Vietnam Journal of Mathematics **38**:4(2010) 395-402

Vietnam Journal of MATHEMATICS

Strong Convergence Theorems of an Implicit Iterative Algorithm with Errors for Asymptotically Quasi-nonexpansive in the Intermediate Sense Mappings

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Received June 11, 2010

Abstract. In this paper, we propose an implicit iterative algorithm with errors for a finite family of asymptotically quasi-nonexpansive in the intermediate sense mappings and establish some strong convergence theorems for said mappings to converge to common fixed points in Banach spaces. The results presented in this paper improve and extend the corresponding results of Liu [5, 6], Shahzad and Udomene [9], Yao and Liou [12] and many others.

2000 Mathematics Subject Classification: 47H09, 47H10.

Key words: Asymptotically quasi-nonexpansive in the intermediate sense mapping, implicit iterative algorithm with errors, common fixed point, Banach space.

1. Introduction

Let C be a nonempty subset of a real Banach space E. Let $T\colon C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a mapping $T\colon C \to C$ is said to be:
(a) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$,
(b) *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|,$$

for all $x \in C$ and $p \in F(T)$,
(c) *asymptotically nonexpansive* if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $r_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + r_n) \|x - y\|, \tag{1}$$

for all $x, y \in C$,
(d) *asymptotically quasi-nonexpansive* if (1) holds for all $x \in C$ and $y \in F(T)$;
(e) *generalized quasi-nonexpansive with respect to* $\{s_n\}$, if there exists a sequence $\{s_n\} \subset [0, 1)$ with $s_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq \|x - p\| + s_n \|x - T^n x\|,$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$,
(f) *generalized asymptotically quasi-nonexpansive* [4] *with respect* to $\{r_n\}$ and $\{s_n\} \subset [0, 1)$ with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq (1 + r_n) \|x - p\| + s_n \|x - T^n x\|,$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 1.1. From the above definition, it is clear that:
(i) a nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;
(ii) a quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;
(iii) an asymptotically nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;
(iv) a generalized quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping.
However, the converse of the above statements is not true.

Recall also that a mapping $T \colon C \to C$ is said to be *asymptotically quasi-nonexpansive* in the intermediate sense [12] provided that T is uniformly continuous and

$$\limsup_{n \to \infty} \sup_{x \in C,\, p \in F(T)} \Big(\|T^n x - p\| - \|x - p\| \Big) \leq 0.$$

Remark 1.2. From the above definition, if $F(T)$ is nonempty, it is easy to see that the generalized asymptotically quasi-nonexpansive mapping must be the asymptotically quasi-nonexpansive in the intermediate sense mapping.

It is well known that the concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has a fixed point. In 1973, Petryshyn and Williamson [8] gave necessary and sufficient conditions for Mann iterative sequence [7] to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [2] extended the results of Petryshyn and Williamson [8] and gave necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

Liu [6] extended results of [2, 8] and gave necessary and sufficient conditions for Ishikawa iterative sequence with errors to converge to fixed point of asymptotically quasi-nonexpansive mappings.

In 2003, Zhou et al. [13] introduced a new class of generalized asymptotically nonexpansive mappings and gave a necessary and sufficient condition for the modified Ishikawa and Mann iterative sequences to converge to fixed points for the class of mappings. Atsushiba [1] studied the necessary and sufficient condition for the convergence of iterative sequences to a common fixed point of the finite family of asymptotically nonexpansive mappings in Banach spaces. Suzuki [10] discussed a necessary and sufficient condition for common fixed points of two nonexpansive mappings and proved some convergence theorems for approximating a common fixed point.

Very recently, Yao and Liou [12] give a necessary and sufficient condition for the iterative sequence to converge to the common fixed points for two asymptotically quasi-nonexpansive in the intermediate sense mappings. More precisely, they proved the following:

Theorem 1.3. (Theorem YL) *Let C be a nonempty closed convex subset of a real Banach space E. Let $T_1, T_2\colon C \to C$ be asymptotically quasi-nonexpansive in the intermediate sense mappings such that $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\omega_n \in C$, $\nu_n \in C$ be two bounded sequences. For any given $x_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be defined by the following:*

$$\begin{aligned} y_n &= (1-\beta_n)\omega_n + \beta_n T_2^n x_n, \\ x_{n+1} &= (1-\alpha_n)\nu_n + \alpha_n T_1^n y_n, \qquad n = 1, 2, \ldots. \end{aligned}$$

Put

$$\begin{aligned} G_n = \max\Big\{ &\sup_{p\in F(T_1)\cap F(T_2)\ \ n\geq 1} \Big(\|T_2^n x_n - p\| - \|x_n - p\| \Big) \vee 0, \\ &\sup_{p\in F(T_1)\cap F(T_2)\ \ n\geq 1} \Big(\|T_1^n y_n - p\| - \|y_n - p\| \Big) \vee 0 \Big\}. \end{aligned}$$

Assume that $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p^* of T_1 and T_2 if and only if

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0,$$

where $d(x, F(T_1) \cap F(T_2))$ *denotes the distance between* x *and the set* $F(T_1) \cap F(T_2)$.

It is our purpose to propose the implicit iterative algorithm with errors for a finite family of asymptotically quasi-nonexpansive in the intermediate sense mappings $\{T_i\}_{i=1}^N$ and give a necessary and sufficient condition for the said iterative sequence and mappings to converge to the common fixed points in Banach spaces. Our results extend the corresponding results of [1, 2, 5, 6, 8, 9, 10, 12, 13] and many others.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Banach space E. Let $T_i \colon C \to C$ $(i \in I = \{1, 2, \ldots, N\})$ be given mappings. For given $x_0 \in C$, the sequence $\{x_n\}$ in C defined iteratively by

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\
x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1^2 x_{N+1} + \gamma_{N+1} u_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 x_{2N} + \gamma_{2N} u_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1^3 x_{2N+1} + \gamma_{2N+1} u_{2N+1}, \\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n, \tag{2}$$

where $n = (k-1)N + i$, $i \in I$ and each $\{u_n\}$ is bounded sequence in C, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three appropriate real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for $n = 1, 2, \ldots$. Process (2) is called the implicit iterative algorithm with errors for a finite family of mappings $\{T_i\}_{i=1}^N$.

In the sequel, we need the following lemma for the main results in this paper.

Lemma 2.1. (see [11]) *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n) a_n + b_n, \qquad n \ge 1.$$

If $\sum_{n=1}^\infty \delta_n < \infty$ *and* $\sum_{n=1}^\infty b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists. In particular, if* $\{a_n\}$ *has a subsequence converging to zero, then* $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Banach space E. Let $T_i\colon C \to C$ $(i \in I = \{1, 2, \ldots, N\})$ be a finite family of asymptotically quasi-nonexpansive in the intermediate sense mappings such that $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let the implicit iterative algorithm with errors $\{x_n\}$ be defined by (2). Put*

$$A_n = \max\left\{\sup_{p\in\mathcal{F},\ n\geq 1}\Big(\|T_i^n x_n - p\| - \|x_n - p\|\Big) \vee 0 : i \in I\right\}. \tag{3}$$

Assume that $\sum_{n=1}^\infty A_n < \infty$, $\sum_{n=1}^\infty \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p^* of the mappings $\{T_i\}_{i=1}^N$ if and only if

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0,$$

where $d(x, \mathcal{F})$ denotes the distance between x and the set $\mathcal{F}$.

Proof. The necessity is obvious and so it is omitted. Now, we prove the sufficiency. For any $p \in \mathcal{F} = \cap_{i=1}^N F(T_i)$, from (2) and (3), we have

$$\begin{aligned}
\|x_n - p\| &= \left\|\alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n - p\right\| \\
&= \left\|\alpha_n(x_{n-1} - p) + \beta_n(T_i^k x_n - p) + \gamma_n(u_n - p)\right\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \left\|T_i^k x_n - p\right\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n(\|x_n - p\| + A_n) \\
&\quad + \gamma_n \|u_n - p\| \\
&= \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n - \gamma_n)\|x_n - p\| + \beta_n A_n \\
&\quad + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)\|x_n - p\| + A_n \\
&\quad + \gamma_n \|u_n - p\|.
\end{aligned} \tag{4}$$

Since $\lim_{n\to\infty} \gamma_n = 0$, there exists a natural number n_1 such that for $n > n_1$, $\gamma_n \leq \frac{s}{2}$. Hence

$$\alpha_n = 1 - \beta_n - \gamma_n \geq 1 - (1 - s) - \frac{s}{2} = \frac{s}{2},$$

for $n > n_1$. Thus, we have from (4) that

$$\alpha_n \|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + A_n + \gamma_n \|u_n - p\|,$$

and

$$\|x_n - p\| \leq \|x_{n-1} - p\| + \frac{A_n}{\alpha_n} + \frac{\gamma_n}{\alpha_n}\|u_n - p\|$$

$$
\begin{aligned}
&\leq \|x_{n-1}-p\| + \frac{2}{s}A_n + \frac{2\gamma_n}{s}\|u_n-p\| \\
&\leq \|x_{n-1}-p\| + \frac{2}{s}A_n + \frac{2\gamma_n}{s}M, \qquad (5)
\end{aligned}
$$

where $M = \sup_{n\geq 1}\{\|u_n-p\|\}$, since $\{u_n\}$ is a bounded sequence in C. This implies that

$$d(x_n,\mathcal{F}) \leq d(x_{n-1},\mathcal{F}) + D_n,$$

where $D_n = \frac{2}{s}A_n + \frac{2\gamma_n}{s}M$. Since by assumption of the theorem, $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, it follows that $\sum_{n=1}^{\infty} D_n < \infty$. Therefore, from Lemma 2.1, we know that $\lim_{n\to\infty} d(x_n,\mathcal{F})$ exists. Because $\liminf_{n\to\infty} d(x_n,\mathcal{F}) = 0$, then

$$\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0.$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in C. It follows from (5) that for any $m \geq 1$, for all $n \geq n_0$ and for any $p \in \mathcal{F}$, we have

$$
\begin{aligned}
\|x_{n+m}-p\| &\leq \|x_{n+m-1}-p\| + \frac{2}{s}A_{n+m} + \frac{2M}{s}\gamma_{n+m} \\
&\leq \|x_{n+m-2}-p\| + \frac{2}{s}[A_{n+m}+A_{n+m-1}] \\
&\quad + \frac{2M}{s}[\gamma_{n+m}+\gamma_{n+m-1}] \\
&\leq \dots \\
&\leq \|x_n-p\| + \frac{2}{s}\sum_{k=n+1}^{n+m} A_k + \frac{2M}{s}\sum_{k=n+1}^{n+m}\gamma_k.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\|x_{n+m}-x_n\| &\leq \|x_{n+m}-p\| + \|x_n-p\| \\
&\leq 2\|x_n-p\| + \frac{2}{s}\sum_{k=n+1}^{n+m} A_k + \frac{2M}{s}\sum_{k=n+1}^{n+m}\gamma_k.
\end{aligned}
$$

Then, we have

$$\|x_{n+m}-x_n\| \leq 2d(x_n,\mathcal{F}) + \frac{2}{s}\sum_{k=n+1}^{n+m} A_k + \frac{2M}{s}\sum_{k=n+1}^{n+m}\gamma_k, \quad \forall n \geq n_0.$$

For any given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$ such that for any $n \geq n_1$,

$$d(x_n,\mathcal{F}) < \frac{\varepsilon}{6}, \qquad \sum_{k=n+1}^{n+m} A_k < \frac{s\varepsilon}{6},$$

and

$$\sum_{k=n+1}^{n+m} \gamma_k < \frac{s\varepsilon}{6M}.$$

Thus, when $n \geq n_1$, we have

$$\|x_{n+m} - x_n\| < 2.\frac{\varepsilon}{6} + \frac{2}{s}.\frac{s\varepsilon}{6} + \frac{2M}{s}.\frac{s\varepsilon}{6M} = \varepsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. Thus, the completeness of E implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = p^*$. Now, we have to show that p^* is a common fixed point of the mappings $\{T_i : i \in I\}$. Indeed, we know that the set $\mathcal{F} = \cap_{i=1}^N F(T_i)$ is closed. From the continuity of $d(x, \mathcal{F})$, $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ and $\lim_{n\to\infty} x_n = p^*$, we get

$$d(p^*, \mathcal{F}) = 0,$$

and so $p^* \in \mathcal{F}$, that is, p^* is a common fixed point of the mappings $\{T_i\}_{i=1}^N$. This completes the proof. ■

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Banach space E. Let $T_i\colon C \to C$ $(i \in I = \{1, 2, \ldots, N\})$ be a finite family of asymptotically quasi-nonexpansive in the intermediate sense mappings such that $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let the implicit iterative algorithm with errors $\{x_n\}$ be defined by* (2). *Put*

$$A_n = \max\Big\{ \sup_{p\in\mathcal{F},\ n\geq 1} \Big(\|T_i^n x_n - p\| - \|x_n - p\| \Big) \vee 0 : i \in I \Big\}.$$

Assume that $\sum_{n=1}^\infty A_n < \infty$, $\sum_{n=1}^\infty \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p^* of the mappings $\{T_i\}_{i=1}^N$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p^*.

Proof. The proof of Theorem 3.2 follows from Lemma 2.1 and Theorem 3.1. ■

Remark 3.3. Theorem 3.1 extends Theorem 3.3 of Yao and Liou [12] to the case of implicit iterative algorithm with errors for a finite family of mappings considered in this paper.

Remark 3.4. Our results also extend, improve and unify the corresponding results of [1, 2, 5, 6, 8, 9, 10, 13]. Especially Theorem 3.1 extends, improves and unifies Theorems 1 and 2 in [6], Theorem 1 in [5] and Theorem 3.2 in [9] in the following ways:

(i) The asymptotically quasi-nonexpansive mapping in [5], [6] and [9] is replaced by asymptotically quasi-nonexpansive in the intermediate sense mappings.

(ii) The usual Ishikawa iteration scheme in [5], the usual modified Ishikawa iteration scheme with errors in [6] and the usual modified Ishikawa iteration scheme with errors for two mappings are extended to the implicit iterative algorithm with errors for a finite family of mappings.

Remark 3.5. Theorem 3.2 extends, improves and unifies Theorem 3 in [6] in the following aspects:

(i) The asymptotically quasi-nonexpansive mapping in [6] is replaced by asymptotically quasi-nonexpansive in the intermediate sense mappings.

(ii) The usual modified Ishikawa iteration scheme with errors in [6] is extended to the implicit iterative algorithm with errors for a finite family of mappings.

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