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On Asymptotic Properties of Toeplitz Operators

Namita Das[1] and Madhusmita Sahoo[2]

[1] *P. G. Department of Mathematics, Utkal University, Vani Vihar, Bhubaneswar-751004, Orissa, India*

[2] *Department of Mathematics, College of Engineering Bhubaneswar, Patia, Bhubaneswar-751024, Orissa, India*

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Abstract. In this paper we derive certain asymptotic properties of Toeplitz operators on Hardy and Bergman spaces. More precisely, we have shown that if $T \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ and $\{T^{*^m}_{\theta I_{n\times n}} T\, T^m_{\theta I_{n\times n}}\}$ converges to an operator L in the strong operator topology for all inner functions $\theta \in H^\infty(\mathbb{T})$ then $L = T_\Phi$, a Toeplitz operator on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ and if T is an operator in the Hankel algebra, the norm closed algebra generated by all Toeplitz and all Hankel operators together on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$, then the sequence $\{T^{*^m}_{\theta I_{n\times n}} T\, T^m_{\theta I_{n\times n}}\}$ converges to a Toeplitz operator on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ in the strong operator topology for all inner functions $\theta \in H^\infty(\mathbb{T})$. As an application, we have shown that if $S \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ and $[S, T_{fI_{n\times n}}]$ is of finite rank for all $f \in H^\infty(\mathbb{T})$ then $S = T_\Phi + F$, where $\Phi \in H^\infty_{M_n}(\mathbb{T}) + R^p_{M_n}$ and F is a finite rank operator. This is an extension of the work done for the scalar valued case in [2] and [12]. Asymptotic properties of Toeplitz operators and Hankel operators defined on the Bergman space were also analysed.

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1. Introduction

Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. Let $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ denote the Hilbert space of $\mathbb{C}^n$-valued, norm square integrable, measurable functions on $\mathbb{T}$

and $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ the corresponding Hardy space of functions in $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ with vanishing negative Fourier coefficients. When endowed with the inner product defined by the equality

$$\langle f, g\rangle = \int_{\mathbb{T}} \langle f(z), g(z)\rangle_{\mathbb{C}^n} dm, \quad f, g \in \mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T}),$$

the space $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ becomes a separable Hilbert space. Here the measure m denotes the normalized Lebesgue measure on $\mathbb{T}$. Notice that $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T}) = L^2(\mathbb{T}) \otimes \mathbb{C}^n$ and $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = H^2(\mathbb{T}) \otimes \mathbb{C}^n$, where the Hilbert space tensor product is used. The set of vectors $\{e_m\}_{m=0}^{\infty} = \{z^m\}_{m=0}^{\infty}$ forms the standard orthonormal basis for $H^2(\mathbb{T})$.

If Φ is a bounded, measurable $M_n = M_n(\mathbb{C})$- valued function (the algebra of $n \times n$ matrices with complex entries) in $L^{\infty}_{M_n}(\mathbb{T}) = L^{\infty}(\mathbb{T}) \otimes M_n$, then T_{Φ} denotes the Toeplitz operator defined on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ by $T_{\Phi} f = P(\Phi f)$ for $f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ where P is the orthogonal projection of $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ onto $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. The operator H_{Φ} denotes the Hankel operator defined in $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ by $H_{\Phi} f = (I - P)(\Phi f)$ for f in $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Let $H^{\infty}_{M_n}(\mathbb{T}) = H^{\infty}(\mathbb{T}) \otimes M_n$. A function $\Theta \in H^{\infty}_{M_n}(\mathbb{T})$ is said to be inner if $\Theta(z)$ is an isometry for almost all $z \in \mathbb{T}$. Given an inner function $\Theta \in H^{\infty}_{M_n}(\mathbb{T})$, it can be shown that [3] there exists an inner function $\Omega \in H^{\infty}_{M_n}(\mathbb{T})$ satisfying the relation $\Theta(\lambda)\Omega(\lambda) = q(\lambda) I_{n\times n}$ for almost every $\lambda \in \mathbb{T}$ where $q = \det \Theta$ (a scalar valued inner function). Hence

$$\Theta^* = \frac{\Theta^* \Theta \Omega}{det\ \Theta} = \frac{\Theta^* \Theta \Omega}{q} = \overline{q}\ \Omega.$$

Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself. Let $\mathcal{LC}(H)$ and $\mathcal{LF}(H)$ denote the set of all compact operators and the set of all finite rank operators in $\mathcal{L}(H)$ respectively. For $A, B \in \mathcal{L}(H)$, let $[A, B] = AB - BA$. In [2], Barria and Halmos introduced the concept of asymptotic Toeplitz operators on the Hardy space $H^2(\mathbb{T})$. The importance of this notion is that it associates with a class of operators a Toeplitz operator where the original operators are not Toeplitz. Thus we are able to assign a symbol to an operator that is not Toeplitz and hence a symbol calculus is obtained. In this paper, we extend the result of [2] to vector-valued Hardy space $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. We showed that all operators in the Hankel algebra defined on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ are asymptotic Toeplitz. We also discuss about some asymptotic properties of Toeplitz operators defined on the Bergman space.

2. Toeplitz Operators on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$

In this section we show that if T belongs to the Hankel algebra, the algebra generated by all Toeplitz and all Hankel operators together defined on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$, then $\{T^{*m}_{qI_{n\times n}} T T^m_{qI_{n\times n}}\}$ converges to a Toeplitz operator $T_{\Phi}, \Phi \in L^{\infty}_{M_n}(\mathbb{T})$ in the strong operator topology for all inner functions $q \in H^{\infty}(\mathbb{T})$ and if we define

$\sigma(T) = \Phi$, then the map σ restricted to the Hankel-algebra $\mathcal{T}^+$ is a contractive, *-homomorphism from $\mathcal{T}^+$ onto $L^\infty_{M_n}(\mathbb{T})$. Further, we have shown that T belongs to the commutator ideal I of the Toeplitz algebra $\mathcal{T}$ if and only if $\{T^{*^m}_{qI_{n\times n}} T T^m_{qI_{n\times n}}\}$ converges to 0 strongly. As applications, we find necessary and sufficient conditions for $S \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ such that $[S, T_\theta I_{n\times n}]$ is of finite rank for all $\theta \in H^\infty(\mathbb{T})$. The following theorem is well-known.

Theorem 2.1. (Douglas - Rudin Theorem [8]) *If $\Phi \in L^\infty_{M_n}(\mathbb{T})$, then for $\epsilon > 0$, there exist an inner function $\Theta \in H^\infty_{M_n}(\mathbb{T})$ and $\Psi \in H^\infty_{M_n}(\mathbb{T})$ such that $\|\Phi - \overline{\Theta}\Psi\|_\infty < \epsilon$.*

Theorem 2.2. *If $T \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ and $\{T^{*^m}_{qI_{n\times n}}\ T\ T^m_{qI_{n\times n}}\}$ converges to an operator L in the strong operator topology for all inner functions $q \in H^\infty(\mathbb{T})$ then $L = T_\Phi$, a Toeplitz operator on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ with symbol $\Phi \in L^\infty_{M_n}(\mathbb{T})$.*

Proof. Let $\sum = \{\overline{q} I_{n\times n}\Psi | q \in H^\infty(\mathbb{T})$ is inner and $\Psi \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\}$. It is shown in [7] and [3] that $\overline{\overline{H^\infty(\mathbb{T})}}^{L^2} = H^2(\mathbb{T})$ and for any inner function $\Theta \in H^\infty_{M_n}(\mathbb{T})$, we have $\Theta^* = \overline{q} I_{n\times n}\Omega$ for some $\Omega \in H^\infty_{M_n}(\mathbb{T})$ and inner function $q \in H^\infty(\mathbb{T})$. Hence it follows from Theorem 2.1 that $\overline{\sum} = \mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$. Let $\xi_{ki} = e_i \otimes v_k$, where $v_k = (0, \ldots, 1, 0 \ldots) \in \mathbb{C}^n$ with 1 in the k-th position. Let $\overline{\xi}_{ki} = \overline{e_i} \otimes v_k, k = 0, \ldots, n,\ i \in \mathbb{Z}_+$. Then $\{\xi_{ki}\}_{k=0}^\infty$ is an orthonormal basis for $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Let $T \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}_n}(\mathbb{T}))$ be such that $T^*_{qI_{n\times n}} T T_{qI_{n\times n}} = T$ for all inner functions $q \in H^\infty(\mathbb{T})$. Define $\Xi : \sum \to \mathbb{C}$ as $\Xi\left(\overline{qI_{n\times n}\xi_{ki}}\Psi\right) = \langle \Psi, TqI_{n\times n}\xi_{ki}\rangle$ for all $k = 1, \ldots, n$ and $i \in \mathbb{Z}_+$, where $q \in H^\infty(\mathbb{T})$ is inner and $\Psi \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Then Ξ is well defined and linear. If $\overline{q_1 I_{n\times n}\xi_{ki}}\Psi_1 = \overline{q_2 I_{n\times n}\xi_{ki}}\Psi_2$ for all $k = 1, \ldots, n$ and $i \in \mathbb{Z}_+$, for inner functions $q_1, q_2 \in H^\infty(\mathbb{T}), \Psi_1, \Psi_2 \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ then

$$\begin{aligned}
\Xi\left(\overline{q_1 I_{n\times n}\xi_{ki}}\Psi_1\right) &= \langle \Psi_1, Tq_1 I_{n\times n}\xi_{ki}\rangle \\
&= \left\langle \Psi_1, T^*_{q_2 I_{n\times n}} T T_{q_2 I_{n\times n}} q_1 I_{n\times n}\xi_{ki}\right\rangle \\
&= \langle q_2 I_{n\times n}\Psi_1, T(q_2 q_1) I_{n\times n}\xi_{ki}\rangle \\
&= \langle q_1 I_{n\times n}\Psi_2, Tq_1 q_2 I_{n\times n}\xi_{ki}\rangle \\
&= \left\langle \Psi_2, T^*_{q_1 I_{n\times n}} T T_{q_1 I_{n\times n}} q_2 I_{n\times n}\xi_{ki}\right\rangle \\
&= \langle \Psi_2, Tq_2 I_{n\times n}\xi_{ki}\rangle \\
&= \Xi\left(\overline{q_2 I_{n\times n}\xi}_{ki}\Psi_2\right).
\end{aligned}$$

Further, $\left|\Xi\left(\overline{qI_{n\times n}\xi}_{ki}\Psi\right)\right| \leq \|T\|\|\Psi\|_2 = \|T\|\|\overline{qI_{n\times n}\xi}_{ki}\Psi\|_2$. Thus Ξ is a bounded, linear functional on $\sum$. Since $\overline{\sum} = \mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$, hence there exists a unique $\Phi \in \mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ such that $\Xi\left(\overline{qI_{n\times n}\xi}_{ki}\Psi\right) = \left\langle \overline{qI_{n\times n}\xi}_{ki}\Psi, \Phi\right\rangle$ and $\left|\left\langle \overline{qI_{n\times n}\xi}_{ki}\Psi, \Phi\right\rangle\right| = \left|\Xi\left(\overline{qI_{n\times n}\xi}_{ki}\Psi\right)\right| \leq \|T\|\ \|\overline{qI_{n\times n}\xi}_{ki}\Psi\|_2$. Thus $\Phi \in L^\infty_{M_n}(\mathbb{T})$ and

$$\begin{aligned}
\langle \Psi, TqI_{n\times n}\xi_{ki}\rangle &= \Xi\left(\overline{qI_{n\times n}\xi}_{ki}\Psi\right) = \left\langle \overline{qI_{n\times n}\xi_{ki}}\Psi, \Phi\right\rangle \\
&= \langle \Psi, \Phi qI_{n\times n}\xi_{ki}\rangle = \langle \Psi, T_\Phi qI_{n\times n}\xi_{ki}\rangle .
\end{aligned}$$

Since the set of all finite linear combinations of inner functions is dense in $H^2(\mathbb{T})$ and $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = H^2(\mathbb{T})\oplus H^2(\mathbb{T})\oplus\ldots\oplus H^2(\mathbb{T})$, we have $T = T_\Phi$.Now let $\theta \in H^\infty(\mathbb{T})$ be an inner function and $T \in \mathcal{L}\left(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\right)$ and suppose $T^{*^m}_{\theta I_{n\times n}} T T^m_{\theta I_{n\times n}} f \to Lf$ for all $f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Then $T^{*^{m+1}}_{\theta I_{n\times n}} T\ T^{m+1}_{\theta I_{n\times n}} f \to T^*_{\theta I_{n\times n}} L T_{\theta I_{n\times n}} f$. Hence $T^*_{\theta I_{n\times n}} L T_{\theta I_{n\times n}} = L$ for all inner functions θ. From the first part, it follows that $L = T_\Phi$ for some $\Phi \in L^\infty_{M_n}(\mathbb{T})$. ■

In Lemma 2.3, we show that if H is a Hankel operator on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$, then $\{T^{*^m}_{qI_{n\times n}} H T^m_{qI_{n\times n}}\}$ converges to 0 in the strong operator topology.

Lemma 2.3. *If $H \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ is a Hankel operator in $\mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$, then $HT^m_{qI_{n\times n}} \to 0$ in the strong operator topology for all inner functions $q \in H^\infty(\mathbb{T})$.*

Proof. From [5], it follows that $HT^m_{qI_{n\times n}} = T^{*^m}_{q^+I_{n\times n}} H$ and $\|T^{*^m}_{q^+I_{n\times n}} Hf\|^2 \le \sum_{i,j=1}^{n} \|T^{*^m}_{q^+} H_{\phi_{ij}} f_j\|^2$ if $H = [H_{\phi_{ij}}]_{1\le i,j\le n}$ and $f = (f_1, f_2, \ldots, f_n) \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Thus from [5], it follows that $T^{*^m}_{q^+I_{n\times n}} H \to 0$ strongly. ■

Let $\mathcal{T}^+$ denote the Hankel algebra, the norm-closed algebra generated by all Toeplitz operators and all Hankel operators together defined on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Let $\mathcal{T}$ denote the Toeplitz algebra, the norm-closed algebra generated by the set of all Toeplitz operators defined on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. In Theorem 2.4, we show that if $T \in \mathcal{T}^+$ then the sequence $\{T^{*^m}_{qI_{n\times n}} T T^m_{qI_{n\times n}}\}$ converges to a Toeplitz operator defined on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ in the strong operator topology for all inner functions $q \in H^\infty(\mathbb{T})$.

Theorem 2.4. *If $T \in \mathcal{T}^+$ then $\{T^{*^m}_{qI_{n\times n}} T T^m_{qI_{n\times n}}\}$ converges to a Toeplitz operator defined on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ for all inner functions $q \in H^\infty(\mathbb{T})$.*

Proof. Notice that if $\Phi \in L^\infty_{M_n}(\mathbb{T})$, then $M_\Phi = PM_\Phi P + PM_\Phi(I-P) + (I-P)M_\Phi P + (I-P)M_\Phi(I-P)$. Let $J\Phi(z) = \Phi(\bar{z})$. Then for all $f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$, $JT_{J\Phi}f = JP((J\Phi)f) = JPJ(\Phi Jf) = (I-P)\Phi Jf = (I-P)\Phi(I-P)Jf$. Hence $JT_{J\Phi}Jf = (I-P)M_\Phi(I-P)f$ for all $f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Similarly, $JH_{J\Phi}f = J(I-P)((J\Phi)f) = J(I-P)J(\Phi Jf) = P\Phi Jf = PM_\Phi(I-P)Jf$ for all $f \in \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$. Hence $JH_{J\Phi}J = PM_\Phi(I-P)$. Since the map J is unitary from $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ onto itself, hence the map M_Φ can be expressed as an operator matrix with respect to the decomposition $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T}) = (\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))^\perp \bigoplus \mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$, as

$$M_\Phi = \begin{pmatrix} T_{J\Phi} & H_\Phi \\ H_{J\Phi} & T_\Phi \end{pmatrix}.$$

If Φ and Ψ are in $L^\infty_{M_n}(\mathbb{T})$, then $M_{\Phi\Psi} = M_\Phi M_\Psi$ and therefore (multiply matrices and compare lower right corners)

$$T_{\Phi\Psi} = T_\Phi T_\Psi + H_{J\Phi} H_\Psi \tag{1}$$

and (compare upper right corners)

$$H_{\Phi\Psi} = T_{J\Phi} H_\Psi + H_\Phi T_\Psi. \tag{2}$$

If $\Phi_1, \dots, \Phi_k$ are in $L^\infty_{M_n}(\mathbb{T})$ and if $\Phi = \Phi_1 \cdots \Phi_k$, then

$$\begin{aligned} T_{\Phi_1} \cdots T_{\Phi_k} - T_{\Phi_1 \cdots \Phi_k} &= T_{\Phi_1} T_{\Phi_2 \cdots \Phi_k} - T_{\Phi_1 (\Phi_2 \cdots \Phi_k)} + T_{\Phi_1} (T_{\Phi_2} T_{\Phi_3 \cdots \Phi_k} \\ &\quad - T_{\Phi_2 (\Phi_3 \cdots \Phi_k)}) + T_{\Phi_1} T_{\Phi_2} (T_{\Phi_3} T_{\Phi_4 \cdots \Phi_k} - T_{\Phi_3 (\Phi_4 \cdots \Phi_k)}) \\ &\quad + \cdots + T_{\Phi_1} T_{\Phi_2} \cdots T_{\Phi_{k-2}} (T_{\Phi_{k-1}} T_{\Phi_k} - T_{\Phi_{k-1} \Phi_k}). \end{aligned}$$

In view of this, equation (1) implies that $T_{\Phi_1} \cdots T_{\Phi_k} - T_{\Phi_1 \cdots \Phi_k} = HH + THH + TTHH + \cdots + TT \cdots THH$, where each T on the right hand side indicates a Toeplitz operator and each H a Hankel operator; since the actual subscripts are not so important, they are omitted. Hence

$$\begin{aligned} &T^{*^m}_{qI_{n\times n}} T_{\Phi_1} \dots T_{\Phi_k} T^m_{qI_{n\times n}} - T^{*^m}_{qI_{n\times n}} T_{\Phi_1 \dots \Phi_k} T^m_{qI_{n\times n}} \\ &= T^{*^m}_{qI_{n\times n}} HH T^m_{qI_{n\times n}} + T^{*^m}_{qI_{n\times n}} THH T^m_{qI_{n\times n}} + \cdots + T^{*^m}_{qI_{n\times n}} TT \dots THH T^m_{qI_{n\times n}}. \end{aligned}$$

By Lemma 2.3, the right side converges strongly to 0 as $n \to \infty$. We next consider a finite product all whose factors are either Toeplitz or Hankel operators, with at least one Hankel factor present. If the rightmost factor is a Hankel operator, then by Lemma 2.3, the product converges strongly to 0. In the remaining cases, the first Hankel factor from the right occurs in a context HT, where, as before, the symbols H and T indicate generic Hankel and Toeplitz operators respectively. In such a case, we shall use (2) to replace HT by $H - TH$, and thus manage to replace the given operator by two others, in each of which the rightmost Hankel factor is one step nearer to the right end; the desired convergence now follows by repeating this process. Now let $\mathcal{T}_0^+$ be the algebra (not closed) consisting of all finite sums of finite products of Toeplitz and Hankel operators. If $T \in \mathcal{T}_0^+$, convergence follows from the strong continuity of operator addition. For norm limits of operators on $\mathcal{T}_0^+$, convergence follows from the standard techniques of analysis. ∎

Thus we have seen that if an operator $T \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ is such that the sequence $\{T^{*^m}_{qI_{n\times n}} T T^m_{qI_{n\times n}}\}$ is strongly convergent then the limit is a Toeplitz operator T_Φ on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ with the symbol $\Phi \in L^\infty_{M_n}(\mathbb{T})$. The function Φ will be called the symbol of T and will be denoted by $\sigma(T)$. In the next theorem we show that the restriction of the symbol map σ to the Hankel algebra is a contractive, *-homomorphism from $\mathcal{T}^+$ onto $L^\infty_{M_n}(\mathbb{T})$.

Theorem 2.5. *The map $\sigma : \mathcal{T}^+ \to L^\infty_{M_n}(\mathbb{T})$ defined by $\sigma(T) = \Phi$ is a contractive, *-homomorphism from $\mathcal{T}^+$ onto $L^\infty_{M_n}(\mathbb{T})$.*

Proof. If $T \in \mathcal{T}^+$, then it follows from Theorem 2.4 that $\{T^{*^m}_{qI_{n\times n}} T T^m_{qI_{n\times n}}\}$ converges strongly for all inner functions $q \in H^\infty(\mathbb{T})$ to a Toeplitz operator T_Φ on $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ and we have defined $\sigma(T) = \Phi$. It follows from [6] that

$\|T_\Phi\| = \|\Phi\|_\infty$. Hence $\|\Phi\|_\infty = \|T_\Phi\| \leq \liminf_{n\to\infty} \|T^{*^m}_{qI_{n\times n}} T T^m_{qI_{n\times n}}\| \leq \|T\|$. Thus the fact that σ preserves sums and products in $\mathcal{T}_0^+$ follows from the main step in the preceeding proof; and that it preserves sums and products for all operators on the Hankel algebra follows from the norm continuity of operator addition and multiplication and the continuity of σ. Now let $T \in \mathcal{T}^+$ and assume $T^{*^m}_{qI_{n\times n}} T T^m_{qI_{n\times n}} \to T_\Phi$ strongly. Then weak continuity of adjunction implies that $T^{*^m}_{qI_{n\times n}} T^* T^m_{qI_{n\times n}} \to T^*_\Phi = T_{\Phi^*}$ weakly. Since $T^* \in \mathcal{T}^+$, the sequence $\{T^{*^m}_{qI_{n\times n}} T^* T^m_{qI_{n\times n}}\}$ converges strongly to some $T_\Psi, \Psi \in L^\infty_{M_n}(\mathbb{T})$. Thus $T_\Psi = T_{\Phi^*}$, and therefore $\sigma(T^*) = \sigma(T)^*$. ∎

In Lemma 2.6, we show that if $K \in \mathcal{LC}\left(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\right)$ then $\{T^*_{\theta I_{n\times n}} K T^m_{\theta I_{n\times n}}\}$ converges to 0 in norm as $m \to \infty$.

Lemma 2.6. *Let $\theta \in H^\infty(\mathbb{T})$ be a nonconstant inner function and $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Then $T^{*^m}_\theta \xrightarrow{s} 0$ on $H^2(\mathbb{D})$ and for each compact operator $K \in \mathcal{LC}\left(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\right), T^{*^m}_{\theta I_{n\times n}} K \longrightarrow 0$ in norm as $m \to \infty$. Hence $K T^m_{\theta I_{n\times n}} \to 0$ in norm as $m \to \infty$.*

Proof. The reproducing kernels $\{g_\lambda\}_{\lambda\in\mathbb{D}}$ of $H^2(\mathbb{D})$ span $H^2(\mathbb{D})$ (see [15]). Let $f = \sum_{i=1}^k c_i g_{\lambda_i}$. Notice that $T^{*^m}_\theta f = T^{*^m}_\theta \left(\sum_{i=1}^k c_i g_{\lambda_i}\right) = \sum c_i \overline{\theta(\lambda_i)}^m g_{\lambda_i}$ and $\|T^{*^m}_\theta f\| \leq \sum |c_i||\theta(\lambda_i)|^m \|g_{\lambda_i}\| \to 0$ as $m \to \infty$. Let $K \in \mathcal{LC}\left(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\right)$. Then

$$K = (K_{ij})_{1\leq i,j\leq n}, K_{ij} \in \mathcal{LC}(H^2(\mathbb{T})).$$

Now $T^{*^m}_\theta K = \left(T_{\overline{\theta}^m} K_{ij}\right)_{1\leq i,j\leq n}$ and $\|T^{*^m}_\theta K\| \leq \sup_{1\leq i,j\leq n} \|T_{\overline{\theta}^m} K_{ij}\|$. For $f, h \in H^2(\mathbb{D})$, $T_{\overline{\theta}^m}(f \otimes h) = \left(T_{\overline{\theta}^m} f\right) \otimes h$. Hence the result follows as $\overline{L\mathcal{F}\left(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\right)} = \mathcal{LC}\left(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\right)$. ∎

Corollary 2.7. *If $T \in \mathcal{T}^+$ is a finite sum of finite products of Toeplitz or Hankel operators and is compact, then the corresponding finite sum of finite products of their symbols is equal to 0 almost everywhere.*

Proof. The corollary follows from Lemma 2.6 and Theorem 2.4. ∎

Theorem 2.8. *If $T \in \mathcal{T}$ then T belongs to the commutator ideal I of $\mathcal{T}$ if and only if $T^{*^m}_{qI_{n\times n}} T T^m_{qI_{n\times n}} \to 0$ strongly. That is, $I = \ker \sigma$.*

Proof. We first show that if T belongs to the algebra $\mathcal{T}_0$ consisting of all finite sums of finite products of Toeplitz operators, and if $\Psi = \sigma(T)$, then $T - T_\Psi \in I$. Let $\Phi_1, \Phi_2, \cdots, \Phi_k \in L^\infty_{M_n}(\mathbb{T})$ and $\Psi = \Phi_1\Phi_2\cdots\Phi_k$ and $T = T_{\Phi_1}T_{\Phi_2}\cdots T_{\Phi_k}$. We claim $T - T_\Psi \in I$. The proof follows by using mathematical induction. For $k = 1$, the assertion is trivial. Now suppose this is true for $k-1$. We prove it for k. Let $\Phi_k = \overline{\eta}\Omega, \eta \in H^\infty(\mathbb{T})$ is an inner function and $\Omega \in H^\infty_{M_n}(\mathbb{T})$. Then

$$\begin{aligned} T - T_{\Psi} &= T_{\Phi_1} \dots T_{\Phi_{k-1}} T_{\bar{\eta}\Omega} - T_{\Phi_1 \Phi_2 \dots \Phi_{k-1} \bar{\eta}\Omega} \\ &= T_{\Phi_1} \dots T_{\Phi_{k-1}} T_{\bar{\eta}} T_{\Omega} - T_{\bar{\eta}} T_{\Phi_1 \dots \Phi_{k-1}} T_{\Omega} \\ &= (T_{\Phi_1} \dots T_{\Phi_{k-1}} T_{\bar{\eta}} - T_{\bar{\eta}} T_{\Phi_1 \dots \Phi_{k-1}}) T_{\Omega} \\ &= ([T_{\Phi_1} \dots T_{\Phi_{k-1}} T_{\bar{\eta}} - T_{\bar{\eta}} T_{\Phi_1} \dots T_{\Phi_{k-1}}] \\ &\quad + [T_{\bar{\eta}} T_{\Phi_1} \dots T_{\Phi_{k-1}} - T_{\bar{\eta}} T_{\Phi_1 \dots \Phi_{k-1}}]) T_{\Omega}. \end{aligned}$$

The first square bracket is a commutator and therefore belongs to I. The second square bracket is T_η times an operator of the same form as $T - T_\Psi$ except with $k-1$ instead of k, and, consequently, by our induction hypothesis it belongs to I. Since every inner function $\Theta \in H^\infty_{M_n}(\mathbb{T})$ can be written as $\bar{\theta}\Omega$ where θ is an inner function in $H^\infty(\mathbb{T})$ and $\Omega \in H^\infty_{M_n}(\mathbb{T})$, it follows from Theorem 2.1 that $\{\bar{\theta}\Omega : \theta \in H^\infty(\mathbb{T}) \text{ is inner}, \ \Omega \in H^\infty_{M_n}(\mathbb{T})\}$ is dense in $L^\infty_{M_n}(\mathbb{T})$. Thus using the above fact we can conclude $T - T_\Psi \in I$ for all $\Phi_k \in L^\infty_{M_n}(\mathbb{T})$. Now suppose that $T = T_1 + \dots + T_p$, where each T_i is a finite product of Toeplitz operators. It follows that $\Psi = \sigma(T) = \Psi_1 + \dots + \Psi_p$, where $\Psi_i = \sigma(T_i), i = 1, \dots, p$. This implies $T - T_\Psi = (T_1 - T_{\Psi_1}) + \dots + (T_p - T_{\Psi_p}) \in I$. Finally, suppose $T \in \mathcal{T}$ with $\sigma(T) = 0$. Let $\{T_n\}$ be a sequence in $\mathcal{T}_0$ such that $\|T_m - T\| \to 0$ as $m \to \infty$. If $\Psi_i = \sigma(T_i)$, then $\Psi_m \to 0$ in $L^\infty_{M_n}(\mathbb{T})$ as $\sigma(T) = 0$. Hence $T_m - T_{\Psi_m} \to T$ in norm. Since we have already shown that $T_m - T_{\Psi_m} \in I$ for each m, it follows that $T \in I$. Thus $\ker \sigma \subset I$. Now since $\mathcal{T} / \ker \sigma$ is commutative, the reverse inclusion is trivial. ■

Using the above asymptotic properties of Toeplitz operators we now proceed to prove that $[S, T_{\theta I_{n\times n}}]$ is of finite rank for all inner functions $\theta \in H^\infty(\mathbb{T})$ if and only if $S = T_\Phi + F$, where F is a finite rank operator and $\Phi \in H^\infty_{M_n}(\mathbb{T}) + R^p_{M_n}$. But first we prove the following lemma.

Lemma 2.9. *Let* $S \in \mathcal{L}\left(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\right)$. *If* $\left[S, T_{\theta I_{n\times n}}\right]$ *is of finite rank for all inner functions* $\theta \in H^\infty(\mathbb{T})$, *then there exists a natural number* M *such that* $\operatorname{rank}\left[S, T_{\theta I_{n\times n}}\right] \le M$.

Proof. Let $\mathcal{C}_n = \left\{\theta \in H^\infty(\mathbb{T}) : \operatorname{rank}\ \left[S, T_{\theta I_{n\times n}}\right] \le n\right\}$. By [12], $\mathcal{C}_n$ is a norm closed subset of $H^\infty(\mathbb{T})$. Since $H^\infty(\mathbb{T}) = \bigcup_n \mathcal{C}_n$ and $H^\infty(\mathbb{T})$ is a Banach space hence we obtain by Baire category theorem [11] that there exists a natural number N such that $\mathcal{C}_N$ contains an open subset of $H^\infty(\mathbb{T})$. That is, the interior of $\mathcal{C}_N = \mathcal{C}^0_N \neq \emptyset$. Thus $\left\{\theta - \Psi : \theta, \Psi \in \mathcal{C}^0_N\right\}$ is a neighbourhood of the function $\theta \equiv 0$. Thus for each $f \in H^\infty(\mathbb{T})$, there exist a real number ϵ and functions $\theta, \Psi \in \mathcal{C}^0_N$ such that $f = \epsilon(\theta - \Psi)$ and therefore

$$\operatorname{rank}\left[S, T_{f I_{n\times n}}\right] \le \operatorname{rank}\left[S, T_{\theta I_{n\times n}}\right] + \operatorname{rank}\left[S, T_{\Psi I_{n\times n}}\right] \le 2N.$$

■

Let $C(\mathbb{T})$ denote the space of continuous, complex-valued functions on $\mathbb{T}$ and $C_{M_n}(\mathbb{T})$ denote the space of continuous, $M_n(\mathbb{C})$-valued functions on $\mathbb{T}$. Let R_p

denote the set of rational functions in $C(\mathbb{T})$ with at most p poles (counting multiplicities) all of which are in the interior of $\mathbb{T}$. Note the following inclusion relations hold:

$$H^\infty \subset H^\infty + R_1 \subset H^\infty + R_2 \subset \ldots \subset H^\infty + C(\mathbb{T}).$$

Let $R^p_{M_n}$ denote the set of all matrices in $C_{M_n}(\mathbb{T})$ whose entries are in R_p.

Theorem 2.10. *If* $\Phi \in L^\infty_{M_n}(\mathbb{T}), T_\Phi \in \mathcal{L}\left(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\right)$ *and* $\left[T_\Phi, T_{\theta I_{n\times n}}\right]$ *is of finite rank for all inner functions* $\theta \in H^\infty(\mathbb{T})$ *then* $\Phi \in H^\infty_{M_n}(\mathbb{T}) + R^p_{M_n}$.

Proof. By Lemma 2.9, there exists a natural number M such that $\left[T_\Phi, T_{\theta I_{n\times n}}\right] \leq M$ for all $\theta \in H^\infty(\mathbb{T})$. Let $\Phi = (\phi_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}, \phi_{ij} \in H^\infty(\mathbb{T})$. Then $T_\Phi = \left(T_{\phi_{ij}}\right)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}$. Now the fact that $\left[T_\Phi, T_{\theta I_{n\times n}}\right]$ is of finite rank for all $\theta \in H^\infty(\mathbb{T})$ implies $[T_{\phi_{ij}}, T_\theta]$ is of finite rank for all $\theta \in H^\infty(\mathbb{T})$ and

$$\begin{aligned}
\text{rank}\left(T_{\overline{\eta}\theta\phi_{ij}} - T_{\overline{\eta}\theta}T_{\phi_{ij}}\right) &= \text{rank}\left(T_{\overline{\eta}}\left(T_{\phi_{ij}}T_\theta - T_\theta T_{\phi_{ij}}\right)\right)\\
&\leq \text{rank}\left(T_{\phi_{ij}}T_\theta - T_\theta T_{\phi_{ij}}\right)\\
&= \text{rank}\ \left[T_{\phi_{ij}}, T_\theta\right]\\
&\leq M.
\end{aligned}$$

Since $\{\overline{\eta}\theta : \eta \in H^\infty(\mathbb{T})$ is an inner function, $\theta \in H^\infty(\mathbb{T})\}$ is [7] dense in $L^\infty(\mathbb{T})$, we obtain $T_{\overline{\phi}_{ij}\phi_{ij}} - T_{\overline{\phi}_{ij}}T_{\phi_{ij}} = H^*_{\overline{\phi}_{ij}}H_{\phi_{ij}}$ is of finite rank. That is, $H_{\phi_{ij}}$ is of finite rank. By Kronecker's theorem [13], this implies $\phi_{ij} \in H^\infty + R_p$ for $1 \leq i \leq n, 1 \leq j \leq n$. Hence $\Phi \in H^\infty_{M_n} + R^p_{M_n}$. ∎

Theorem 2.11. *Given* $S \in \mathcal{L}\left(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})\right), [S, T_{\theta I_{n\times n}}]$ *is of finite rank for all inner functions* $\theta \in H^\infty(\mathbb{T})$ *if and only if* $S = T_\Phi + F$, *where* F *is a finite rank operator and* $\Phi \in H^\infty_{M_n}(\mathbb{T}) + R^p_{M_n}$.

Proof. The sequence $\left\{T^{*^m}_{\theta I_{n\times n}} S T^m_{\theta I_{n\times n}}\right\}$ is a bounded sequence and hence it has a subsequence which converges to an operator L in the weak operator topology. Without loss of generality, we assume the sequence $\left\{T^{*^m}_{\theta I_{n\times n}} S T^m_{\theta I_{n\times n}}\right\}$ converges to L in weak operator topology (WOT, for short). Now $S - T^{*^m}_{\theta I_{n\times n}} S T^m_{\theta I_{n\times n}} \xrightarrow{WOT} S - L$. By Lemma 2.9, there exists $M \geq 0$ such that rank $[S, T_\eta I_{n\times n}] \leq M$ for all $\eta \in H^\infty$ and thus

$$\begin{aligned}
\text{rank}\left(S - T^{*^m}_{\theta I_{n\times n}} S\ T^m_{\theta I_{n\times n}}\right) &= \text{rank}\left(T^{*^m}_{\theta I_{n\times n}}\left(T^m_{\theta I_{n\times n}} S - S\ T^m_{\theta I_{n\times n}}\right)\right)\\
&\leq \text{rank}\left(T_{\theta^m I_{n\times n}} S - S\ T_{\theta^m I_{n\times n}}\right)\\
&\leq M.
\end{aligned}$$

Hence by [12], rank$(S-L) \leq M$. Let $F = S-L$. Then F is a finite rank operator. For each inner function $q \in H^\infty(T)$,

$$\begin{aligned} T_q^* L T_q &= (\mathrm{WOT})\ \lim T_q^* T^*_{\theta^m I_{n\times n}} S\ T_{\theta^m I_{n\times n}} T_q \\ &= (\mathrm{WOT})\ \lim T^*_{\theta^m I_{n\times n}} T_q^* S\ T_q T_{\theta^m I_{n\times n}} \\ &= (\mathrm{WOT})\ \lim T^*_{\theta^m I_{n\times n}} T_q^* T_q S T_{\theta^m I_{n\times n}} \\ &\quad + (\mathrm{WOT})\ \lim T_q^* T^*_{\theta^m I_{n\times n}} \left(S T_q - T_q S\right) T_{\theta^m I_{n\times n}}. \end{aligned}$$

Since $[S, T_{qI_{n\times n}}]$ is a finite rank operator, by Lemma 2.6,

$$(\mathrm{WOT})\ \lim T_q^* T^*_{\theta^m I_{n\times n}} [S, T_q] T_{\theta^m I_{n\times n}} = 0.$$

Thus it follows that $T_q^* L T_q = (\mathrm{WOT})\ \lim T^*_{\theta^m I_{n\times n}} T_q^* T_q S T_{\theta^m I_{n\times n}} = L$. Hence L is a Toeplitz operator and $L = T_\Phi$ for some $\Phi \in L^\infty_{M_n}(\mathbb{T})$. By Theorem 2.10, $\Phi \in H^\infty_{M_n}(\mathbb{T}) + R^p_{M_n}$. ■

3. Toeplitz Operators on Bergman Space

In this section we discuss about asymptotic properties of Toeplitz operators defined on the Bergman space. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi}\ r dr d\theta$ be the normalized Lebesgue measure on $\mathbb{D}$. We define

$$A^2(\mathbb{D}) = \left\{ f \in L^2(\mathbb{D}, dA) : f \ \text{ is analytic on } \mathbb{D} \right\}.$$

This is a closed subspace of $L^2(\mathbb{D}, dA)$. It is a Hilbert space with the inner product it inherits from $L^2(\mathbb{D}, dA)$. The space $A^2(\mathbb{D})$ is called the Bergman space. The set $\{u_n(z)\}_{n\geq 0} = \left\{\sqrt{n+1} z^n\right\}_{n\geq 0}$ is an orthonormal basis for $A^2(\mathbb{D})$. Let $\phi \in L^\infty(\mathbb{D})$. We define the multiplication operator $M_\phi : L^2(\mathbb{D}) \longrightarrow L^2(\mathbb{D})$, the Toeplitz operator $B_\phi : A^2(\mathbb{D}) \longrightarrow A^2(\mathbb{D})$, the Hankel operator $H_\phi : A^2(\mathbb{D}) \longrightarrow L^2(\mathbb{D}) \ominus A^2(\mathbb{D})$ and the little Hankel operator $S_\phi : A^2(\mathbb{D}) \longrightarrow A^2(\mathbb{D})$ with symbol ϕ, respectively, by the formulas $M_\phi f = \phi f$, $B_\phi f = \widetilde{P}(\phi f)$, $H_\phi f = \left(I - \widetilde{P}\right)(\phi f)$, $S_\phi f = \widetilde{P}(J(\phi f))$ where $\widetilde{P}$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$ and $J : L^2(\mathbb{D}) \longrightarrow L^2(\mathbb{D})$ is defined by $Jf(z) = f(\overline{z})$. These operators are clearly linear, bounded, their norms do not exceed $\|\phi\|_\infty$. The Toeplitz operator B_z is called the Bergman shift operator. Define a map from $H^2(\mathbb{D})$ onto $A^2(\mathbb{D})$ as $Wz^n = \sqrt{n+1} z^n$. Notice that W is a unitary operator. Let $\mathcal{A}(B_z) = \{T \in \mathcal{L}(A^2(\mathbb{D})) : TB_z - B_z\ T \ \text{ is compact}\}$. For $\phi \in L^\infty(\mathbb{D}), B_\phi \in \mathcal{A}(B_z)$. This can be verified as follows:

$$\begin{aligned} B_\phi - B_z^* B_\phi B_z &= B_\phi - B_{\overline{z}} B_\phi B_z \\ &= B_{\phi - \overline{z}\phi z} \\ &= B_{(1-|z|^2)\phi} \end{aligned}$$

which is compact by [15].

Proposition 3.1. *If $T \in \mathcal{L}(A^2(\mathbb{D}))$ and $T_q^* W^* T W T_q = W^* T W$ for all inner functions $q \in H^\infty(\mathbb{T})$ then $T = W T_{\widetilde{\phi}} W^*$ for some $\widetilde{\phi} \in L^\infty(\mathbb{T})$ and $T \in \mathcal{A}(B_z)$. Further, if $\widetilde{\phi} \geq 0$ then T is positive.*

Proof. The proposition follows from [4] and [12]. ∎

Proposition 3.2. *If $S \in \mathcal{L}(A^2(\mathbb{D})), [W^* S W, T_f]$ is of finite rank for all $f \in H^\infty(\mathbb{T})$ then $S = W T_g W^* + \widetilde{F}$ where $\widetilde{F}$ is a finite rank operator and $g \in H^\infty(\mathbb{T}) + R_p$.*

Proof. The proof follows from Theorem 2.11. ∎

Theorem 3.3. *Let $T \in \mathcal{L}(A^2(\mathbb{D}))$. Then there exists a sequence $\{\Psi_m\} \in L^\infty(\mathbb{D})$ such that $B_{\Psi_m} \to T$ in the strong operator topology (SOT, for short) and $B^*_{\Psi_m} \longrightarrow T^*$ in the SOT.*

Proof. Let $T \in \mathcal{L}(A^2(\mathbb{D}))$. By [9], the set $\{B_\phi : \phi \in L^\infty(\mathbb{D})\}$ is strongly dense in $\mathcal{L}(A^2(\mathbb{D}))$. Hence there exists a sequence $\{\phi_m\}$ in $L^\infty(\mathbb{D})$ such that $B_{\phi_m} \overset{SOT}{\longrightarrow} T$. By [10], there exists $\Psi_m = \sum_{k=1}^{r_m} C_k \phi_k, C_k \geq 0, \sum_{k=1}^{r_m} C_k = 1$, such that $B_{\Psi_m} \overset{SOT}{\longrightarrow} T$ and $B^*_{\Psi_m} \overset{SOT}{\longrightarrow} T^*$. ∎

Theorem 3.4. *Let $\{\Psi_m\}$ be a sequence in $\overline{H^\infty(\mathbb{D})}$. Then there exists a sequence $\{\phi_m\}$ in $L^\infty(\mathbb{D})$ and a subsequence $\{\Psi_{m_k}\}$ of $\{\Psi_m\}$ such that for all $f \in A^2(\mathbb{D})$, $\lim_{m_k, m \to \infty} \|B_{\phi_m} f - S_{\Psi_{m_k}} f\| = 0$ and $\lim_{m_k, m \to \infty} \|B^*_{\phi_m} f - S^*_{\Psi_{m_k}} f\| = 0$.*

Proof. Let $\Psi \in L^\infty(\mathbb{D})$. By Theorem 3.3, for $S_\Psi \in \mathcal{L}(A^2)$, there exists a sequence of Toeplitz operators $\{B_{\phi_m}\}, \phi_m \in L^\infty(\mathbb{D})$ such that $\lim_{m \to \infty} \|B_{\phi_m} f - S_\Psi f\| = 0$ and $\lim_{m \to \infty} \|B^*_{\phi_m} f - S^*_\Psi f\| = 0$ for all $f \in A^2(\mathbb{D})$. From [1], it follows that there exists a subsequence $\{S_{\Psi_{m_k}}\}$ of $\{S_{\Psi_m}\}$ such that $\lim_{m_k \to \infty} \|S_{\Psi_{m_k}} f - S_\Psi f\| = 0$ and $\lim_{m_k \to \infty} \left\|S^*_{\Psi_{m_k}} f - S^*_\Psi f\right\| = 0$ for all $f \in A^2(\mathbb{D})$. Thus $\lim_{m_k, m \to \infty} \|B_{\phi_m} f - S_{\Psi_{m_k}} f\| = 0$ and $\lim_{m_k, m \to \infty} \|B^*_{\phi_m} f - S^*_{\Psi_{m_k}} f\| = 0$ for all $f \in A^2(\mathbb{D})$. ∎

Definition 3.5. A function $G \in A^2(\mathbb{D})$ is called an *inner function* in $A^2(\mathbb{D})$ if $\int_\mathbb{D} (|G(z)|^2 - 1) g(z) dA(z) = 0$ for all $g \in H^\infty(\mathbb{D})$.

For $a \in \mathbb{D}, z \in \mathbb{D}$, let $\phi_a(z) = \dfrac{a - z}{1 - \overline{a} z}$. It can be verified easily that
(i) $\phi_a o \phi_a(z) = z$,
(ii) $\phi_a(0) = a, \phi_a(a) = 0$,
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

Let $K(z,\omega) = \dfrac{1}{(1-z\overline{\omega})^2} = \sum_{m=1}^{\infty} u_m(z)\,\overline{u_m(\omega)}$, the reproducing kernel of $A^2(\mathbb{D})$. It is holomorphic in z and anti-holomorphic in ω, and

$$\int_{\mathbb{D}} |K(z,\omega)|^2 dA(\omega) = K(z,z) > 0, \text{ for all } z \in \mathbb{D}.$$

Thus we define for each $\omega \in \mathbb{D}$, a unit vector k_ω in $A^2(\mathbb{D})$ by

$$k_\omega(z) = \frac{K(z,\omega)}{\sqrt{K(\omega,\omega)}} = \frac{(1-|\omega|^2)}{(1-z\overline{\omega})^2}.$$

We shall write $K(z,\omega) = \overline{K_z(\omega)}$, for $z, \omega \in \mathbb{D}$.

Theorem 3.6. *Let G be a nonconstant inner function in $A^2(\mathbb{D})$. Then $B_{\overline{G}^m} \xrightarrow{SOT} 0$ on $A^2(\mathbb{D})$ and $B_{\overline{G}^m} K \longrightarrow 0$ in norm for each compact operator $K \in \mathcal{L}(A^2)$.*

Proof. Notice that $\|G\|_{A^2(\mathbb{D})} = 1$ and $|G(z)| < 1$ for all $z \in \mathbb{D}$. It is well known [15] that the reproducing kernels $\{K_\lambda\}_{\lambda \in \mathbb{D}}$ span $A^2(\mathbb{D})$. Let $f = \sum_{i=1}^{m} b_i K_{\lambda_i}$. Then

$$B_{\overline{G}^m}\left(\sum_{i=1}^{m} b_i K_{\lambda_i}\right) = \sum_{i=1}^{m} b_i \overline{G(\lambda_i)}^m K_{\lambda_i}.$$

Hence

$$\left\| B_{\overline{G}^m}\left(\sum_{i=1}^{m} b_i K_{\lambda_i}\right)\right\| \le \sum_{i=1}^{m} |b_i||G(\lambda_i)|^m \|K_{\lambda_i}\| \to 0$$

as $m \to \infty$. Thus $B_{\overline{G}^m} \longrightarrow 0$ in the strong operator topology. Consider the rank one operator $f \otimes g$ defined by $(f \otimes g)(h) = \langle h, g\rangle f$. Notice that $B_{\overline{G}^m}(f \otimes g) = (B_{\overline{G}^m} f) \otimes g$. Since $\overline{\mathcal{LF}(A^2(\mathbb{D}))} = \mathcal{LC}(A^2(\mathbb{D}))$, we obtain $\|B_{\overline{G}^m} K\| \longrightarrow 0$ as $m \to \infty$. ∎

Given $\lambda \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define two functions $C_\lambda f$ and $U_\lambda f$ on $\mathbb{D}$ by $C_\lambda f(z) = f(\phi_\lambda(z))$ and $U_\lambda f(z) = k_\lambda(z) f(\phi_\lambda(z))$. The map C_λ is a composition operator on various spaces. For example, C_λ is a bounded composition operator on $L^2(\mathbb{D}, dA)$ and $A^2(\mathbb{D})$ for all $\lambda \in \mathbb{D}$. Since $|k_\lambda|^2$ is the real Jacobian determinant of the mapping ϕ_λ, the operator U_λ is easily seen to be a unitary operator on $L^2(\mathbb{D}, dA)$ and $A^2(\mathbb{D})$. It is also easy to check that $U_\lambda^* = U_\lambda$, thus U_λ is a self adjoint unitary operator. Now since $k_a(z) k_a(\phi_a(z)) = 1$ for each $a \in \mathbb{D}$, hence $U_a k_a = 1$ and $U_a k_{m_a} = k_{m_a}$, where m_a is the geodesic midpoint between 0 and a (see [14]), i.e, $m_a = \dfrac{1-\sqrt{1-|a|^2}}{|a|^2} a$. Further $U_z k_\omega = k_{\phi_z(\omega)}$.

Theorem 3.7. *Let $z \in \mathbb{D}$. Suppose $M_z = \{k_{m_z}(go\phi_{m_z}) : g$ even, $g \in A^2(\mathbb{D})\}$ is a reducing subspace of the operator $T \in \mathcal{L}(A^2(\mathbb{D}))$. Then $U_z T = T U_z$ and hence $U_z^m\ T\ U_z^m \to T$ in the strong operator topology.*

Proof. Let $U_z = P_z - P_z^+$ be the spectral decomposition of U_z. From [14], it follows that M_z is the range space of P_z and $U_z f = f$ if and only if $P_z f = f$ for all $f \in A^2(\mathbb{D})$. That is, $U_z f = f$ if and only if $f \in M_z$. Hence $TU_z = U_z T$ if and only if $TP_z = P_z T$. This is true if and only if M_z is a reducing subspace of T. Since $U_z^2 = I$, the result follows. ■

Theorem 3.8. *Let $z \in \mathbb{D}$, and m_z be the geodesic midpoint between 0 and z. Let $\phi = go\phi_{m_z}$ where g is an even function in $L^\infty(\mathbb{D})$. Then $U_z^m B_\phi U_z^m \to B_\phi$ in the strong operator topology.*

Proof. From [14], it follows that $\phi o \phi_z = \phi$ as $\phi = go\phi_{m_z}$, g is even. Since $U_z B_\phi U_z = B_{\phi o \phi_z} = B_\phi$ and $U_z^2 = I$; the result follows. ■

For $T \in \mathcal{L}(A^2(\mathbb{D}))$, let $\widehat{T} = \int_{\mathbb{D}} U_a T U_a dA(a)$, where the integral is taken in the sense that $\left\langle \left(\int_{\mathbb{D}} U_a\ T\ U_a dA(a)\right) f, g \right\rangle = \int_{\mathbb{D}} \langle U_a T U_a f, g\rangle\, dA(a)$. For $\phi \in L^\infty(\mathbb{D})$, we can define a function $\widehat{\phi}$ on $\mathbb{D}$ as follows:

$$\widehat{\phi}(z) = \int_{\mathbb{D}} \phi(\phi_\omega(z)) dA(\omega).$$

Theorem 3.9. *If $\phi \in L^\infty(\mathbb{D})$ is such that $\widehat{\phi} = \phi$ then $C_a^m B_\phi C_a^m \longrightarrow B_\phi$ in the strong operator topology.*

Proof. Suppose $\phi \in L^\infty(\mathbb{D})$ and $\widehat{\phi} = \phi$. Then we have $\widehat{B}_\phi = B_{\widehat{\phi}} = B_\phi$. Hence by [14], $C_a B_\phi = B_\phi C_a$ for all $a \in \mathbb{D}$. Since $C_a^2 = I$ on $A^2(\mathbb{D})$, hence $C_a^m B_\phi C_a^m \to B_\phi$ in the strong operator topology. ■

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