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Existence and Uniqueness of Periodic Solutions for a Kind of Rayleigh Type p-Laplacian Equation

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Abstract. By using topological degree theory and some analysis skill, we obtain some sufficient conditions for the existence and uniqueness of periodic solutions for Rayleigh type p-Laplacian differential equation.

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1. Introduction

In recent years, periodic solutions to Rayleigh type p-Laplacian differential equation have been extensively studied in the literature (see, for example, [1, 2, 6, 7, 5]). However, to the best of our knowledge, most authors have only considered the existence of periodic solutions, and few results exist concerning both existence and uniqueness of periodic solutions to this equation.

In this paper we deal with the existence and uniqueness of T-periodic solutions of the Rayleigh type p-Laplacian differential equation of the form:

$$(\varphi_p(x'(t)))' + f(t, x'(t)) + g(t, x(t)) = e(t), \tag{1}$$

where $p > 1$ and $\varphi_p : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, f and g are continuous functions defined on $\mathbb{R}^2$, e is a continuous periodic function defined on $\mathbb{R}$ with period T, f and g are T-periodic in the first argument, $f(\cdot, 0) = 0$ and $T > 0$.

By using topological degree theory and some analysis skill, we establish some sufficient conditions for the existence and uniqueness of T-periodic solutions of equation (1). Our results are different from those of bibliographies listed above. In particular, an example is also given to illustrate the effectiveness of our results.

2. Preliminary Results

For convenience, let us denote

$$C_T^1 := \{x \in C^1(\mathbb{R},\ \mathbb{R}) : x \ \text{ is } T\text{-periodic}\},$$

which is a Banach space endowed with the norm $||.||$ defined by $||x|| = |x|_\infty + |x'|_\infty$, for all x, and

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|, \quad |x|_k = \left(\int_0^T |x(t)|^k dt \right)^{1/k}.$$

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = \widetilde{f}(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T) \tag{2}$$

where $\widetilde{f}$ is a continuous function and T-periodic in the first variable, we have the following result.

Lemma 2.1. [4] *Let Ω be an open bounded set in C_T^1, if the following conditions hold:*

(i) *For each $\lambda \in (0, 1)$ the problem*

$$(\varphi_p(x'(t)))' = \lambda\widetilde{f}(t, x, x'),\ x(0) = x(T), \quad x'(0) = x'(T)$$

has no solution on $\partial\Omega$.

(ii) *The equation*

$$F(a) := \frac{1}{T}\int_0^T \widetilde{f}(t, a, 0)\, dt = 0,$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) *The Brouwer degree of F*

$$\deg(F,\ \Omega \cap \mathbb{R},\ 0) \neq 0.$$

Then the periodic boundary value problem (2) *has at least one T-periodic solution on $\overline{\Omega}$.*

Set

$$y(t) = \varphi_p(x'(t)). \tag{3}$$

We can rewrite equation (1) in the following form

$$\begin{cases} x'(t) = |y(t)|^{q-1}\mathrm{sgn}(y(t)), \\ y'(t) = -f(t, |y(t)|^{q-1}\mathrm{sgn}(y(t))) - g(t, x(t)) + e(t), \end{cases} \tag{4}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.2. *Suppose that the following condition holds.*
(A_1) $(x_1 - x_2)((g(t, x_1) - g(t, x_2)) < 0$ *for all* $t, x_1, x_2 \in \mathbb{R}, x_1 \neq x_2$.
Then equation (1) *has at most one* T*-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of equation (1). Then, from (4), we obtain

$$\begin{cases} x_i'(t) = |y_i(t)|^{q-1}\mathrm{sgn}(y_i(t)), \\ y_i'(t) = -f(t, |y_i(t)|^{q-1}\mathrm{sgn}(y_i(t))) - g(t, x_i(t)) + e(t), \end{cases} \quad i = 1, 2. \tag{5}$$

Set

$$u(t) = x_1(t) - x_2(t), \quad v(t) = y_1(t) - y_2(t), \tag{6}$$

it follows from equation (5) that

$$\begin{cases} u'(t) = |y_1(t)|^{q-1}\mathrm{sgn}(y_1(t)) - |y_2(t)|^{q-1}\mathrm{sgn}(y_2(t)), \\ v'(t) = -[f(t, |y_1(t)|^{q-1}\mathrm{sgn}(y_1(t))) - f(t, |y_2(t)|^{q-1}\mathrm{sgn}(y_2(t)))] \\ \qquad - [g(t, x_1(t)) - g(t, x_2(t))]. \end{cases} \tag{7}$$

Now, we prove that

$$u(t) \leq 0 \quad \text{for all} \ \ t \in \mathbb{R}.$$

Contrarily, in view of $u \in C^1[0,\ T]$ and $u(t+T) = u(t)$ for all $t \in \mathbb{R}$, we obtain

$$\max_{t \in \mathbb{R}} u(t) > 0.$$

Then, there must exist $t^* \in \mathbb{R}$ (for convenience, we can choose $t^* \in (0,\ T)$) such that

$$u(t^*) = \max_{t \in [0,\ T]} u(t) = \max_{t \in \mathbb{R}} u(t) > 0,$$

which implies that

$$\begin{aligned} &u'(t^*) = |y_1(t^*)|^{q-1}\mathrm{sgn}(y_1(t^*)) - |y_2(t^*)|^{q-1}\mathrm{sgn}(y_2(t^*)) = 0, \\ &v(t^*) = y_1(t^*) - y_2(t^*) = 0, \end{aligned} \tag{8}$$

and

$$x_1(t^*) - x_2(t^*) > 0. \tag{9}$$

In view of (7), we have

$$\begin{aligned} v'(t^*) &= -[f(t^*, |y_1(t^*)|^{q-1}\mathrm{sgn}(y_1(t^*))) - f(t^*, |y_2(t^*)|^{q-1}\mathrm{sgn}(y_2(t^*)))] \\ &\quad -[g(t^*, x_1(t^*)) - g(t^*, x_2(t^*))] \\ &= -[g(t^*, x_1(t^*)) - g(t^*, x_2(t^*))] > 0, \end{aligned} \tag{10}$$

and there exists $\varepsilon^* > 0$ such that $v'(t) > 0$ for all $t \in (t^* - \varepsilon^*, t^*]$. Therefore, $v(t) = y_1(t) - y_2(t)$ is strictly increasing for $t \in (t^* - \varepsilon^*, t^*]$, which implies that

$$v(t) = y_1(t) - y_2(t) < v(t^*) = 0 \text{ for all } t \in (t^* - \varepsilon^*, t^*),$$

and

$$u'(t) = |y_1(t)|^{q-1}\mathrm{sgn}(y_1(t)) - |y_2(t)|^{q-1}\mathrm{sgn}(y_2(t)) < 0 \quad \text{for all} \quad t \in (t^* - \varepsilon^*, t^*).$$

This contradicts the definition of t^*. Thus,

$$u(t) = x_1(t) - x_2(t) \leq 0 \quad \text{for all} \quad t \in \mathbb{R}.$$

By using a similar argument, we can also show that

$$x_2(t) - x_1(t) \leq 0 \quad \text{for all} \quad t \in \mathbb{R}.$$

Therefore, we obtain

$$x_2(t) \equiv x_1(t) \quad \text{for all} \quad t \in \mathbb{R}.$$

Hence, equation (1) has at most one T-periodic solution. The proof is now complete. ■

3. Main Results

By using Lemmas 2.1, 2.2, we obtain our main results:

Theorem 3.1. *Let* (A_1) *hold. Moreover, assume that the following conditions are satisfied.*
(A_2) *There exists a positive constant* d *such that* $x(g(t,x) - e(t)) < 0$ *for* $|x| > d, t, x \in \mathbb{R}$
(A_3) *There exist nonnegative constants* m_1 *and* m_2 *such that*

$$m_1 < \frac{2}{T}, \quad |f(t,x)| \leq m_1|x|^{p-1} + m_2, \quad \textit{for} \quad \textit{all } t, x \in \mathbb{R}.$$

Then equation (1) *has a unique* T*-periodic solution.*

Proof. Consider the homotopic equation of equation (1) as follows

$$(\varphi_p(x'(t)))' + \lambda f(t, x'(t)) + \lambda g(t, x(t)) = \lambda e(t), \quad \lambda \in (0,1). \tag{11}$$

By Lemma 2.2, together with (A_1) , it is easy to see that equation (1) has at most one T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that equation (1) has at least one T-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible T-periodic solutions of equation (11) is bounded.

Let $x(t) \in C_T^1$ be an arbitrary solution of equation (11) with period T. As $x(0) = x(T)$, there exists $t_0 \in [0, T]$ such that $x'(t_0) = 0$, while $\varphi_p(0) = 0$ we see that

$$\begin{aligned} |\varphi_p(x'(t))| &= \left| \int_{t_0}^{t} (\varphi_p(x'(s)))'\, ds \right| \\ &\le \lambda \int_0^T |f(t, x'(t))|\, dt + \lambda \int_0^T |g(t, x(t))|\, dt + \lambda \int_0^T |e(t)|\, dt, \end{aligned} \tag{12}$$

where $t \in [t_0, t_0 + T]$.

Let $\bar{t}$ and $\underline{t}$ be, respectively, the global maximum point and global minimum point of $x(t)$ on $[0, T]$, then $x'(\bar{t}) = 0$ and we claim that

$$(\varphi_p(x'(\bar{t})))' = (|x'(\bar{t})|^{p-2} x'(\bar{t}))' \le 0. \tag{13}$$

Assume, by way of contradiction, that (13) does not hold. Then

$$(\varphi_p(x'(\bar{t})))' = (|x'(\bar{t})|^{p-2} x'(\bar{t}))' > 0,$$

and there exists $\varepsilon > 0$ such that $(\varphi_p(x'(t)))' = (|x'(t)|^{p-2} x'(t))' > 0$ for $t \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon)$. Therefore, $\varphi_p(x'(t)) = |x'(t)|^{p-2} x'(t)$ is strictly increasing for $t \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon)$, which implies that $x'(t)$ is strictly increasing for $t \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon)$. This contradicts the definition of $\bar{t}$. Thus, (13) is true. From $f(\cdot, 0) = 0$, (11) and (13), we have

$$g(\bar{t}, x(\bar{t})) - e(\bar{t}) \ge 0. \tag{14}$$

Similarly, we get

$$g(\underline{t}, x(\underline{t})) - e(\underline{t}) \le 0. \tag{15}$$

In view of (A_2), (14) and (15) imply that

$$x(\bar{t}) < d \qquad \text{and} \qquad x(\underline{t}) > 0. \tag{16}$$

Since $x(t)$ is a continuous T-periodic function on $\mathbb{R}$, it follows that there exists a constant $\bar{\xi} \in [0,\ T]$ such that

$$|x(\bar{\xi})| \le d.$$

Then, we have

$$|x(t)| = |x(\bar{\xi}) + \int_{\bar{\xi}}^{t} x'(s)ds| \le d + \int_{\bar{\xi}}^{t} |x'(s)|ds,\ \ t \in [\bar{\xi},\ \bar{\xi} + T],$$

and

$$|x(t)| = |x(t-T)| = |x(\bar{\xi}) - \int_{t-T}^{\bar{\xi}} x'(s)ds| \le d + \int_{t-T}^{\bar{\xi}} |x'(s)|ds, t \in [\bar{\xi},\ \bar{\xi} + T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x|_\infty &= \max_{t\in[0,T]} |x(t)| = \max_{t\in[\bar{\xi},\ \bar{\xi}+T]} |x(t)| \\ &\le \max_{t\in[\bar{\xi},\ \bar{\xi}+T]} \left\{ d + \frac{1}{2} \left(\int_{\bar{\xi}}^{t} |x'(s)|ds + \int_{t-T}^{\bar{\xi}} |x'(s)|ds \right) \right\} \\ &\le d + \frac{1}{2} \int_0^T |x'(s)|ds. \end{aligned} \tag{17}$$

Denote

$$E_1 = \{t : t \in [0,T], |x(t)| > d\}, \quad E_2 = \{t : t \in [0,T], |x(t)| \le d\}.$$

Since $x(t)$ is T-periodic, multiplying $x(t)$ and (12) and then integrating it from 0 to T, in view of (A_2) and (A_3), we get

$$\begin{aligned} \int_0^T |x'(t)|^p dt &= -\int_0^T (\varphi_p(x'(t)))' x(t)dt \\ &= \lambda \int_0^T f(t,x'(t))x(t)dt + \lambda \int_0^T g(t,x(t))x(t)dt - \lambda \int_0^T e(t)x(t)dt \\ &= \lambda \int_0^T f(t,x'(t))x(t)dt + \lambda \int_{E_1} [g(t,x(t)) - e(t)]x(t)dt \\ &\quad + \lambda \int_{E_2} [g(t,x(t)) - e(t)]x(t)dt \\ &\le |x|_\infty \int_0^T [m_1|x'(t)|^{p-1} + m_2]dt \end{aligned}$$

$$+\int_0^T \max\{|g(x(t))-e(t)| : t\in\mathbb{R}, |x(t)|\le d\}|x(t)|dt$$
$$\le m_1|x|_\infty \int_0^T |x'(t)|^{p-1}dt + (D+m_2)T|x|_\infty, \tag{18}$$

where $D=\max\{|g(t,x)-e(t)| : |x|\le d,\ t\in\mathbb{R}\}$.

For $x(t)\in C(\mathbb{R},\ \mathbb{R})$ with $x(t+T)=x(t)$, and $0<r\le s$, by using Hölder inequality, we obtain

$$\left(\frac{1}{T}\int_0^T |x(t)|^r dt\right)^{1/r} \le \left(\frac{1}{T}\left(\int_0^T (|x(t)|^r)^{\frac{s}{r}}dt\right)^{\frac{r}{s}}\left(\int_0^T 1dt\right)^{\frac{s-r}{s}}\right)^{1/r}$$
$$= \left(\frac{1}{T}\int_0^T |x(t)|^s dt\right)^{1/s},$$

this implies that

$$|x|_r \le T^{\frac{s-r}{rs}}|x|_s,\ \text{for}\quad 0<r\le s. \tag{19}$$

Then,

$$\int_0^T |x'(t)|dt \le T^{\frac{p-1}{p}}|x'(t)|_p, \qquad \int_0^T |x'(t)|^{p-1}dt \le T^{\frac{1}{p}}|x'(t)|_p^{p-1}. \tag{20}$$

In view of (17), (18) and (20), we can get

$$|x'(t)|_p^p = \int_0^T |x'(t)|^p dt$$
$$\le m_1|x|_\infty \int_0^T |x'(t)|^{p-1}dt + (D+m_2)T|x|_\infty$$
$$\le m_1\left(d+\frac{1}{2}T^{\frac{p-1}{p}}|x'(t)|_p\right)T^{\frac{1}{p}}|x'(t)|_p^{p-1} + (D+m_2)T\left(d+\frac{1}{2}T^{\frac{p-1}{p}}|x'(t)|_p\right)$$
$$\le \frac{1}{2}Tm_1|x'(t)|_p^p + m_1 dT^{\frac{1}{p}}|x'(t)|_p^{p-1}$$
$$+(D+m_2)T\frac{1}{2}T^{\frac{p-1}{p}}|x'(t)|_p + (D+m_2)Td. \tag{21}$$

Since $p>1$ and $m_1<\frac{2}{T}$, (21) yields that we can choose a positive constant M_1 such that

$$|x'(t)|_p < M_1, \qquad \int_0^T |x'(t)|dt \le T^{\frac{p-1}{p}} |x'(t)|_p < M_1,$$

and

$$|x|_\infty \le d + \frac{1}{2}\int_0^T |x'(s)|ds \le M_1.$$

In view of (12) and (20), we have

$$\begin{aligned}
|x'|_\infty^{p-1} &= \max_{t\in[0,T]}\{|\varphi_p(x'(t))|\} \\
&= \max_{t\in[t_0,t_0+T]}\left\{\left|\int_{t_0}^t (\varphi_p(x'(s)))'\,ds\right|\right\} \\
&\le \int_0^T |f(t,x'(t))|dt + \int_0^T |g(t,x(t))|\,dt + \int_0^T |e(t)|\,dt \\
&\le \int_0^T [m_1|x'(t)|^{p-1} + m_2]dt + \int_0^T |g(t,x(t))|\,dt + \int_0^T |e(t)|\,dt \\
&\le m_1 T^{\frac{1}{p}}|x'(t)|_p^{p-1} + T[m_2 + \max\{|g(t,x)| : |x| \le M_1, t\in\mathbb{R}\} + |e|_\infty] \\
&\le m_1 T^{\frac{1}{p}} M_1^{p-1} + T[m_2 + \max\{|g(t,x)| : |x| \le M_1, t\in\mathbb{R}\} + |e|_\infty]. \quad (22)
\end{aligned}$$

Thus, we can get some positive constant $M_2 > M_1 + 1$ such that for all $t \in \mathbb{R}$,

$$|x'(t)| \le M_2.$$

Set

$$\Omega = \{x \in C_T^1 : |x|_\infty \le M_2 + 1, |x'|_\infty \le M_2 + 1\},$$

then we know that equation (12) has no T-periodic solution on $\partial\Omega$ as $\lambda \in (0,1)$ and when $x(t) \in \partial\Omega \cap \mathbb{R}$, $x(t) = M_2 + 1$ or $x(t) = -M_2 - 1$, from (A_2) , we can see that

$$\frac{1}{T}\int_0^T [-g(t,M_2+1) + e(t)]\,dt = -\frac{1}{T}\int_0^T [g(t,M_2+1) - e(t)]\,dt > 0,$$

$$\frac{1}{T}\int_0^T [-g(t,-M_2-1) + e(t)]\,dt = -\frac{1}{T}\int_0^T [g(t,-M_2-1) - e(t)]\,dt < 0,$$

so condition (ii) is also satisfied. Set

$$H(x,\mu) = \mu x - (1-\mu)\frac{1}{T}\int_0^T [g(t,x) - e(t)]\,dt,$$

and when $x \in \partial\Omega \cap \mathbb{R},\ \mu \in [0,1]$ we have

$$xH(x,\mu) = \mu x^2 - (1-\mu)x\frac{1}{T}\int_0^T [g(t,x) - e(t)]\,dt > 0.$$

Thus $H(x,\mu)$ is a homotopic transformation and

$$\begin{aligned}\deg\{F, \Omega\cap\mathbb{R}, 0\} &= \deg\left\{-\frac{1}{T}\int_0^T \{g(t,x) - e(t)\}\,dt, \Omega\cap\mathbb{R}, 0\right\}\\ &= \deg\{x, \Omega\cap\mathbb{R}, 0\} \neq 0.\end{aligned}$$

so condition (iii) is satisfied. In view of the previous Lemma 2.1, there exists at least one solution with period T. This completes the proof. ■

Remark 3.2. If $p = 2$, we can find that the main results of [8] are special ones of Theorem 3.1.

4. An Example

As an application, let us consider the following equation:

$$(\varphi_p x'(t))' + \frac{1}{100}e^{\sin(t)}\cos(x'(t))(x'(t))^3 - (x^9(t) + x(t) - 12) = \cos^2 t, \qquad (23)$$

where $p = \sqrt{15}$. We can easily check the conditions (A_1), (A_2) and (A_3) hold. By Theorem 3.1, Equation (23) has a unique 2π-periodic solution.

Remark 4.1. Since $p = \sqrt{15}$, one can easily see that all the results in [1–8] and the references therein can not be applicable to (23) to obtain the existence and uniqueness of 2π-periodic solutions. This implies that the results of this paper are essentially new.

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