^\triangle\right)$ this implies $R_x^\triangle = (R_x)^\triangle \neq 0$ implies $R_x \neq 0$. Hence $\operatorname{sp}\left(R^\triangle\right) \subseteq \operatorname{sp}(R)$ and, we get $\operatorname{sp}\left(R^\triangle\right) = \operatorname{sp}(R)$.

Lemma 3.19. *Let R be a graded semiring.*

(i) *If R is graded semiprime, then the grading on R is non-degenerate.*

(ii) *If R_1 is prime, then for any two homogeneous nonzero elements a, b of R, $aR_hb \neq 0$, for $h \in \operatorname{sp}(R)$.*

Proof. (i) Assume that R is graded semiprime and $a_x \in R_x$ a nonzero homogeneous element. Then a_xR and Ra_x are nonzero graded right and left ideals of R and these are nonnilpotent by assumption. Also from Lemma 3.2, $0 \neq (a_xR)_1 = \left(a_x \sum_{h\in G} R_h\right)_1 = a_xR_{x^{-1}}$, where $a_x \in R_x$ and $0 \neq (Ra_x)_1 = \left(\sum_{h\in G} R_h\, a_x\right)_1 = R_{x^{-1}}a_x$. In other words R is nondegenerate.

(ii) We know if R_1 is prime, then $R_1$is semiprime. Let $a,\ b$ be two nonzero homogeneous elements of R and say $a \in R_{x^{-1}}, b \in R_{y^{-1}}$. Then by (i), $R_xaR_1 \neq 0$ and $R_1bR_y \neq 0$ and both are ideals of R_1. Since R_1 is prime we have $(R_xaR_1)(R_1bR_y) \neq 0$ and in particular

$$aR_1b \neq 0. \tag{1}$$

Again, let a, b be nonzero homogeneous elements of R and $h \in \operatorname{sp}(R)$ (i.e. $R_h \neq 0$). Now, by (1), $aR_h = aR_1R_h \neq 0$. Furthermore since aR_h is homogeneous we conclude that $aR_hb = aR_hR_1b \neq 0$. ■

Finally, we compute Γ_R in terms of $\operatorname{sp}(R)$ when R_1 is prime.

Theorem 3.20. *Let R be yoked and graded prime. If R_1 is prime, then $\Gamma_R = \operatorname{sp}(R)$ and the converse follows if R is strongly graded.*

Proof. First we assume that R_1 is prime. It is obvious that $\Gamma_R \subseteq \mathrm{sp}(R)$, for the other inclusion, let $B = AL$ with $0 \neq A \in GrA, 0 \neq L \in GrL$. Since A and L are nonnilpotent, and by Lemma 3.3, we can choose nonzero elements $a \in A_1$ and $b \in L_1$. Let $h \in \mathrm{sp}(R)$. Then $R_h \neq 0$ so $aR_hb \neq 0$ (c.f. Lemma 3.19 (ii)). But $aR_hb \subseteq AL \cap R_h = B_h$, so $B_h \neq 0$ and Lemma 3.14, implies that $h \in \Gamma_R$ as required. Conversely, let $\Gamma_R = \mathrm{sp}(R)$ and R be a graded prime. Then by Lemma 2.2, $R^{\triangle}$ is graded prime. Also by Remark 3.18, $\mathrm{sp}\left(R^{\triangle}\right) = \mathrm{sp}(R) \subseteq \Gamma_R^{\triangle}$ and by Lemma 3.16, for a strongly graded semiring, we have $\Gamma_R \subseteq \Gamma_R^{\triangle}$ and obviously $\Gamma_R^{\triangle} \subseteq \mathrm{sp}\left(R^{\triangle}\right)$. Hence $\Gamma_R^{\triangle} = \mathrm{sp}\left(R^{\triangle}\right)$. So, by [4, Proposition 2.11]), $\left(R^{\triangle}\right)_1 \cong (R_1)^{\triangle}$ is prime. Therefore by Lemma 2.2 (i), R_1 is prime. ■

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