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Existence Results for Impulsive Quasilinear Integrodifferential Equations in Banach Spaces

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Abstract. We study the existence and uniqueness of mild solutions for a quasilinear integrodifferential equation with nonlocal and impulsive conditions in Banach spaces. The results are obtained by using the Schauder fixed point theorem. Examples are provided to illustrate the theory.

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Key words: Quasilinear integrodifferential equation, mild solution, fixed point theorem, impulsive conditions.

1. Introduction

Many physical systems are characterized by the fact that at certain moments of time they experience a sudden change of their state. For example, when a mass on a spring is given a blow by a hammer, it experiences a sharp change of velocity, and a pendulum is a mechanical clock undergoes a drastic increase of momentum everytime when it crosses its equilibrium position. These systems are subject to short-term perturbations which are often assumed to be in the form of impulses in the modeling process. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus differential equations involving impulsive effects appear as a natural description of observed evolution phenomena of several real world problems. Recently, the study of impulsive differential and integrodifferential equations has attracted a great deal of attention. The theory of impulsive differential equations is an important branch of differential equations [7, 9, 11, 12, 14, 15].

Another direction is to consider the nonlocal condition

$$u(0) + g(u) = u_0, \tag{1}$$

where g is a mapping from some space of functions so that it constitutes a nonlocal conditions have better effects in applications than traditional initial value problems [1, 2, 3, 4, 6, 9, 15]. Kato [8] studied the nonhomogeneous evolution equations in Banach spaces. Several authors have studied the existence of solutions of abstract quasilinear evolution equations in Banach spaces [2, 8, 13].

In this paper, we study the following nonlocal impulsive problems for quasilinear integrodifferential equation of the form

$$u'(t) + A(t,u)u(t) = f\left(t, u(t), \int_0^t h(t,s,u(s))ds\right), \tag{2}$$

$$u(0) + g(u) = u_0, \quad t \in [0,a] \tag{3}$$

$$\Delta u(t_i) = I_i(u(t_i)), \quad i = 1, \dots, m,\ 0 < t_1 <, \dots t_m < a. \tag{4}$$

Let $A(t,u)$ be the generator of a strongly continuous semigroup $U_u(\cdot)$ in a Banach space $(X; ||\cdot||)$. Let $\mathcal{PC}([0,a]; X)$ consist of functions u from $[0,a]$ into X, such that $u(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$, and the right limit $u(t_i^+)$ exists for $i = 1, 2, 3, \dots, m$. Evidently $\mathcal{PC}([0,a], X)$ is a Banach space with the norm

$$\|u\|_{\mathcal{PC}} = \sup_{t\in[0,a]} \|u(t)\|.$$

Let $u_0 \in X$, $f : [0,a] \times X \times X \to X$, $h : \Omega \times X \to X$, $g : \mathcal{PC}([0,a] : X) \to X$ and $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitutes an impulsive condition. Here $\Omega = \{(t,s) : 0 \le s \le t \le a\}$.

2. Preliminaries

We shall make the following assumptions: Let $B(X)$ be the Banach space of all bounded linear operators on X. Denote

$$M_0 = \sup_{t\in[0,a]} \|U_u(t,s)\|_{B(X)},$$

which is a finite number.

(A_1) $f : [0,a] \times X \times X \to X$ is continuous and there exist constants $F_A > 0$ and $F_0 > 0$ such that

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le F_A\Big(\|u_1-u_2\| + \|v_1-v_2\|\Big), \quad u,v \in X,$$
$$F_0 = \max_{t\in[0,a]} \|f(t,0,0)\|.$$

(A_2) $h : [0,a]\times[0,a]\times X \to X$ is continuous and there exist constants $H_A > 0$ and $H_0 > 0$ such that

$$\int_0^t \|h(t,s,u) - h(t,s,v)\|\, ds \le H_A\|u-v\|,$$
$$H_0 = \max\left\{\int_0^t \|h(t,s,0)\|ds : (t,s) \in [0,a]\right\}.$$

(A_3) $g : \mathcal{PC}([0,a] : X)$ is Lipschitz continuous in X and there exists a constant $G_A > 0$ such that

$$\|g(u) - g(v)\| \le G_A\|u-v\|_{\mathcal{PC}}, \quad u,v \in \mathcal{PC}([0,a]:X).$$

(A_4) $I_i : X \to X$ is continuous and there exists a constant $l_i > 0$, $i = 1,2,\ldots,m$ such that

$$\|I_i(u) - I_i(v)\| \le l_i\|u-v\| \quad u,v \in X.$$

(A_5) Let $\rho = \Big\{K_0 a\Big[\|u_0\| + G_A r + \|g(0)\| + a\Big(F_A\Big[r(1+H_A) + H_0\Big] + F_0\Big) + \sum_{i=1}^m (l_i r + \|I_i(0)\|)\Big] + M_0\Big[G_A + a\Big(F_A(1+H_A)\Big) + \sum_{i=1}^m l_i\Big]\Big\}$ be such that $0 < \rho < 1$.

From [13], we know that for any fixed $u \in \mathcal{PC}([0,a]:X)$ there exists a unique continuous function $U_u : [0,a]\times[0,a] \to B(X)$ defined on $[0,a]\times[0,a]$ such that

$$U_u(t,s) = I + \int_s^t A_u(w)U_u(w,s)dw,$$

where $B(X)$ denote the Banach space of bounded linear operators from X to X with the norm $||F|| = \sup\{||Fu|| : ||u|| = 1\}$, and I stands for the identity operator on X, $A_u(t) = A(t,u(t))$. Using the semigroup properties [13], we have

$$U_u(t,t) = I,\ U_u(t,s)U_u(s,r) = U_u(t,r), \quad (t,s,r) \in [0,a]\times[0,a]\times[0,a],$$
$$\frac{\partial U_u(t,s)}{\partial t} = A_u(t)U_u(t,s) \quad \text{for almost all} \quad t,s \in [0,a]. \quad (5)$$

Further there exists a constant K_0 such that for every $u, v \in \mathcal{PC}([0,a]:X)$ and every $y \in X$ we have, $\|U_u(t,s)y - U_v(t,s)y\| \leq K_0 a \|y\|_X \|u - v\|_{\mathcal{PC}}$.

For details of the above mentioned results, we refer the reader to Theorem 6.4.3 and Lemma 6.4.4 in [13].

3. Existence Results

Definition 3.1. A function $u \in \mathcal{PC}([0,a]:X)$ is a *mild solution* of equations (2) - (4) if it satisfies

$$u(t) = U_u(t,0)u_0 - U_u(t,0)g(u) + \int_0^t U_u(t,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) ds + \sum_{0<t_i<t} U_u(t,t_i) I_i(u(t_i)), \quad 0 \leq t \leq a. \tag{6}$$

Under the above assumptions, we can prove the following result.

Theorem 3.2. *Let $u_0 \in X$ and let $\mathcal{B}_r = \{u \in \mathcal{PC}([0,a]:X); \|u\| \leq r\}$, $r > 0$. If the assumptions $(A_1) - (A_5)$ are satisfied, then (2) - (4) has a unique mild solution.*

Proof. Let $u_0 \in X$ be fixed. Define an operator $\mathcal{F}$ on $\mathcal{PC}([0,a]:X)$ by

$$(\mathcal{F}u)(t) = U_u(t,0)[u_0 - g(u)] + \int_0^t U_u(t,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) ds + \sum_{0<t_i<t} U_u(t,t_i) I_i(u(t_i)).$$

Then it is clear that $\mathcal{F} : \mathcal{PC}([0,a]:X) \to \mathcal{PC}([0,a]:X)$. Also we have

$$\begin{aligned}
&\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \\
&\leq \|U_u(t,0)u_0 - U_v(t,0)u_0\| + \|U_u(t,0)g(u) - U_v(t,0)g(v)\| \\
&\quad + \int_0^t \Big\| U_u(t,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big)\Big\| ds + \sum_{0<t_i<t} \|U_u(t,t_i) I_i(u(t_i))\| \\
&\quad - \int_0^t \Big\| U_v(t,s) f\Big(s, v(s), \int_0^s h(s,\tau,v(\tau))d\tau\Big)\Big\| ds - \sum_{0<t_i<t} \|U_v(t,t_i) I_i(v(t_i))\|.
\end{aligned}$$

Using the assumptions $(A_1) - (A_4)$, one can get

$$
\begin{aligned}
&\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \\
&\le K_0 a\|u_0\|\|u-v\|_{\mathcal{PC}} + K_0 a[G_A\|u\| + \|g(0)\|]\|u-v\|_{\mathcal{PC}} + M_0 G_A\|u-v\|_{\mathcal{PC}} \\
&\quad + K_0 a\|u-v\|_{\mathcal{PC}} \int_0^t \Big\| f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) - f(s,0,0) + f(s,0,0)\Big\| ds \\
&\quad + M_0 \int_0^t \Big[\Big\| f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) - f\Big(s, v(s), \int_0^s h(s,\tau,v(\tau))d\tau\Big)\Big\|\Big] ds \\
&\quad + \sum_{0<t_i<t} \|U_u(t,t_i) - U_v(t,t_i)\|\, \|I_i(u(t_i))\| + \sum_{0<t_i<t} \|U_v(t,t_i)\|\|I_i(u(t_i)) - I_i(v(t_i))\| \\
&\le \Big\{ K_0 a\Big[\|u_0\| + G_A r + \|g(0)\| + a\Big(F_A\Big[r(1+H_A) + H_0\Big] + F_0\Big) \\
&\quad + \sum_{i=1}^m (l_i r + \|I_i(0)\|)\Big] + M_0\Big[G_A + a\Big(F_A(1+H_A)\Big) + \sum_{i=1}^m l_i\Big]\Big\}\|u-v\|_{\mathcal{PC}} \\
&= \rho\ \|u-v\|_{\mathcal{PC}}, \quad u, v \in \mathcal{PC}([0,a]:X).
\end{aligned}
$$

From this inequality it follows that for any $t \in [0,a]$,

$$
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \le \rho\|u-v\|_{\mathcal{PC}}.
$$

■

Remark 3.3. In the case where $I_i's$ are constants, one has $l_i = 0,\ i = 1, 2, \ldots, m$. So we only need

$$
K_0 a\Big[\|u_0\| + G_A r + \|g(0)\| + a\Big(F_A[r(1+H_A) + H_0] + F_0\Big)\Big] + M_0\Big[G_A + a\Big(F_A(1+H_A)\Big)\Big] < 1
$$

in assumption (A_5).

Remark 3.4. We will derive mild solutions under the following assumptions.

(E_1) The function f is continuous on $[0,a]$ and there exists a constant $F_C > 0$ such that

$$
\Big\| f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big)\Big\| \le F_C.
$$

(E_2) $g : \mathcal{PC}([0,a] : X) \to X$ and $I_i : X \to X, i = 1, 2, \ldots, m$ are compact operators and $U_u(\cdot)$ is also compact ($U_u(t,s)$ is a compact operator for any $t > 0$).

(E_3) For each $u_0 \in X$, there exists a positive constant $r > 0$ such that

$$
M_0\Big\{\|u_0\| + \sup_{\phi\in\mathcal{B}_r}\|g(\phi)\| + aF_C + \sup_{\phi\in\mathcal{B}_r}\sum_{i=1}^m \|I_i(\phi(t_i))\|\Big\} \le r,
$$

where $\mathcal{B}_r := \{\phi \in \mathcal{PC}([0,a]:X) : ||\phi(t)|| \le r \text{ for } t \in [0,a]\}$.

Theorem 3.5. *Let $(E_1) - (E_3)$ be satisfied. Then for every $u_0 \in X$, the problem* (2) - (4) *has at least a mild solution.*

Proof. Let $u_0 \in X$ be fixed. Define an operator $\mathcal{H}$ on $\mathcal{PC}([0,a]:X)$ by

$$\begin{aligned}(\mathcal{H}u)(t) &= U_u(t,0)[u_0 - g(u)] + \int_0^t U_u(t,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big)ds \\ &\quad + \sum_{0<t_i<t} U_u(t,t_i) I_i(u(t_i)), \\ (\mathcal{H}u)(t) &= (\mathcal{H}_1 u)(t) + (\mathcal{H}_2 u)(t),\end{aligned}$$

where

$$\begin{aligned}(\mathcal{H}_1 u)(t) &= U_u(t,0)[u_0 - g(u)] + \int_0^t U_u(t,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big)ds \\ (\mathcal{H}_2 u)(t) &= \sum_{0<t_i<t} U_u(t,t_i) I_i(u(t_i)), \quad 0 \le t \le a.\end{aligned}$$

From our assumptions, $\mathcal{H}$ is a continuous mapping from $\mathcal{B}_r$ to $\mathcal{B}_r$. Thus, for applying Schauder's fixed point theorem, we have to prove that $\mathcal{H}$ is a compact operator, or $\mathcal{H}_1$ and $\mathcal{H}_2$ are both compact operators.

Let

$$\begin{aligned}(\mathcal{H}_2 u)(t) &= \sum_{0<t_i<t} U_u(t,t_i) I_i(u(t_i)) \\ &= \begin{cases} 0 & \text{if } \ t \in [0,t_1], \\ U_u(t,t_1) I_1(u(t_1)) & \text{if } \ t \in (t_1,t_2], \\ \dots & \\ \sum_{i=1}^m U_u(t,t_i) I_i(u(t_i)) & \text{if } \ t \in (t_m,a] \end{cases}\end{aligned}$$

and that the interval $[0,a]$ is divided into finite subintervals by $t_i, \ i = 1,2,3,\dots,m$, so that we only need to prove that

$$\mathcal{Z} = \{U_u(\cdot,t_1) I_1(u(t_1)) : \cdot \in [t_1,t_2], u \in \mathcal{B}_r\}$$

be precompact in $\mathcal{PC}([t_1,t_2]:X)$, as the cases for other subintervals are the same.

From the above assumption, we see that for each $t \in [t_1,t_2]$, the set

$$\{U_u(t,t_1) I_1(u(t_1)) : u \in \mathcal{B}_r\}$$

is precompact in X. Using semigroup property, $t_1 \le s \le t \le t_2$ and the equation (5) and we get

$$\begin{aligned} &\|U_u(t,t_1)I_1(u(t_1)) - U_u(s,t_1)I_1(u(t_1))\| \\ &= \|U_u(s,t_1)[U_u(t,s) - U_u(t,t)]I_1(u(t_1))\| \\ &\le M_0\|[U_u(t,s) - U_u(t,t)]I_1(u(t_1))\| \end{aligned} \tag{7}$$

for any $t, s \in [0,a]$. Thus, the functions in $\mathcal{Z}$ are equicontinuous due to the compactness of I_1 and the strong continuity of $U_u(\cdot)$. An application of the Arzela-Ascoli's theorem justifies the precompactness of $\mathcal{Z}$. Therefore, $\mathcal{H}_2$ is a compact operator.

The same argument can be used to prove the compactness of $\mathcal{H}_1$. That is, for each $t \in [0,a]$, the set

$$\{U_u(t,0)[u_0 - g(u)] : u \in \mathcal{B}_r\}$$

is precompact in X, since g is compact. In [13], using the semigroup properties (5), we have

$$\begin{aligned} &\left\{ \int_0^{t-\varepsilon} U_u(t,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) ds : u \in \mathcal{B}_r \right\} \\ &= \left\{ \int_0^{t-\varepsilon} U_u(t,t-\varepsilon) U_u(t-\varepsilon,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) ds : u \in \mathcal{B}_r \right\} \\ &= \left\{ U_u(t,t-\varepsilon) \int_0^{t-\varepsilon} U_u(t-\varepsilon,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) ds : u \in \mathcal{B}_r \right\} \end{aligned}$$

is precompact in X, since U_u is compact. Then

$$\begin{aligned} &U_u(t,t-\varepsilon) \int_0^{t-\varepsilon} U_u(t-\varepsilon,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) ds \\ &\to \int_0^{t} U_u(t,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) ds, \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

We conclude that $\left\{ \int_0^t U_u(t,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) ds : u \in \mathcal{B}_r \right\}$ is precompact in X, using the total boundeness. Therefore, for each $t \in [0,a]$, $\{(\mathcal{H}_1 u)(t) : u \in \mathcal{B}_r\}$ is precompact in X.

Next we show the equicontinuity of $\mathcal{W} = \{(\mathcal{H}_1 u)(\cdot) : \cdot \in [0,a],\ u \in \mathcal{B}_r\}$. The equicontinuity of

$$\{U_u(\cdot, s)[u_0 - g(u)] : \cdot \in [0, a], \quad u \in \mathcal{B}_r\}$$

can be shown using the condition (7). For the second term in $\mathcal{W}$. For any $0 \leq s_1 < s_2 \leq a$, we have

$$\begin{aligned}
&\|(\mathcal{H}_1 u)(s_2) - (\mathcal{H}_1 u)(s_1)\| \\
&\leq \| \int_0^{s_2} U_u(s_2, s) f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)ds \\
&\quad - \int_0^{s_1} U_u(s_1, s) f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)ds\| \\
&\leq \| \int_0^{s_1} [U_u(s_2, s) - U_u(s_1, s)] f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)ds \\
&\quad + \int_{s_1}^{s_2} U_u(s_2, s) f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)ds\| \\
&\leq \| \int_0^{s_1} [U_u(s_2, s) - U_u(s_1, s)] f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)ds \\
&\quad + M_0 \int_{s_1}^{s_2} \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\|ds. \qquad (8)
\end{aligned}$$

If $s_1 = 0$, then the right-hand side of (8) can be made small when s_2 is small independently of $u \in \mathcal{B}_r$. If $s_1 > 0$, then we can find a small number $\eta > 0$ so that if $s_1 \leq \eta$, then the right-hand side of (8) can be estimated as

$$\begin{aligned}
&\|(\mathcal{H}_1 u)(s_2) - (\mathcal{H}_1 u)(s_1)\| \\
&\leq \int_0^{s_1} \|U_u(s_2, s) - U_u(s_1, s)\| \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\|ds \\
&\quad + M_0 \int_{s_1}^{s_2} \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\|ds \\
&\leq 2\eta M_0 \max_{s \in [0,a]} \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\| \\
&\quad + M_0 \int_{s_1}^{s_2} \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\|ds,
\end{aligned}$$

which can be made small when $s_2 - s_1$ is small independently of $u \in \mathcal{B}_r$. If $s_1 > \eta$, then the right-hand side of (8) can be estimated as

$$\int_0^{s_1} \|U_u(s_2, s) - U_u(s_1, s)\| \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\| ds$$

$$+M_0 \int_{s_1}^{s_2} \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\| ds.$$

$$\le \int_0^{s_1-\eta} \|U_u(s_2, s) - U_u(s_1, s)\| \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\| ds$$

$$+ \int_{s_1-\eta}^{s_1} \|U_u(s_2, s) - U_u(s_1, s)\| \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\| ds$$

$$+M_0 \int_{s_1}^{s_2} \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\| ds$$

$$\le \int_0^{s_1-\eta} F_C \|U_u(s_2, s) - U_u(s_1, s)\| ds + 2\eta M_0 F_C + M_0 F_C (s_1 - s_2).$$

Now, as $U_u(\cdot)$ is compact, $U_u(t, 0)$ is operator norm continuous for $t > 0$. Thus $U_u(t, 0)$ is operator norm continuous uniformly for $t \in [\eta, a]$. Therefore, $\|U_u(s_2, s) - U_u(s_1, s)\|$ and hence

$$\int_0^{s_1-\eta} \|U_u(s_2, s) - U_u(s_1, s)\| \|f\Big(s, u(s), \int_0^s h(s, \tau, u(\tau))d\tau\Big)\| ds$$

can be made small when $s_2 - s_1$ is small independently of $u \in \mathcal{B}_r$. Accordingly, we see that the functions in $\mathcal{W}$ are equicontinuous. Therefore, $\mathcal{H}_1$ is a compact operator by the Arzela-Ascoli theorem, and hence $\mathcal{H}$ is also a compact operator. Now, Schauder's fixed point theorem implies that $\mathcal{H}$ has a fixed point, which gives rise to a mild solution. This completes the proof. ■

4. Existence results without compactness on *g*

In many studies of nonlocal Cauchy problems, the mapping g is given by

$$g(t_1, \ldots, t_p, u(t_1, \ldots, u(t_p)), \tag{9}$$

where $0 < t_1 < t_2 <, \ldots, t_p \le a$. For example, see [5, 6, 12], where in [6],

$$g(t_1,\dots,t_p,u(t_1,\dots,u(t_p)) = \sum_{i=1}^{p} c_i u(t_i) \tag{10}$$

(for some given constants c_i) is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In these cases, $u(0) + g(t_1,\dots,t_p, u(t_1,\dots,u(t_p))$ allows the measurements at $t = 0, t_1,\dots,t_p$, rather than just at $t = 0$. So more information is available. Note that such a g is completely determined on $[\mu, a]$ for some small $\mu > 0$; i.e., such a g ignores $t = 0$. We will generalize this idea and formulate a related condition for a general mapping g. Here, we will establish mild solutions under the following assumptions.

(H_1) $I_i : X \to X$, $i : 1, 2, \dots, m$ are compact operators and $U_u(\cdot)$ is also compact ($U_u(t,s)$ is a compact operator for any $t > 0$).

(H_2) $g : \mathcal{PC}([0,a] : X) \to X$ is continuous, maps $\mathcal{B}_r$ into a bounded set, and there is a $\mu = \mu(r) \in (0, t_1)$ such that $g(\phi) = g(\chi)$ for any ϕ, $\chi \in \mathcal{B}_r$ with $\phi(s) = \chi(s), s \in [\mu, a]$.

(H_3) For each $u_0 \in X$, there exists a positive constant $r > 0$ such that

$$M_0\Big\{\|u_0\| + \sup_{\phi\in\mathcal{B}_r}\|g(\phi)\| + a\sup_{s\in[0,a]\ \phi\in\mathcal{B}_r}\Big\|f(s,u(s),\int_0^s h(s,\tau,u(\tau))d\tau)\Big\| + \sup_{\phi\in\mathcal{B}_r}\sum_{i=1}^m \|I_i(\phi(t_i))\|\Big\} \le r,$$

(H_4) There exists a constant $k > 0$ such that

$$M_0\Big[\|u_0 + \sup_{\phi\in\mathcal{B}_r}\|g(\phi)\| + F_A H_0 a + F_0 a + \sum_{i=1}^m l_i[\|v\| + \|I_i(0)\|]\Big] L_0 < k,$$

where $L_0 = e^{M_0 F_A (1+H_A)a}$.

Under these assumptions, we can prove the following result.

Theorem 4.1. *Let* (A_1) *and* $(H_1) - (H_4)$ *be satisfied. Then for every* $u_0 \in X$, *the problem* (2) - (4) *has at least a mild solution.*

Proof. For $\mu = \mu(r) \in (0, t_1)$, set

$$\mathcal{B}_c(\mu) := \mathcal{PC}([\mu,a],X) = \text{ restrictions of functions in } \mathcal{PC}([0,a]:X) \text{ on } [\mu,a],$$

$$\mathcal{B}_r(\mu) := \{\phi \in \mathcal{B}_c(\mu) : \|\phi(t)\| \le r \text{ for } t \in [\mu,a]\}.$$

For $v \in \mathcal{B}_r(\mu)$ fixed, we define a mapping $\mathcal{Q}_u$ on $\mathcal{B}_r$ by

$$(\mathcal{Q}_v\phi)(t) = U_v(t,0)[u_0 - g(\tilde{v})] + \int_0^t U_v(t,s) f\Big(s,\phi(s),\int_0^s h(s,\tau,\phi(\tau))d\tau\Big)ds + \sum_{0<t_i<t} U_v(t,t_i) I_i(v(t_i)), \quad t \in [0,a],$$

where

$$\tilde{v}(t) = \begin{cases} v(t) & \text{if} \quad t \in [\mu, a], \\ v(\mu) & \text{if} \quad t \in [0, \mu]. \end{cases}$$

By assumptions (H_3), the mapping $\mathcal{Q}_u$ maps $\mathcal{B}_r$ into itself. Moreover, by (H_1) we deduce inductively that for $m \in N$,

$$\|\mathcal{Q}_v^m \phi(t) - \mathcal{Q}_v^m \chi(t)\| \leq \frac{[M_0 F_A a(1 + H_A)]^m}{m!} \max_{s \in [0,a]} \|\phi(s) - \chi(s)\|,$$

$$t \in [0, a], \ \phi, \chi \in \mathcal{B}_r, \ m = 1, 2, \ldots$$

Hence, we infer that for m large enough, the mapping $\mathcal{Q}_v^m$ is a contractive mapping.

Thus, by a well-known extension of the Banach contraction principle, $\mathcal{Q}_v$ has a unique fixed point $\phi_v \in \mathcal{B}_r$, i.e.,

$$\phi_v(t) = U_v(t, 0)[u_0 - g(\tilde{v})] + \int_0^t U_v(t, s) f\Big(s, \phi_v(s), \int_0^s h(s, \tau, \phi_v(\tau)) d\tau\Big) ds$$
$$+ \sum_{0 < t_i < t} U_v(t, t_i) I_i(v(t_i)), \quad t \in [0, a]. \tag{11}$$

This implies by $(A_0) - (A_4)$ and (H_2) that for each $t \in [0, a]$, we have $\|v(t)\| \leq \|\phi_v(t)\|$.

$$\|\phi_v(t)\| = \|U_v(t, 0)[u_0 - g(\tilde{v})] + \int_0^t U_v(t, s) f\Big(s, \phi_v(s), \int_0^s h(s, \tau, \phi_v(\tau)) d\tau\Big) ds$$
$$+ \sum_{0 < t_i < t} U_v(t, t_i) I_i(v(t_i))\|.$$
$$\leq M_0 \Big[\|u_0\| + \sup_{\phi \in \mathcal{B}_r} \|g(\phi)\| + F_A H_0 a + F_0 a + \sum_{i=1}^m l_i \|v\| + \|I_i(0)\|\Big]$$
$$+ M_0 F_A [1 + H_A] \int_0^t \|\phi_v(s)\| ds.$$

Making use of the Gronwall-Bellman's inequality, such that

$$\sup_{t \in [0,a]} \|\phi_v(t)\| \leq \Big[M_0 \Big(\|u_0\| + \sup_{\phi \in \mathcal{B}_r} \|g(\phi)\| + F_A H_0 a + F_0 a + \sum_{i=1}^m l_i [\|v\| + \|I_i(0)\|]\Big)\Big]$$
$$\times e^{M_0 F_A (1 + H_A) a}$$

and the above inequality holds, consequently

$$\|v\| \leq \Big[M_0\Big(\|u_0\|+\sup_{\phi\in\mathcal{B}_r}\|g(\phi)\|+F_AH_0a+F_0a+\sum_{i=1}^{m}l_i[\|v\|+\|I_i(0)\|]\Big)\Big]e^{M_0F_A(1+H_A)a}.$$

and therefore

$$\frac{\|v\|}{M_0\Big[\|u_0\|+\sup_{\phi\in\mathcal{B}_r}\|g(\phi)\|+F_AH_0a+F_0a+\sum_{i=1}^{m}l_i[\|v\|+\|I_i(0)\|]\Big]L_0} \leq 1,$$

where $L_0 = e^{M_0F_A(1+H_A)a}$.

From (H_4), there exists k such that $\|v\| \neq k$. Set

$$\mathcal{B}_r^* = \{v \in \mathcal{PC}([\mu,a]:X);\ \sup_{\mu\leq t\leq a}\|v(t)\| < k\}.$$

Based on this fact, we define a mapping $\mathcal{P}$ from $\mathcal{B}_r(\mu)$ into itself by

$$(\mathcal{P}v)(t) = \phi_v(t) = v(t_i), \quad t \in [\mu,a]$$

$$\phi_{v_1}(t) = U_v(t,0)[u_0 - g(\tilde{v_1})] + \int_0^t U_v(t,s)f\Big(s,\phi_{v_1}(s),\int_0^s h(s,\tau,\phi_{v_1}(\tau))d\tau\Big)ds + \sum_{0<t_i<t}U_v(t,t_i)I_i(v_1(t_i))$$

and

$$\phi_{v_2}(t) = U_v(t,0)[u_0 - g(\tilde{v_2})] + \int_0^t U_v(t,s)f\Big(s,\phi_{v_2}(s),\int_0^s h(s,\tau,\phi_{v_2}(\tau))d\tau\Big)ds + \sum_{0<t_i<t}U_v(t,t_i)I_i(v_2(t_i)), \quad t\in[0,a].$$

From (11), we deduce that for $t \in [\mu,a]$, $v_1, v_2 \in \mathcal{B}_r(\mu)$. We show that $\mathcal{P}$ is continuous.

$$\begin{aligned}\|\ \phi_{v_1}(t) - \phi_{v_2}(t)\| &\\ \leq \|U_v(t,0)(g(\tilde{v_1}) - g(\tilde{v_2}))\| &+ \Big\|\int_0^t U_v(t,s)\Big[f\Big(s,\phi_{v_1}(s),\int_0^s h(s,\tau,\phi_{v_1}(\tau))d\tau\Big)\\ &-f\Big(s,\phi_{v_2}(s),\int_0^s h(s,\tau,\phi_{v_2}(\tau))d\tau\Big)\Big]ds\Big\|\\ &+\sum_{0<t_i<t}\Big\|U_v(t,t_i)\Big[I_i(v_1(t_i)) - I_i(v_2(t_i))\Big]\Big\|\end{aligned}$$

$$\leq M_0\|(g(\tilde{v_1}) - g(\tilde{v_2}))\| + M_0 \int_0^t F_A\Big[\|\phi_{v_1}(s) - \phi_{v_2}(s)\|$$

$$+\|\int_0^s [h(s,\tau,\phi_{v_1}(\tau)) - h(s,\tau,\phi_{v_2}(\tau))]d\tau\|\Big]ds + M_0\sum_{i=1}^m l_i\Big[\|(v_1(t_i)) - (v_2(t_i))\|\Big]$$

$$\leq M_0\|(g(\tilde{v_1}) - g(\tilde{v_2}))\| + M_0F_A\int_0^t \|\phi_{v_1}(s) - \phi_{v_2}(s)\|ds$$

$$+M_0F_AH_A\int_0^t \|\phi_{v_1}(s) - \phi_{v_2}(s)\|ds + M_0\sum_{i=1}^m l_i\|(v_1(t_i)) - (v_2(t_i))\|.$$

This gives, by Gronwall-Bellman's inequality, that for t, v_1 and v_2 as above

$$\|\phi_{v_1}(t) - \phi_{v_2}(t)\| \leq M_0 e^{M_0[F_A a(1+H_A)+\sum_{i=1}^m l_i]}\|(g(\tilde{v_1}) - g(\tilde{v_2}))\|.$$

Therefore

$$\|(\mathcal{P}v_1)(t) - (\mathcal{P}v_2)(t)\| \leq M_0 e^{M_0[F_A a(1+H_A)+\sum_{i=1}^m l_i]}\|(g(\tilde{v_1}) - g(\tilde{v_2}))\|,$$
$$t \in [\mu, a], \quad v_1, v_2 \in \mathcal{B}_r(\mu). \tag{12}$$

This means that $\mathcal{P}$ is continuous. Next we show that $\mathcal{P}$ maps $\mathcal{B}_r(\mu)$ into a precompact subset of $\mathcal{B}_r(\mu)$. By the compactness and the norm continuity of $\{U_v(t,0)\}_{0<t\leq a}$, we deduce that for each $t \in [\mu, a]$ and $\varepsilon \in (0,t)$, the set

$$\{U_v(t,0)(u_0 - g(\tilde{v})) : v \in \mathcal{B}_r(\mu)\}$$

is precompact in X and that the family of functions on $[\mu, a]$

$$\{U_v(\cdot,0)(u_0 - g(\tilde{v})) : v \in \mathcal{B}_r(\mu)\} \tag{13}$$

is equicontinuous because the set $\{g(\tilde{v}) : v \in \mathcal{B}_r(\mu)\}$ is bounded by assumption (H_2). Moreover, by (A_2),

$$\left\{ f\Big(t, \phi(t), \int_0^t h(t,s,\phi(s))ds : t \in [0,a], \quad \phi \in \mathcal{B}_r \right\}$$

is bounded. Thus the set

$$\left\{ \int_0^t U_v(t,s) f\Big(s, \phi_v(t), \int_0^t h(s,\tau,\phi_v(\tau))d\tau ds : t \in [0,a] \quad \phi \in \mathcal{B}_r(\mu) \right\}$$

is precompact in X and so is the set $\{(\mathcal{P}v)(t) : v \in \mathcal{B}_r(\mu)\}$ due to (11) and the definition of $\mathcal{P}$. On the other hand, for each $\delta > 0$, there exists $\sigma > 0$ such that

$$\|U_v(t,0) - U_v(s,0))(u_0 - g(\tilde{v})\| < \frac{1}{5}\delta$$

is valid for all $v \in \mathcal{B}_r(\mu)$, $\quad t, s \in [\mu, a]$ with $|t-s| < \sigma$, by assertion (13). Also, there exists an $\varepsilon \in (0,\mu)$ such that

$$2\varepsilon M_0 \sup_{[0,a],\ \phi\in\mathcal{B}_r} \Big\| f(t, \phi_v(t), \int_0^s h(s,\tau,\phi_v(\tau))d\tau \Big\| < \frac{1}{5}\delta$$

and

$$\sum_{i=1}^{m} \Big\| I_i(v(t_i)) \Big[U_v(t,t_i) - U_v(s,t_i)) \Big] \Big\| < \frac{1}{5}\delta.$$

Thus, from the norm continuity of $\{U_u(t,0)\}_{0<t\le a}$ it follows that there exists an $\eta \in (0, \min\{\varepsilon,\sigma\})$ such that

$$\begin{aligned}
&\|(\mathcal{P}v)(t) - (\mathcal{P}v)(s)\| \\
\le\ & \|(U_v(t,0) - U_v(s,0))(u_0 - g(\tilde{v}))\| \\
&+ \Big\| \int_0^t U_v(t,\tau) f\Big(\tau, \phi_v(\tau), \int_0^\tau h(\tau,\gamma,\phi_v(\gamma))d\gamma\Big) d\tau \\
&- \int_0^s U_v(s,\tau) f(\tau, \phi_v(\tau), \int_0^\tau h(\tau,\gamma,\phi_v(\gamma))d\gamma\Big) d\tau \Big\| \\
&+ \sum_{i=1}^{m} \Big\| I_i(v(t_i)) \Big[U_v(t,t_i) - U_v(s,t_i)) \Big] \Big\| \\
\le\ & \|(U_v(t,0) - U_v(s,0))(u_0 - g(\tilde{v}))\| \\
&+ \Big\| \int_0^t \Big[U_v(t,\tau) - U_v(s,\tau) \Big] f\Big(\tau, \phi_v(\tau), \int_0^\tau h(\tau,\gamma,\phi_v(\gamma))d\gamma\Big) d\tau \Big\| \\
&+ \Big\| \int_t^s U_v(t,\tau) f\Big(\tau, \phi_v(\tau), \int_0^\tau h(\tau,\gamma,\phi_v(\gamma))d\gamma\Big) d\tau \Big\| \\
&+ 2M_0 \sum_{i=1}^{m} l_i[\|v\| + \|I_i(0)\|] \\
\le\ & \frac{1}{5}\delta + \int_0^{t-\varepsilon} \Big\| \Big[U_v(t,\tau) - U_v(s,\tau) \Big] f\Big(\tau, \phi_v(\tau), \int_0^\tau h(\tau,\gamma,\phi_v(\gamma))d\gamma\Big) \Big\| d\tau \\
&+ 2M_0 \int_{t-\varepsilon}^{t} \Big\| f\Big(\tau, \phi_v(\tau), \int_0^\tau h(\tau,\gamma,\phi_v(\gamma))d\gamma\Big) \Big\| d\tau
\end{aligned}$$

$$+M_0 \int_t^s \Big\| f\Big(\tau, \phi_v(\tau), \int_0^\tau h(\tau,\gamma,\phi_v(\gamma))d\gamma\Big) \Big\| d\tau$$

$$+2M_0 \sum_{i=1}^m l_i[\|v\| + \|I_i(0)\|]$$

$$< \delta$$

for every $v \in \mathcal{B}_r(\mu)$, $t, s \in [\mu, a]$ with $0 \le s - t < \eta$ that is, the family of functions $\{(\mathcal{P}v)(\cdot) : v \in \mathcal{B}_r(\mu)\}$ on $[\mu, a]$ is equicontinuous. Now an application of Arzela-Ascoli's theorem justifies the precompactness of $\mathcal{P}(\mathcal{B}_r(\mu))$. It is clear that $\mathcal{B}_r(\mu)$ is a bounded closed convex subset of $\mathcal{B}_c(\mu)$. Therefore, we can make use of Schauder's fixed point theorm to conclude that $\mathcal{P}$ has a fixed point $v_\alpha \in \mathcal{B}_r(\mu)$.

Put $u = \phi_{v_\alpha}$. Then

$$u(t) = U_u(t,0)[u_0 - g(\tilde{v}_\alpha)] + \int_0^t U_u(t,s) f\Big(s, u(s), \int_0^s h(s,\tau,u(\tau))d\tau\Big) ds$$
$$+ \sum_{0<t_i<t} U_u(t,t_i) I_i(v_\alpha), \quad t \in [0,a]. \tag{14}$$

But

$$g(\tilde{v}_\alpha) = g(u) \text{ and } v_\alpha(t_i) = u(t_i).$$

Since

$$v_\alpha(t_i) = (\mathcal{P}v_\alpha)(t) = \phi_{v_\alpha}(t) = u(t), \quad t \in [\mu, a],$$

by the definition of $\mathcal{P}$. This concludes, together with (12), that $u(t)$ is a mild solution of the problem (2)-(4). This completes the proof. ■

5. Examples

Finally, we give two examples to illustrate our abstract results above.

Example 5.1. Consider the partial integrodifferential equation

$$\frac{\partial}{\partial t} z(t,y) + \frac{\partial^3}{\partial y^3} z(t,y) + z(t,y) \frac{\partial}{\partial y} z(t,y) = a_1(y) \sin z(t,y) + a_2 \int_0^t e^{-z(s,y)} ds, \tag{15}$$

$$z(0,y) + \int_0^a m(s) \log(1 + |z(s,y)|) ds = z_0(y), \quad z \in \mathcal{PC}([0,a] : X). \tag{16}$$

$$\Delta z|_{t=t_i} = I_i(z(y)) = \int_\Omega d_i(y,s) \cos^2 u(s) ds \quad u \in X, \quad 1 \le i \le n, \tag{17}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $0 < s_1 < s_2 < \ldots, < s_q < a$, $0 < t_1 < t_2 < \ldots, < t_p < a$, $c_j \in \mathbb{R}$ $(i = 0, 1, 2, \ldots, p)$, $m(\cdot) \in L^1([0, a]; \mathbb{R})$ and $d_i \in C(\bar{\Omega} \times \bar{\Omega}, \mathbb{R})$ for each $i = 1, 2, \ldots, n$.

Let H^s be the Hilbert space introduced in [13]. Take $X = L^2(R) = H^0(R)$ and $Y = H^s(R)$, $s \geq 3$. Define an operator A_0 by $D(A_0) = H^3(R)$ and $A_0 z = D^3 z$ for $z \in D(A_0)$ where $D = d/dy$. Then A_0 is the infinitesimal generator of a C_0-group of isometries on X. Next we define for every $v \in Y$ an operator $A_1(v)$ by $D(A_1(v)) = H^1(R)$ and $z \in D(A_1(v)), A_1(v)z = vDz$. Then we have for every $v \in Y$ the operator $A(v) = A_0 + A_1(v)$ is the infinitesimal generator of C_0 semigroup $U_v(t, 0)$ on X satisfying $\|U_v(t, 0)\| \leq e^{\beta t}$ for every $\beta \geq c_0 \|v\|_s$ where c_0 is a constant independent of $v \in Y$. Let B_r be the ball of radius $r > 0$ in Y and it is proved that the family of operators $A(v), v \in B_r$ satisfies the conditions [13].

Assume that the function $a_1(\cdot)$ is continuous on $[0, a]$ and $a_2 > 0$. Here we define the functions $f : [0, a] \times X \times X \to X$, $h : [0, a] \times [0, a] \to X$, $g : \mathcal{PC}([0, a] : X) \to X$ and $I_i : X \to X$ by

$$f\Big(t, z(t), \int_0^t h(t, s, z(s))ds\Big)(y) = a_1(y) \sin z(t, y) + a_2 \int_0^t e^{-z(s,y)} ds,$$

$$g(z(t, y)) = \int_0^a m(s) \log(1 + |z(s, y)|)ds, \quad z \in \mathcal{PC}([0, a] : X)$$

and

$$I_i(z(y)) = \int_\Omega d_i(y, s) \cos^2 u(s)ds, \quad u \in X, \quad 1 \leq i \leq n.$$

With this choice of $A(u)$, I_i, f, g,h we see that the equation (15) - (17) is an abstract formulation of (2)–(4).

Further other conditions $(E_1) - (E_3)$ are obviously satisfied and it is possible to choose a_1 and a_2 in such a way that the constant $\rho < 1$. Hence by Theorem 3.2 the equation (15) - (17) has a unique mild solution on $[0, a]$.

Example 5.2. Let f be as in Example 5.1 and $m(\cdot) \in L^1([0, a] : \mathbb{R})$ and $d_i \in C(\bar{\Omega} \times \bar{\Omega} : \mathbb{R})$ for each $i = 1, 2, \ldots, n$. Let

$$z(0, y) + \sum_{j=1}^q c_j z^{\frac{1}{3}}(s_j, y) = z_0(y), \quad z \in \mathcal{PC}([0, a] : X). \tag{18}$$

Put

$$u(t) = z(t, \cdot), \quad g(z(t, y)) = \sum_{j=1}^q c_j z^{\frac{1}{3}}(s_j, y) \text{ and } I_i(z(y)) = \int_\Omega d_i(y, s) \cos^2 u(s)ds.$$

With these choice of $A(u)$, I_i, f, g,h, the assumptions $(H_1) - (H_4)$ of Theorem 4.1 are satisfied. We can represent the system (15), (17) and (18) by the abstract impulsive Cauchy problem (2) - (4). This implies that the corresponding (6) has at least one solution in $\mathcal{PC}([0, a] : X)$.

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