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Category of Noncommutative CW Complexes

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Abstract. We review the notion of noncommutative CW (NCCW) complexes, define noncommutative (NC) mapping cylinder and NC mapping cone, and prove a noncommutative Approximation Theorem. We also formulate and derive a long exact homotopy sequences associated with arbitrary morphisms.

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1. Introduction

Classical algebraic topology was fruitfully developed on the category of topological spaces with CW complex structure, see e.g. [10]. Our goal is to show that an equally succesful theory can be developed in the framework of noncommutative topology.

In noncommutative geometry the notion of topological spaces is substituted by the notion of C*-algebras, motivating the consideration of spectra of C*-algebras as noncommutative spaces. In [8] and [9] the notion of noncommutative CW (NCCW) was introduced and some elementary properties of NCCW complexes are proved. We continue this line in proving some basic noncommutative results. In this paper we aim to explore the same properties of NCCW complexes, as those of CW complexes from algebraic topology. In particular, we prove an NC Cellular Approximation Theorem and the existence of homotopy exact sequences associated with morphisms. In the work [4] we introduced the notion of NC Serre fibrations (NCSF) and studied cyclic theories for the (co)homology of these NCCW complexes. In [5] we studied the Leray-Serre spectral sequences related with cyclic theories: periodic cyclic homology and KK-theory. In [6] and [7] we computed some noncommutative Chern characters. Some deep properties should be related with the Busby invariant, studied in [2, 3].

Let us describe in more detail the content of the paper. In Sec. 2 we expose the pullback and pushout diagrams of Pedersen [9] on categories of C*-algebras. In Sec. 3 we introduce NCCW complexes following S. Eilers, T. A. Loring and G. K. Pedersen, etc. We prove in Sec. 4 a noncommutative Cellular Approximation Theorem. In Sec. 5 we formulate and derive a long exact homotopy sequences associated with morphisms of C*-algebras.

2. Constructions in Categories of C*-algebras

In this section we review the pullback and pushout constructions of Eilers, Loring, and Pedersen [8] and of Pedersen [9] on categories of C*-algebras, and after that we define mapping cylinders and mapping cones associated with arbitrary morphisms.

Let us introduce some general notations. Denote by $\mathbf{I} = [0,1]$ the closed interval from 0 to 1 on the real line of real numbers. It is easy to construct a homeomorphism $\mathbf{I}^n \approx \mathbf{B}^n$ between the n-cube and the n dimensional closed ball. Denote also the interior of the cube $\mathbf{I}^n$ by $\mathbf{I}^n_0 = (0,1)^n = (\mathbf{I}^n)^\circ$. It is easy to show that the boundary $\partial\mathbf{I}^n = \mathbf{I}^n \setminus \mathbf{I}^n_0$ is homotopic to the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}$. Denote the space of all continuous functions on $\mathbf{I}^n$ with values in a C*-algebra A by $\mathbf{I}^n = \mathbf{C}([0,1]^n, A)$, and by analogy by $\mathbf{I}^n_0 A := \mathbf{C}_0((0,1)^n, A)$ the space of all continuous functions with compact support with values in A, and finally, by $\mathbf{S}^n A = \mathbf{C}(\mathbf{S}^n, A)$ the space of all continuous maps from $\mathbf{S}^n$ to A.

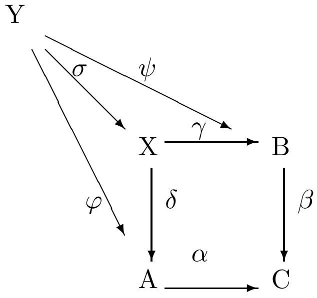
Definition 2.1. [Pullback diagram] A commutative diagram of C*-algebras and *-homomorphisms

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & C \end{array}$$

is a *pullback*, if $\ker\gamma \cap \ker\delta = 0$ and if

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & B \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & C \end{array}$$

is another commutative diagram, then there exists a unique morphism $\sigma : Y \to X$ such that $\varphi = \delta \circ \sigma$ and $\psi = \gamma \circ \sigma$, i.e. we have the so called *pullback diagram*



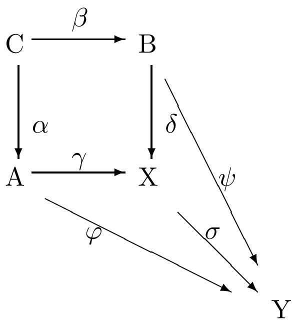
Definition 2.2. [Pushout diagram] A commutative diagram of C*-algebras and *-homomorphisms

$$\begin{array}{ccc} C & \xrightarrow{\beta} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ A & \xrightarrow{\delta} & X \end{array}$$

is a *pushout*, if X is generated by $\gamma(B) \cup \delta(A)$ and if

$$\begin{array}{ccc} C & \xrightarrow[\beta]{} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \psi} \\ A & \xrightarrow{\varphi} & Y \end{array}$$

is another commutative diagram, then there exists a unique morphism $\sigma : X \to Y$ such that $\varphi = \sigma \circ \gamma$ and $\psi = \sigma \circ \delta$, i.e. we have the so called *pushout diagram*



Definition 2.3. [NC cone] For C*-algebras the NC cone of A is defined as the tensor product with $\mathbf{C}_0((0,1])$, i.e.

$$\mathrm{Cone}(A) := \mathbf{C}_0((0,1]) \otimes A.$$

Definition 2.4. [NC suspension] For C*-algebras the NC suspension of A is defined as the tensor product with $\mathbf{C}_0((0,1))$, i.e.

$$\mathbf{S}(A) := \mathbf{C}_0((0,1)) \otimes A.$$

Remark 2.5. If A admits a NCCW complex structure, the same have the cone $\mathrm{Cone}(A)$ of A and the suspension $\mathbf{S}(A)$ of A.

We refer the reader to work of Pedersen [9] and others for the standard notions such as NC mapping cylinder, mapping cone, etc.

Remark 2.6. It is easy to show that A is included in $\mathrm{Cyl}(f : A \to B)$ as $\mathbf{C}\{0\} \otimes A \subset \mathrm{Cyl}(f : A \to B)$ and B is included in also $B \subset \mathrm{Cyl}(f : A \to B)$.

Remark 2.7. It is easy to show that B is included in $\mathrm{Cone}(f : A \to B)$.

Proposition 2.8. [9] *Both the mapping cylinder and mapping cone satisfy the pullback diagrams*

$$\begin{array}{ccc} \mathrm{Cone}(\varphi) & \xrightarrow{pr_1} & \mathbf{C}_0(0,1] \otimes A \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \varphi o \mathrm{ev}(1)} \\ B & \xrightarrow{id} & B \end{array} \qquad \begin{array}{ccc} \mathrm{Cyl}(\varphi) & \xrightarrow{pr_1} & \mathbf{C}[0,1] \otimes A \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \varphi o \mathrm{ev}(1)} \\ B & \xrightarrow{id} & B \end{array}$$

where $ev(1)$ is the map of evaluation at the point $1 \in [0, 1]$.

Remark 2.9. The pullback diagrams in Proposition 2.8 can be used as the initial definition of mapping cyliner and mapping cone. The previous definitions are therefore the existence of those universal objects.

3. The Category NCCW

In this section we introduce NCCW complexes following Cuntz [1] and following Eilers, Loring, and Pedersen, [8] etc.

Definition 3.1. A *dimension 0 NCCW* complex is defined, following [9] as a finite sum of C*-algebras of finite linear dimension, i.e. a sum of finite dimensional matrix algebras

$$A_0 = \bigoplus_k \mathbf{M}_{n(k)}.$$

In dimension n, an *NCCW complex* is defined as a sequence $\{A_0, A_1, \ldots, A_n\}$ of C*-algebras A_k obtained each from the previous one by the pullback construction

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_0^k F_k & \longrightarrow & A_k & \xrightarrow{\pi} & A_{k-1} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \rho_k} & & \downarrow{\scriptstyle \sigma_k} & & \\ 0 & \longrightarrow & I_0^k F_k & \longrightarrow & I^k F_k & \xrightarrow{\partial} & \mathbf{S}^{k-1} F_k & \longrightarrow & 0, \end{array}$$

where F_k is some C*-algebra of finite linear dimension, ∂ the restriction morphism, σ_k the connecting morphism, ρ_k the projection on the first coordinates and π the projection on the second coordinates in the presentation

$$A_k = \mathbf{I}^k F_k \bigoplus_{\mathbf{S}^{k-1}F_k} A_{k-1}.$$

Proposition 3.2. *If the algebras A and B admit a NCCW complex structure, then the same has the NC mapping cylinder* $\mathrm{Cyl}(f : A \to B)$.

Proof. Let us remember from [9] that the interval $\mathbf{I}$ admits a structure of a NCCW complex. Next, tensor product of two NCCW complex [9] is also a NCCW complex and finally the quotient of a NCCW complex is also a NCCW complex, loc. cit.. ∎

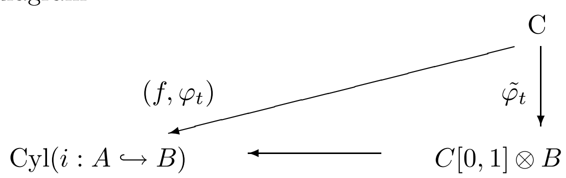
Proposition 3.3. *If the algebras A and B admit a NCCW complex structure, then the same has the NC mapping cone* $\mathrm{Cone}(f : A \to B)$.

Proof. The same argument as in proof of Proposition 3.2. ∎

4. Approximation Theorem

We prove in this section a noncommutative analog of the well-known Cellular Approximation Theorem. First we introduce the so called noncommutative homotopy extension property (NC HEP).

Definition 4.1. [NC HEP] For a given (f, φ_t) and a C*-algebra C, we say that $\tilde{h} = \tilde{\varphi}_t$ is a solution of the extension problem if we have the commutative homotopy extension diagram



Definition 4.2. [NC NDR] We say that the pair of algebras (B, A) is a NCNDR pair, if there are continuous morphisms $u : \mathbf{C}[0,1] \to B$ and $\varphi : B \to \mathbf{C}[0,1] \otimes B \cong \mathbf{C}(I, B)$ such that

1. $u^{-1}(A) = 0$;
2. If $\varphi(b) = (x(t), b')$ and $x(t) = 0 \in \mathbf{C}(\mathbf{I})$ then $b' = b$, $\forall b \in B$;
3. $\varphi(a) = (x(t), a), \forall a \in A, x(t) \in \mathbf{C}(\mathbf{I})$;
4. $\varphi(b) = (x(t), b')$ and if $x(t) = 1 \in \mathbf{C}(\mathbf{I})$ then $b' \in A$ for all $b \in B$ such that $u(b) \neq 1$.

The following proposition is easily to prove.

Proposition 4.3. *The assertion that NC HEP has solution for every φ_t and C is equivalent to the property that (B, A) is a NC NDR pair.*

Proof. If (B, A) has NC HEP, we can for every C, construct $\tilde{\varphi} : B \to \mathbf{C}[0,1] \otimes B$ satisfying the NC HEP diagram. Choose $C = B$ and $f = \mathrm{id}$ we have the function φ and then choose $(D, C) = (\mathbf{C}[0,1], 0)$ in the definition of NC NDR pair we have the function u.

Conversely, if (B, A) is a NC NDR pair, we can define

$$h = \varphi : B \to \mathbf{C}[0,1] \otimes B,$$

the composition of which with $f : C \to B$ satisfy the NC HEP diagram. ■

Theorem 4.4 (Extension). *Suppose that* $B = \mathbf{I}^n F_n \oplus_{\mathbf{S}^{n-1} F_n} A$ *and* (C, D) *is a NC NDR pair. Every relative morphism of pairs of* C*-*algebras*

$$f : (D, C) \to (\mathrm{Cyl}(i : A \hookrightarrow B), \mathbf{C}\{1\} \otimes A)$$

can be up-to homotopy extended to a relative morphism of pairs of C*-*algebras*

$$F : (D, C) \to (\mathbf{C}(\mathbf{I}) \otimes B, \mathbf{C}\{1\} \otimes B).$$

Proof. By the assumption that (D, C) is a NC NDR pair, there is a natural extension

$$f_1 : (D, C) \to (\mathbf{C}(\mathbf{I}) \otimes B, \mathbf{C}\{1\} \otimes A).$$

Composing f_1 with the map, evaluating the value at 1 give a morphism

$$\mathrm{ev}(1) \circ f_1 : (D, C) \to (\mathbf{C}\{1\} \otimes B, \mathbf{C}\{1\} \otimes A).$$

Therefore, there exists a natural extension f_2 from the pair (D, C) to the pair

$$(\mathbf{C}\{0\} \otimes \mathbf{C}(\mathbf{I}) \otimes B + \mathbf{C}(\mathbf{I}) \otimes \mathbf{C}\{1\} \otimes B, \mathrm{Cyl}(\mathbf{C}\{1\} \otimes A \hookrightarrow \mathbf{C}\{1\} \otimes B)).$$

Once again, there is a natural extension f_3 from the pair to the pair

$$(\mathbf{C}\{0\} \otimes \mathbf{C}(\mathbf{I}) \otimes B + \mathbf{C}\{1\} \otimes \mathbf{C}(\mathbf{I}) \otimes B + \mathbf{C}(\mathbf{I}) \otimes \mathbf{C}(\mathbf{I}) \otimes A, Cyl(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B)) =$$

$$= (\mathbf{C}\{0\} \otimes \mathrm{Cyl}(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B) + \mathbf{C}\{1\}\ otimes\, \mathrm{Cyl}(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B), \mathrm{Cyl}(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B)).$$

And finally, there is a natural extension f_4 from the pair (D, C) to the pair

$$\mathbf{C}(\mathbf{I}) \otimes \mathbf{C}(\mathbf{I}) \otimes B, \mathrm{Cyl}(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B)).$$

We define the desired extension

$$F : (D, C) \to (\mathbf{C}(\mathbf{I}) \otimes B, \mathbf{C}\{1\} \otimes B)$$

as

$$F(t, x) := f_4(t, 0, x).$$

■

Theorem 4.5. *Let $\{A_0, A_1, \ldots, A_n\}$ and $\{B_0, B_1, \ldots, B_m\}$ be two NCCW complexes and $f : A = A_n \to B_m = B$ an algebraic homomorphism (map). Then f is homotopic to a cellular NCCW complex map $h : A \to B$.*

Proof. We construct a sequence of maps

$$g_p : \mathbf{A}_p \to C(\mathbf{I}) \otimes B_p,$$

with 4 well-known properties:

1. $g_p(x) = (0, f(x))$, for all $x \in A_p$
2. If $g_p(b) = (x(t), f(b))$ and if $x(t) = 0 \in \mathbf{C}(\mathbf{I})$, then $f(b) = b$, for all $b \in B$.
3. $\mathrm{ev}(1) \circ g_p = g_{p-1}$,
4. $g_p(A_p) \subset \mathbf{C}\{1\} \otimes B_p$.

Indeed, following the definition of a NCCW complex structure, we have

$$A_0 = F_0 \otimes A, \qquad F_0 = \bigoplus_{j_0} \mathbf{M}_n(j_0)$$

a finite system of quantum points, i.e. a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{I}_0^1 F_1 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow \rho_1 & & \downarrow \sigma_1 & & \\
0 & \longrightarrow & \mathbf{I}_0^1 F_1 & \longrightarrow & \mathbf{I}^1 F_1 & \underset{\alpha_1}{\longrightarrow} & \mathbf{S}^0 F_1 & \longrightarrow & 0
\end{array}$$

in which the second square is a pullback diagram,

$$F_1 = \bigoplus_{j_1} \mathbf{M}_{n(j_1)} = \bigoplus_{j_1} \mathrm{Mat}_{n(j_1)},$$

and we can present A_1 as

$$A_1 \approx \mathbf{I}^1 F_1 \bigoplus_{S^0 F_1} A_0.$$

Following the compression theorems [10] and the previous Extension Theorem 4.4, the function g_0 can be naturally extended to a function g_1 with properties 1. - 4. and now we have again following the definition of a NCCW complex,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{I}_0^2 F_2 & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow \rho_2 & & \downarrow \sigma_2 & & \\
0 & \longrightarrow & \mathbf{I}_0^2 F_2 & \longrightarrow & \mathbf{I}^2 F_2 & \underset{\alpha_2}{\longrightarrow} & \mathbf{S}^1 F_2 & \longrightarrow & 0
\end{array}$$

$$F_2 = \bigoplus_{j_2} \mathbf{M}_{n(j_2)} = \bigoplus_{j_2} \mathrm{Mat}_{n(j_2)},$$

and we can present A_2 as

$$A_2 \approx \mathbf{I}^2 F_2 \bigoplus_{S^1 F_2} A_1.$$

Following the compression theorems [10] and the previous Extension Theorem 4.4, the function g_1 can be naturally extended to a function g_2 with properties 1. - 4. The procedure is continued for all p. Once these functions g_p were defined, the function $g : A \to \mathbf{C}(\mathbf{I}) \otimes B$ which is continuous and g is a homotopy of f to h, where

$$h(x) := \mathrm{ev}(1) \circ g(x).$$

Because of property 4. the function $h : A \to B$ is a cellular NCCW complex map. ∎

5. Homotopy of NCCW Complexes

We prove in this section the standard long exact homotopy sequences.

Let us first recall the definition of homotopic morphisms.

Definition 5.1. A homotopy between two morphisms $\varphi, \psi : A \to B$ is a morphism $\Phi : A \to \mathbf{C}(\mathbf{I}) \otimes B$, such that $\Phi(0, .) = \varphi$ and $\Phi(1, .) = \psi$.

Proposition 5.2. *There is a natural homotopy* $\mathrm{Cyl}(\varphi : A \to B) \simeq B$ *and* $\mathrm{Cone}(\varphi : A \to B) \simeq B/A$*, if the last one* B/A *is defined.*

Theorem 5.3. *For every morphism* $\varphi : A \to B$*, there is a natural long exact homotopy sequence*

$$\ldots \longrightarrow \mathbf{S}^2(A) \longrightarrow \mathbf{S}(\mathrm{Cone}(\varphi : A \to B)) \longrightarrow$$
$$\mathbf{S}(\mathrm{Cyl}(\varphi : A \to B)) \longrightarrow S(A) \longrightarrow \mathrm{Cone}(\varphi : A \to B) \longrightarrow$$
$$\mathrm{Cyl}(\varphi : A \to B) \longrightarrow A \xrightarrow{\varphi} B.$$

Proof. Put $A_0 = B, A_1 = A$ and $\varphi_0 = \varphi$ we have

$$A_0 \xrightarrow{\varphi_0 = \varphi} A_1.$$

Because of Proposition 5.2 we have

$$A_0 = B \xleftarrow{\varphi_0} A_1 = A \xleftarrow{\varphi_1} A_2 = \mathrm{Cyl}(\varphi) \xleftarrow{\varphi_2} A_3 = \mathrm{Cone}(\varphi). \quad (1)$$

Because of the exact sequence

$$\mathbf{S}(A) \longleftarrow \mathrm{Cyl}(\varphi) \longleftarrow \mathrm{Cone}(\varphi),$$

we have

$$A_0 = B \xleftarrow{\varphi_0} A_1 = A \xleftarrow{\varphi_1} A_2 = \mathrm{Cyl}(\varphi) \xleftarrow{\varphi_2} A_3 = \mathrm{Cone}(\varphi) \xleftarrow{\varphi_3}$$
$$\xleftarrow{\varphi_3} A_4 = \mathbf{S}(A) = \mathbf{C}_0((0,1)) \otimes A.$$

Because the tensor product $\mathbf{C}_0((0,1)) \otimes .$ is a left exact functor and because of (1), we have

$$A_0 = B \xleftarrow{\varphi=\varphi_0} A_1 = A \xleftarrow{\varphi_1} A_2 = \mathrm{Cyl}(\varphi) \xleftarrow{\varphi_2} A_3 = \mathrm{Cone}(\varphi) \xleftarrow{\varphi_3}$$
$$\xleftarrow{\varphi_3} \mathbf{S}(A) \xleftarrow{\varphi_4} \mathbf{S}(\mathrm{Cyl}(\varphi)) \xleftarrow{\varphi_5} A_5 = \mathbf{S}(\mathrm{Cone}(\varphi)) \xleftarrow{\varphi_5}$$
$$\xleftarrow{\varphi_5} A_6 = \mathbf{S}^2(A) \xleftarrow{\varphi_6} \dots,$$

etc. ■

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