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Certain Subclasses of Analytic Functions Involving Sălăgean-Ruscheweyh Operator

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Abstract. The objective of the present paper is to define the class $\mathcal{T}_\lambda^n(\alpha,\gamma)$ using the differential operator D_λ^n. For functions belonging to this class we obtain coefficient estimates and many more properties. We also determine the extreme points.

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1. Introduction

Let $\mathcal{A}$ denote the class of all analytic univalent functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1}$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$.

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathcal{U}$, consisting of analytic functions whose non-zero coefficients from the second onwards are negative. That is, an analytic function $f \in \mathcal{T}$ if it has a Taylor expansion of the form

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m \quad (a_m \geq 0) \tag{2}$$

which are univalent in the open disc $\mathcal{U}$.

For $f \in \mathcal{A}$, Sălăgean [6], introduced the following operator S^n which is called the Sălăgean differential operator.

$$S^0 f(z) = f(z), \quad S^1 f(z) = zf'(z),$$

$$S^n f(z) = S(S^{n-1} f(z)), \quad (n \in \mathbb{N}_0).$$

We note that for $f \in \mathcal{A}$,

$$S^n f(z) = z + \sum_{m=2}^{\infty} m^n a_m z^m, \quad (n \in \mathbb{N}_0).$$

For $n \in \mathbb{N}_0$ and $\lambda \geq 0$, let D_λ^n [3] denote the linear operator defined by

$$D_\lambda^n : A \to A$$

such that

$$D_\lambda^n f(z) = (1-\lambda) S^n f(z) + \lambda R^n f(z), \quad z \in \mathcal{U},$$

where S^n is the Sălăgean differential operator, and R^n is the Ruscheweyh differential operator [5] defined by

$$R^0 f(z) = f(z), \quad R^1 f(z) = zf'(z),$$

with recurrence relation given by

$$(n+1) R^{n+1} f(z) = z[R^n f(z)]' + nR^n f(z) \quad (z \in \mathcal{U}).$$

For $f \in \mathcal{A}$ given by (1)

$$R^n f(z) = z + \sum_{m=2}^{\infty} \binom{n}{n+m-1} a_m z^m.$$

Notice that D_λ^n is a linear operator and for $f \in \mathcal{A}$ of the form (1), we have

$$D_\lambda^n f(z) = z + \sum_{m=2}^{\infty} B_\lambda(m,n) a_m z^m, \tag{3}$$

where

$$B_\lambda(m,n) = \left[(1-\lambda) m^n + \lambda \binom{n}{n+m-1} \right].$$

It is observed that for $n = 0$,

$$D_\lambda^0 f(z) = (1-\lambda)S^0 f(z) + \lambda R^0 f(z) = f(z) = S^0 f(z) = R^0 f(z),$$

and for $n = 1$,

$$D_\lambda^1 f(z) = (1-\lambda)S^1 f(z) + \lambda R^1 f(z) = zf'(z) = S^1 f(z) = R^1 f(z).$$

Now using D_λ^n linear operator, we define the following subclass of $\mathcal{T}$.

$\mathcal{T}_\lambda^n(\alpha,\gamma)$ is the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\Re\left\{\frac{z(D_\lambda^n f)'}{\gamma z(D_\lambda^n f)' + (1-\gamma)D_\lambda^n f}\right\} > \alpha,$$

for some α, γ, $(0 \le \alpha, \gamma < 1)$ and $n \in \mathbf{N}_0$.

For different parametric values of λ, n, we get the classes studied by Mostafa [4].

2. Main Results

Theorem 2.1. *A function f defined by* (2) *is in the class $\mathcal{T}_\lambda^n(\alpha,\gamma)$ if and only if*

$$\sum_{m=2}^{\infty} B_\lambda(m,n)a_m[m - \alpha + \alpha\gamma - \alpha\gamma m] < 1 - \alpha, \tag{4}$$

where α, γ $(0 \le \alpha, \gamma < 1)$ *and* $n \in \mathbf{N}_0$.

Proof. Suppose $f \in \mathcal{T}_\lambda^n(\alpha,\gamma)$. Then

$$\Re\left\{\frac{z(D_\lambda^n f)'}{\gamma z(D_\lambda^n f)' + (1-\gamma)D_\gamma^n f}\right\} > \alpha,$$

$$\Re\left\{\frac{z - \sum_{m=2}^{\infty} B_\lambda(m,n) m a_m z^m}{\gamma\left[z - \sum_{m=2}^{\infty} B_\lambda(m,n) m a_m z^m\right] + (1-\gamma)\left[z - \sum_{m=2}^{\infty} B_\lambda(m,n) a_m z^m\right]}\right\} > \alpha.$$

$$\Re\left\{\frac{z - \sum_{m=2}^{\infty} m B_\lambda(m,n) a_m z^m}{z - \sum_{m=2}^{\infty} B_\lambda(m,n) a_m z^m\left[\gamma(m-1)+1\right]}\right\} > \alpha.$$

Letting $z \to 1$ then we get

$$1 - \sum_{m=2}^{\infty} B_\lambda(m,n) a_m m > \alpha\left\{1 - \sum_{m=2}^{\infty} B_\lambda(m,n) a_m\left[\gamma(m-1)+1\right]\right\},$$

$$\sum_{m=2}^{\infty} B_\lambda(m,n)a_m m - \alpha \sum_{m=2}^{\infty} B_\lambda(m,n)a_m \left[\gamma(m-1)+1\right] < (1-\alpha),$$

$$\sum_{m=2}^{\infty} B_\lambda(m,n)a_m \left[m - \alpha\gamma m + \alpha\gamma - \alpha\right] < (1-\alpha).$$

Conversely, assume that (4) is true. We have to show that (3) is satisfied or equivalently

$$\left| \left\{ \frac{z(D_\lambda^n f)'}{\gamma z(D_\lambda^n f)' + (1-\gamma)D_\lambda^n f} \right\} - 1 \right| < 1-\alpha.$$

But

$$\left| \left\{ \frac{z - \sum_{m=2}^{\infty} m B_\lambda(m,n)a_m z^m}{z - \sum_{m=2}^{\infty} B_\lambda(m,n)a_m z^m \left[\gamma(m-1)+1\right]} \right\} - 1 \right|$$

$$= \left| \frac{\sum_{m=2}^{\infty} B_\lambda(m,n)a_m (m-1)(\gamma-1) z^m}{z - \sum_{m=2}^{\infty} B_\lambda(m,n)a_m [\gamma(m-1)+1] z^m} \right|$$

$$\le \frac{\sum_{m=2}^{\infty} B_\lambda(m,n)a_m (m-1)(\gamma-1)|z^m|}{|z| - \sum_{m=2}^{\infty} B_\lambda(m,n)a_m [\gamma(m-1)+1]|z^m|}$$

$$\le \frac{\sum_{m=2}^{\infty} B_\lambda(m,n)a_m (m-1)(\gamma-1)}{1 - \sum_{m=2}^{\infty} B_\lambda(m,n)a_m [\gamma(m-1)+1]}.$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{m=2}^{\infty} B_\lambda(m,n)a_m (m-1)(\gamma-1) \le (1-\alpha)\left(1 - \sum_{m=2}^{\infty} B_\lambda(m,n)a_m [\gamma(m-1)+1]\right),$$

or

$$\sum_{m=2}^{\infty} B_\lambda(m,n)a_m [m - \alpha + \alpha\,\gamma - \alpha\,\gamma m] < 1-\alpha,$$

which is true by hypothesis. This completes the assertion of Theorem 2.1. ■

For parametric value $\lambda = 0$ we get the following result studied by Dileep and Latha [2].

Corollary 2.2. *A function f defined by (2) is in the class $\mathcal{T}_n(\alpha,\gamma)$ if and only if*

$$\sum_{m=2}^{\infty} m^n a_m[m-\alpha+\alpha\,\gamma-\alpha\,\gamma m] < 1-\alpha,$$

where $\alpha,\ \gamma \quad (0 \le \alpha,\gamma < 1)$ and $n \in \mathbf{N}_0$.

For parametric values $n = 0$ and $n = 1$ in Corollary 2.2, we have the following result of Mostafa [4] respectively.

Corollary 2.3. (a) *A function $f(z)$ defined by (2) is in the class $\mathcal{T}(\gamma,\alpha)$ if and only if*

$$\sum_{m=2}^{\infty}(m-\gamma\alpha m-\alpha+\gamma\alpha)a_m \le 1-\alpha.$$

(b) *A function $f(z)$ defined by (2) is in the class $\mathcal{C}(\gamma,\alpha)$ if and only if*

$$\sum_{m=2}^{\infty} m(m-\gamma\alpha m-\alpha+\gamma\alpha)a_m \le 1-\alpha.$$

Corollary 2.4. *If $f \in \mathcal{T}_\lambda^n(\alpha,\gamma)$ then*

$$|a_m| \le \frac{1-\alpha}{B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]}.$$

Theorem 2.5. *Let $0 \le \alpha < 1$, $0 \le \gamma_1 \le \gamma_2 < 1$, $n \in \mathbb{N}_0$, then*

$$\mathcal{T}_\lambda^n(\alpha,\gamma_2) \subset \mathcal{T}_\lambda^n(\alpha,\gamma_1).$$

Proof. From Theorem 2.1,

$$\sum_{m=2}^{\infty} B_\lambda(m,n)\,[m-\alpha\gamma_2 m+\alpha\gamma_2-\alpha]\,a_m \le \sum_{m=2}^{\infty} B_\lambda(m,n)\,[m-\alpha\gamma_1 m+\alpha\gamma_1-\alpha]\,a_m \le (1-\alpha),$$

for $f(z) \in \mathcal{T}_\lambda^n(\alpha,\gamma_2)$. Hence $f(z) \in \mathcal{T}_\lambda^n(\alpha,\gamma_1)$. ∎

Theorem 2.6. *Let $f(z) \in \mathcal{T}_\lambda^n(\alpha,\gamma)$. Define $f_1(z) = z$ and*

$$f_m(z) = z + \frac{1-\alpha}{B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]}z^m, \quad m = 2,3,\ldots,$$

for some $\alpha,\ \gamma\ (0 \le \gamma < 1)$, $n \in \mathbb{N}_0$ and $z \in \mathcal{U}$. $f \in \mathcal{T}_\lambda^n(\alpha,\gamma)$ if and only if f can be expressed as $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$, where $\mu_m \ge 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

Proof. If $f(z)=\sum_{m=1}^{\infty}\mu_m f_m(z)$ with $\sum_{m=1}^{\infty}\mu_m=1,\ \mu_m\geq 0$, then

$$\sum_{m=2}^{\infty}\frac{B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]\,\mu_m}{B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]}(1-\alpha)\sum_{m=2}^{\infty}\mu_m(1-\alpha)=(1-\mu_1)(1-\alpha)$$
$$\leq(1-\alpha).$$

Hence $f\in\mathcal{T}_\lambda^n(\alpha,\gamma)$.

Conversely, let $f(z)=z-\sum_{m=2}^{\infty}a_m z^m\in\mathcal{T}_\lambda^n(\alpha,\gamma)$, define

$$\mu_m=\frac{B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]\,|a_m|}{(1-\alpha)},\quad m=2,3,\ldots,$$

and define $\mu_1=1-\sum_{m=2}^{\infty}\mu_m$. From Theorem 2.1, $\sum_{m=2}^{\infty}\mu_m\leq 1$ and so $\mu_1\geq 0$. Since $\mu_m f_m(z)=\mu_m f+a_m z^m$, $\sum_{m=1}^{\infty}\mu_m f_m(z)=z-\sum_{m=2}^{\infty}a_m z^m=f(z)$. ■

Theorem 2.7. *The class $\mathcal{T}_\lambda^n(\alpha,\gamma)$ is closed under convex linear combination.*

Proof. Let $f,\ g\in\mathcal{T}_\lambda^n(\alpha,\gamma)$ and let

$$f(z)=z-\sum_{m=2}^{\infty}a_m z^m,\quad g(z)=z-\sum_{m=2}^{\infty}b_m z^m.$$

For η such that $0\leq\eta\leq 1$, it suffices to show that the function defined by $h(z)=(1-\eta)f(z)+\eta g(z)$, $z\in\mathcal{U}$ belongs to $\mathcal{T}_\lambda^n(\alpha,\gamma)$. Now

$$h(z)=z-\sum_{m=2}^{\infty}[(1-\eta)a_m+\eta b_m]z^m.$$

Applying Theorem 2.1 to $f,\ g\in\mathcal{T}_\lambda^n(\alpha,\gamma)$, we have

$$\begin{aligned}&\sum_{m=2}^{\infty}B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]\,[(1-\eta)a_m+\eta b_m]\\&=(1-\eta)\sum_{m=2}^{\infty}B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]\,a_m\\&\quad+\eta\sum_{m=2}^{\infty}B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]\,b_m\\&\leq(1-\eta)(1-\alpha)+\eta(1-\alpha)=(1-\alpha).\end{aligned}$$

This implies that $h \in \mathcal{T}_\lambda^n(\alpha,\gamma)$. ■

Corollary 2.8. *If f_1, f_2 are in $\mathcal{T}_\lambda^n(\alpha,\gamma)$ then the function defined by $g(z) = \frac{1}{2}[f_1(z)+f_2(z)]$ is also in $\mathcal{T}_\lambda^n(\alpha,\gamma)$.*

Theorem 2.9. *Let for $j = 1,2,\ldots,m$, $f_j(z) = z - \sum_{m=2}^{\infty} a_{m,j} z^m \in \mathcal{T}_\lambda^n(\alpha,\gamma)$ and $0 < \gamma_j < 1$ such that $\sum_{j=1}^{m} \gamma_j = 1$, then the function $F(z)$ defined by*

$$F(z) = \sum_{j=1}^{m} \gamma_j f_j(z)$$

is also in $\mathcal{T}_\lambda^n(\alpha,\gamma)$.

Proof. For each $j \in \{1,2,3,\ldots,m\}$ we obtain

$$\sum_{m=2}^{\infty} B_\lambda(m,n)\,[m - \alpha\gamma m + \alpha\gamma - \alpha]\,|a_m| < (1-\alpha).$$

Since

$$F(z) = \sum_{j=1}^{m} \gamma_j (z - \sum_{m=2}^{\infty} a_{m,j} z^m) = z - \sum_{m=2}^{\infty} (\sum_{j=1}^{m} \gamma_j a_{m,j}) z^m,$$

$$\begin{aligned}
&\sum_{m=2}^{\infty} B_\lambda(m,n)\,[m - \alpha\gamma m + \alpha\gamma - \alpha] \left[\sum_{j=1}^{m} \gamma_j a_{m,j}\right] \\
&= \sum_{j=1}^{m} \gamma_j \left[\sum_{m=2}^{\infty} B_\lambda(m,n)\,[m - \alpha\gamma m + \alpha\gamma - \alpha]\right] \\
&< \sum_{j=1}^{m} \gamma_j (1-\alpha) \\
&< (1-\alpha).
\end{aligned}$$

Therefore $F(z) \in \mathcal{T}_\lambda^n(\alpha,\gamma)$. ■

For $f \in \mathcal{U}$ we define the integral transform

$$\mathcal{V}_\mu(f)(z) = \int_0^1 \mu(t) \frac{f(tz)}{t} dt,$$

where μ is a real valued nonnegative weight function normalized so that $\int_0^1 \mu dt = 1$. Some special cases of $\mu(t)$ are particularly interesting as $\mu(t) = (1+c)t^c$, $c > -1$, for which $\mathcal{V}_\mu$ is known as the Bernardi operator and

$$\mu(t) = \frac{(c+1)^\eta}{\mu(\eta)} t^c \left(\log \frac{1}{t}\right)^{\eta-1}, \ c > -1, \quad \eta \geq 0,$$

which gives Komato operator.

Theorem 2.10. *Let $f(z) \in \mathcal{T}_\lambda^n(\alpha, \gamma)$. Then $\mathcal{V}_\mu(f) \in \mathcal{T}_\lambda^n(\alpha, \gamma)$.*

Proof. By definition, we have

$$\begin{aligned}\mathcal{V}_\mu(f)(z) &= \frac{(c+1)^\eta}{\mu(\eta)} \int_0^1 (-1)^{\eta-1} t^c \left(\log \frac{1}{t}\right)^{\eta-1} \left(z - \sum_{m=2}^{\infty} a_m z^m t^{m-1}\right) dt \\ &= \frac{(-1)^{\eta-1}(c+1)^\eta}{\mu(\eta)} \lim_{\epsilon \to 0} \left[\int_\epsilon^1 t^c (\log t)^{\eta-1} \left(z - \sum_{m=2}^{\infty} a_m z^m t^{m-1}\right) dt\right] \\ &= z - \sum_{m=2}^{\infty} \left(\frac{c+1}{c+m}\right)^\eta a_m z^m.\end{aligned}$$

We need to prove that

$$\sum_{m=2}^{\infty} \left(\frac{c+1}{c+m}\right)^\eta B_\lambda(m,n)[m - \alpha\gamma m + \alpha\gamma - \alpha] a_m < 1 - \alpha. \tag{5}$$

On the other hand, from Theorem 2.1 we have

$$\sum_{m=2}^{\infty} B_\lambda(m,n)[m - \alpha\gamma m + \alpha\gamma - \alpha] a_m < 1 - \alpha.$$

Hence $\left(\frac{c+1}{c+m}\right)^\eta < 1$. Therefore, (5) holds. Hence the proof is complete. ■

Theorem 2.11. *Let $f \in \mathcal{T}_\lambda^n(\alpha, \gamma)$, then $\mathcal{V}_\mu(f)$ is starlike of order β, $\beta \in [0,1]$ in $|z| < R_1$, where*

$$R_1 = \inf_n \left[\left(\frac{c+m}{c+1}\right)^\eta \frac{(1-\beta)[m - \alpha\gamma m + \alpha\gamma - \alpha] B_\lambda(m,n)}{(m-\beta)(1-\alpha)}\right]^{\frac{1}{m-1}}.$$

Proof. It is sufficient to prove

$$\left|\frac{z(\mathcal{V}_\mu f(z))'}{\mathcal{V}_\mu f(z)}-1\right| = \left|\frac{\sum_{m=2}^{\infty}(1-m)\left(\frac{c+1}{c+m}\right)^{\eta}a_m z^{m-1}}{1-\sum_{m=2}^{\infty}\left(\frac{c+1}{c+m}\right)^{\eta}a_m z^{m-1}}\right| \leq \frac{\sum_{m=2}^{\infty}(m-1)\left(\frac{c+1}{c+m}\right)^{\eta}a_m |z|^{m-1}}{1-\sum_{m=2}^{\infty}\left(\frac{c+1}{c+m}\right)^{\eta}a_m |z|^{m-1}}.$$

The last expression is less than $1-\beta$, since

$$|z|^{m-1} \leq \left(\frac{c+m}{c+1}\right)^{\eta}\frac{(1-\beta)[m-\alpha\gamma m+\alpha\gamma-\alpha]B_\lambda(m,n)}{(m-\beta)(1-\alpha)}.$$

Hence the proof is finished. ■

Using the fact that f is convex if and only if zf' is starlike, we obtain the following:

Theorem 2.12. *Let $f \in \mathcal{T}_\lambda^n(\alpha,\gamma)$, then $\mathcal{V}_\mu(f)$ is convex of order β, $\beta \in [0,1]$ in $|z| < R_2$ where*

$$R_2 = \inf_n \left[\left(\frac{c+m}{c+1}\right)^{\eta}\frac{(1-\beta)[m-\alpha\gamma m+\alpha\gamma-\alpha]B_\lambda(m,n)}{m(m-\beta)(1-\alpha)}\right]^{\frac{1}{m-1}}.$$

Theorem 2.13. *Let $f \in \mathcal{T}_\lambda^n(\alpha,\gamma)$, then for every $0 \leq \delta < 1$ the function*

$$\mathcal{H}_\delta(z) = (1-\delta)f(z) + \delta\int_0^z \frac{f(t)}{t}dt$$

belongs to $\mathcal{T}_\lambda^n(\alpha,\gamma)$.

Proof. We have $\mathcal{H}_\delta(z) = z - \sum_{m=2}^{\infty}\left(1+\frac{\delta}{m}-\delta\right)a_m z^m$. Since $\left(1+\frac{\delta}{m}-\delta\right) < 1$, $m \geq 2$, so by Theorem 2.1,

$$\begin{aligned}&\sum_{m=2}^{\infty}\left(1+\frac{\delta}{m}-\delta\right)B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]\,a_m\\ &< \sum_{m=2}^{\infty}B_\lambda(m,n)\,[m-\alpha\gamma m+\alpha\gamma-\alpha]\,a_m\\ &< (1-\alpha).\end{aligned}$$

Therefore $\mathcal{H}_\delta(z) \in \mathcal{T}_\lambda^n(\alpha,\gamma)$. ■

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