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A Unified Proof of Classical Ramanujan's Identities and Jacobi's Four Square Theorem

Bhaskar Srivastava

Department of Mathmatics and Astronomy, Lucknow University, Lucknow - 226 007, India

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Abstract. By using a simple theta function identity we have tried to give a unified proof of Ramanujan's classical identities and Jacobi's famous four square theorem.

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1. Introduction

As the title says, we give a unified proof of some identities of Ramanujan and the Jacobi's four square theorem by using a single simple theta function identity (5) stated later in the section.

In the literature [1, 3, 4, 5, 7, 8, 9, 11] different proofs are available, but I feel my proof is simpler and as pointed out, unified, as they all emanate from a simple theta function identity. Moreover, my endeavour is just to show the efficacy of this simple identity and have all the proofs at one place.

I shall prove the following identities:

$$\frac{f^9(-q)}{f^3(-q^3)} = 1 - 9\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{n^2q^n}{1-q^n}, \tag{1}$$

([2, Entry 18.2.10, eq. (18.2.11), p. 403]).

$$\frac{f^8(-q)}{f^4(-q^2)} = 1 - 8\sum_{n=0}^{\infty}\left[\frac{(4n+1)q^{4n+1}}{1-q^{4n+1}} - \frac{2(4n+2)q^{4n+2}}{1-q^{4n+2}} + \frac{(4n+3)q^{4n+3}}{1-q^{4n+3}}\right], \quad (2)$$

$$\frac{f^5(-q)}{f(-q^5)} = 1 - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}, \quad (3)$$

([2, Entry 18.2.22, p. 406]), and

$$k(\tau)\,(8 + 49h(\tau)) = 8 - 7\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{n^2q^n}{1-q^n}, \quad (4)$$

where $h(\tau) = \frac{\eta^4(7\tau)}{\eta^4(\tau)}$, $k(\tau) = \frac{\eta^7(\tau)}{\eta(7\tau)}$ and $\left(\frac{n}{m}\right)$ denotes the Legendre symbol.

The second identity is the well known classical identity due to Jacobi, and the result was also stated by Ramanujan.

This identity is a slight modification of the Jacobi's four square theorem. Hirschhorn [5, 7] gave a simple proof of this identity. In [12], I used Bailey's ${}_6\psi_6$ summation formula to prove this identity.

In proving these identities (1)-(4) the following simple identity [10, Eq.(8.1), p. 117] and its corollary are used:

$$\cot^2 y - \cot^2 x + 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}(\cos 2nx - \cos 2ny)$$
$$= \theta_1^{'}(0|q)^2\,\frac{\theta_1(x-y|q)\theta_1(x+y|q)}{\theta_1^2(x|q)\theta_1^2(y|q)}. \quad (5)$$

There is a slight misprint which I have corrected.

Corollary 1.1. *Differentiating partially* (5) *with respect to x and then putting $y = x$, we have*

$$2\cot x\,\mathrm{cosec}^2\,x - 16\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}\sin 2nx = \theta_1^{'}(0|q)^3\,\frac{\theta_1(2x|q)}{\theta_1^4(x|q)}. \quad (6)$$

2. Some Basic Results

We shall use the following standard q-notation, $|q| < 1$:

$$(a;q^k)_n = (1-a)(1-aq^k)\ldots(1-aq^{k(n-1)}), n \geq 1$$
$$(a;q^k)_0 = 1.$$

Jacobi theta function $\theta_1(z|q)$ is defined by [13, p. 464]:

$$\theta_1(z|q) = -iq^{\frac{1}{8}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} e^{(2n+1)iz}$$
$$= 2q^{\frac{1}{8}} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \sin(2n+1)z,$$

where $q = e^{2\pi i\tau}$ and $\text{Im}(\tau) > 0$.

The function $\theta_1(z|q)$ can also be expressed in terms of an infinite product

$$\theta_1(z|q) = 2q^{\frac{1}{8}} \sin z(q;q)_\infty(qe^{2iz};q)_\infty(qe^{-2iz};q)_\infty \tag{7}$$
$$= iq^{\frac{1}{8}} e^{-iz}(q;q)_\infty(e^{2iz};q)_\infty(qe^{-2iz};q)_\infty.$$

From the definition of $\theta_1(z|q)$

$$\theta_1(z+n\pi|q) = (-1)^n\theta_1(z|q). \tag{8}$$

Taking $z = \frac{\pi}{2}$ in (7), we have

$$\theta_1\left(\frac{\pi}{2}|q\right) = 2q^{\frac{1}{8}}(q;q)_\infty(-q;q)^2_\infty. \tag{9}$$

Taking $z = \frac{\pi}{5}$ and $\frac{2\pi}{5}$, respectively, in (7), and using elementary identity

$$\sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4},$$

we have

$$\theta_1\left(\frac{\pi}{5}|q\right)\theta_1\left(\frac{2\pi}{5}|q\right) = \sqrt{5}q^{\frac{1}{4}}(q;q)_\infty(q^5;q^5)_\infty.$$

Differentiating both sides of (7) with respect to z and then putting $z = 0$, we have the identity

$$\theta_1^{'}(0|q) = 2q^{\frac{1}{8}}(q;q)^3_\infty. \tag{10}$$

Ramanujan defined general theta function $f(a,b)$ as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, |ab| < 1,$$

and

$$f(-q) = (q;q)_\infty, |q| < 1.$$

The Dedekind eta-function:

$$\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty. \tag{11}$$

3. Proof of (1)

Setting $x = \frac{\pi}{3}$ in (6), we have

$$\frac{8}{3\sqrt{3}} - 16\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}\left(\sin\frac{2n\pi}{3}\right) = 8q^{\frac{3}{8}}(q;q)_\infty^9\frac{\theta_1\left(\frac{2\pi}{3}|q\right)}{\theta_1^4\left(\frac{\pi}{3}|q\right)} = \frac{8q^{\frac{3}{8}}(q;q)_\infty^9}{\theta_1^3\left(\frac{\pi}{3}|q\right)}. \quad (12)$$

By simple calculation and using the definition of θ_1-function as given in (7), we get

$$\theta_1\left(\frac{\pi}{3}|q\right) = \sqrt{3}q^{\frac{1}{8}}(q^3;q^3)_\infty. \quad (13)$$

Hence (12) can be written as

$$1 - 9\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{n^2q^n}{1-q^n} = \frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^3} = \frac{f^9(-q)}{f^3(-q^3)}, \quad (14)$$

which proves (1).

4. Proofs of (2) and (3)

Proof of (2). Setting $x = \frac{\pi}{4}$ and $y = \frac{2\pi}{4}$ in (5), we have

$$\begin{aligned}-1 + 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\left(\cos\frac{n\pi}{2} - \cos n\pi\right) &= 4q^{\frac{1}{4}}(q;q)_\infty^6\frac{\theta_1(-\frac{\pi}{4}|q)\theta_1(\frac{3\pi}{4}|q)}{\theta_1^2(\frac{\pi}{4}|q)\theta_1^2(\frac{\pi}{2}|q)}\\ &= -\frac{4q^{\frac{1}{4}}(q;q)_\infty^6}{\theta_1^2(\frac{\pi}{2}|q)}\\ &= -\frac{4q^{\frac{1}{4}}(q;q)_\infty^6}{4q^{\frac{1}{4}}(q;q)_\infty^2(-q;q)_\infty^4}\\ &= -\frac{(q;q)_\infty^4}{(-q;q)_\infty^4},\end{aligned}$$

here we have used (8), (9) to simplify the right-hand side.

Writing n-modulo 4, we have

$$\begin{aligned}&1 - 8\sum_{n=0}^{\infty}\left[\frac{(4n+1)q^{4n+1}}{1-q^{4n+1}} - \frac{2(4n+2)q^{4n+2}}{1-q^{4n+2}} + \frac{(4n+3)q^{4n+3}}{1-q^{4n+3}}\right]\\ &= \frac{(q;q)_\infty^4}{(-q;q)_\infty^4} = \frac{f^8(-q)}{f^4(-q^2)},\end{aligned}$$

which proves (2). ∎

Proof of (3). Setting $x = \frac{\pi}{5}$ and $y = \frac{2\pi}{5}$ in (5), we have

$$\cot^2\frac{2\pi}{5}-\cot^2\frac{\pi}{5}+8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\left(\cos\frac{2n\pi}{5}-\cos\frac{4n\pi}{5}\right)$$

$$=\theta_1^{'}(0|q)^2\,\frac{\theta_1(-\frac{\pi}{5}|q)\theta_1(\frac{3\pi}{5}|q)}{\theta_1^2(\frac{\pi}{5}|q)\theta_1^2(\frac{2\pi}{5}|q)}. \tag{15}$$

By elementary calculation we have the following trigonometrical identities

$$\cot^2\frac{\pi}{5}-\cot^2\frac{2\pi}{5}=\frac{4}{\sqrt{5}}$$

and

$$\cos\frac{2n\pi}{5}-\cos\frac{4n\pi}{5}=\frac{\sqrt{5}}{2}\left(\frac{n}{5}\right),$$

where $\left(\frac{n}{5}\right)$ is the Legendre symbol.
Putting these in (15) and using (8) and (10)

$$-\frac{4}{\sqrt{5}}+4\sqrt{5}\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}=-4q^{\frac14}(q;q)_\infty^6\frac{1}{\theta_1(\frac{\pi}{5}|q)\theta_1(\frac{2\pi}{5}|q)}$$

$$=-\frac{4q^{\frac14}(q;q)_\infty^6}{\sqrt{5}q^{\frac14}(q;q)_\infty(q^5;q^5)_\infty},$$

which simplifies to

$$1-5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}=\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}=\frac{f^5(-q)}{f(-q^5)},$$

which is (3). ■

Liu [9, Eq. (7.20), p. 146] has given another proof using another identity [9, Eq. (7.1), p. 143].

5. Proof of (4)

Liu [10, p.117-118] proved this identity using (6). For completeness, I give a brief outline of his proof. There are slight misprints, which have been corrected.

Taking $x=\frac{\pi}{7},\frac{2\pi}{7}$ and$-\frac{3\pi}{7}$, respectively, in (6) we have

$$s-16\sum_{n=1}^{\infty}s(n)\frac{n^2q^n}{1-q^n}=\theta_1^{'}(0|q)^3\left(\frac{\theta_1(\frac{2\pi}{7}|q)}{\theta_1^4(\frac{\pi}{7}|q)}-\frac{\theta_1(\frac{\pi}{7}|q)}{\theta_1^4(\frac{3\pi}{7}|q)}+\frac{\theta_1(\frac{3\pi}{7}|q)}{\theta_1^4(\frac{2\pi}{7}|q)}\right), \tag{16}$$

where

$$s=2\cot\left(\frac{\pi}{7}\right)\operatorname{cosec}^2\left(\frac{\pi}{7}\right)+2\cot\left(\frac{2\pi}{7}\right)\operatorname{cosec}^2\left(\frac{2\pi}{7}\right)-2\cot\left(\frac{3\pi}{7}\right)\operatorname{cosec}^2\left(\frac{3\pi}{7}\right)$$

and

$$s(n) = \sin\left(\frac{2n\pi}{7}\right) + \sin\left(\frac{4n\pi}{7}\right) - \sin\left(\frac{6n\pi}{7}\right).$$

Setting $q = 0$ in (16) and using [10, Eq. (1.32), p. 107]

$$s = \frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^4\left(\frac{\pi}{7}\right)} - \frac{\sin\left(\frac{\pi}{7}\right)}{\sin^4\left(\frac{3\pi}{7}\right)} + \frac{\sin\left(\frac{3\pi}{7}\right)}{\sin^4\left(\frac{2\pi}{7}\right)} = \frac{64}{7}\sqrt{7}. \tag{17}$$

From [9, Eq. (7.18), p. 145] we know that

$$s(n) = \sin\left(\frac{2n\pi}{7}\right) + \sin\left(\frac{4n\pi}{7}\right) - \sin\left(\frac{6n\pi}{7}\right) = \frac{\sqrt{7}}{2}\left(\frac{n}{7}\right). \tag{18}$$

Putting (17) and (18) in (16), we have (4).

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