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A Proximal Point-Type Algorithm for Multivalued Variational Inequalities

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Abstract. We present a new method for solving multivalued variational inequalities. The method is based on a proximal point algorithm with M which is a positive definite matrix but not necessarily symmetric. We first solve monotone multivalued variational inequalities satisfying a nonlipschitzian assumption and prove the convergence of the proposed algorithm. Next, we couple this technique with the Banach contraction method for the multivalued variational inequalities. Finally some preliminary computational results are given.

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1. Introduction

Let C be a closed convex subset in a real Euclidean space $\mathbb{R}^n$ and F be a multivalued mapping from C into subsets of $\mathbb{R}^n$. We consider the following multivalued variational inequalities (shortly (MVI)), which is to find points $x^* \in C$ and $w^* \in F(x^*)$ such that

$$\langle w^*, x - x^* \rangle \geq 0 \quad \forall x \in C,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard dot product in $\mathbb{R}^n$. Denote by S^* the set of solutions of (MVI) and assume that $S^* \neq \emptyset$.

The multivalued variational inequalities have many important applications in economics equilibrium, nonlinear analysis, engineering design and have been studied by many researchers (see [8, 9, 12, 14, 15, 18]). In general, these methods assume the underlying mapping to be strongly monotone, Lipschitz continuous and hence cannot be applied directly to (MVI) where F is multivalued. With (MVI), most of these methods require that F is either Lipschitz continuous with respect to the Hausdorff distance or strongly monotone on C. However, both Lipschitz constant and strongly monotone constant are not easy to compute.

There exist several methods for solving (MVI) with F being a monotone multivalued function. We can name as projection methods (see [3, 5, 6, 7, 11, 17]) or the interior-quadratic regularization methods (see [1, 14]). The interior quadratic regularization technique has been used to develop proximal iterative algorithm for variational inequalities (see [2, 10, 13]). In [4], we proposed a cutting hyperplane method for generalized monotone nonlipschitzian multivalued variational inequalities. We first construct an appropriate hyperplane which separates the current iterative point from the solution set of the problem. The next iterate is obtained as the projection of the current iterative point onto the intersection of the feasible set with the halfspace containing the solution set.

A classical method for solving (MVI) is the *proximal point algorithm* (PPA) (see [16, 17]). For given $x^k \in C$, the new iterate x^{k+1} generated by (PPA) is the unique solution to the following auxiliary multivalued variational inequality:

$$\text{Find} \quad x \in C, w \in F(x) \text{ such that } \langle w + M(x - x^k), y - x \rangle \geq 0 \quad \forall y \in C, \quad (1)$$

where M is a symmetric positive-definite matrix and $M(x - x^k)$ is called the proximal function in (PPA). This linear proximal function is the gradient of a quadratic function, namely,

$$M(x - x^k) = \bigtriangledown \left(\frac{1}{2} \|x - x^k\|_M^2 \right).$$

Recently, a number of authors have concentrated on the generalization of (PPA) by replacing the linear proximal function $M(x - x^k)$ with some nonlinear functions $d(x, x^k)$ arising from appropriately formular Bregman functions (see [10, 13]).

In this paper, we propose method (PPA) for solving (MVI) with M being a positive definite matrix but not necessarily symmetric and F a monotone multivalued mapping which is not assumed to be Lipschitz continuous on C. In Sec. 3, we prove their convergence and in Sec. 4 we couple the proximal point algorithm with the Banach contraction method for solving (MVI) and present some numerical experiments to illustrate the behavior of the proposed algorithms.

2. Proposed Methods

Let C be a closed convex subset in a real Euclidean space $\mathbb{R}^n$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. We list some well known definitions and the projection which will be required in our following analysis.

Definition 2.1. (i) A function $F : C \to 2^{\mathbb{R}^n}$ is said to be monotone on C, if for each $x, x^{'} \in C$, we have

$$\langle w - w^{'}, x - x^{'}\rangle \geq 0 \quad w \in F(x), w^{'} \in F(x^{'}).$$

(ii) A matrix $M_{n\times n}$ is said to be positive definite, if there exists a $\tau > 0$ such that

$$\langle Mx, x\rangle \geq \tau\|x\|^2 \quad \forall x \in \mathbb{R}^n. \tag{2}$$

Denote by $P_C(\cdot)$ the metric projection of $x \in \mathbb{R}^n$ on C, i.e.,

$$P_C(x) = \text{argmin}\{\|y - x\| \mid y \in C\}.$$

Throughout the paper, we assume that the mapping F is monotone, closed and upper semicontinuous on C.

Algorithm 1. *Step 0.* Given an $n \times n$ positive definite matrix $M, \varepsilon > 0, x^0 \in C, w^0 \in F(x^0)$. Set $k = 0$.

Step 1. Find a proximal point $(\bar{x}^k, w^k)$ which is the solution of the following:

$$\textit{Find } \bar{x} \in C, \bar{w} \in F(\bar{x}) \quad \textit{such that} \quad \langle \bar{w} + M(\bar{x} - x^k), y - \bar{x}\rangle \geq 0 \ \ \forall y \in C. \tag{3}$$

If $\|x^k - \bar{x}^k\| < \epsilon$, then terminate.
Otherwise, go to Step 2.

Step 2. Compute

$$x^{k+1} = x^k - \alpha_k M(x^k - \bar{x}^k), \tag{4}$$

with

$$\alpha_k = \gamma\frac{\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle}{\|M(x^k - \bar{x}^k)\|^2}, \gamma \in [1, 2). \tag{5}$$

Increase k by 1 and go back to Step 1.

Remark 2.2. Subproblem (3) in Step 1 is similar to subproblem (1) of classical (PPA), the only difference is that matrix M in (3) is positive definite but not necessarily symmetric. Thus, the proposed method is a general form of (PPA)-based methods with a linear proximal function. Note that we can not take $\bar{x}^k$ as the new iterate unless M is symmetric and positive definite.

Remark 2.3. In comparison with solving problem (3), the additional computational load for updating x^{k+1} via (4)-(5) is slight.

The solution of (3), $\bar{x}^k$ is the proximal point in the k-th iteration. It is viewed as a test vector because it is a solution of (MVI) if and only if $x^k = \bar{x}^k$. In fact,

$\|x^k - \bar{x}^k\|$ can be viewed as an error bound function which measures how much $\bar{x}^k$ fails to be a solution point of (MVI). Therefore, we have the following.

Proposition 2.4. *If* $\|x^k - \bar{x}^k\| \leq \epsilon$, *then* x^k *is an* ϵ*-solution to* (MVI).

The following lemma gives important inequalities which are useful tool in the next discussion.

Lemma 2.5. (see [14]) *For any* $x, y \in \mathbb{R}^n$, *the mapping* P_C *satisfies*

$$\big\langle P_C(x) - x, y - P_C(x)\big\rangle \geq 0 \quad \forall y \in C,$$

$$\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2 \quad \forall y \in C.$$

3. Convergence Analysis of the Proposed Method

As is known, for any $x^* \in S^*, (x^k - x^*)$ is the gradient of the unknown distance function $\frac{1}{2}\|x - x^*\|^2$ at the point $x^k \notin S^*$. A direction d is called a *descent direction* of $\frac{1}{2}\|x - x^*\|^2$ at the point x^k if

$$\langle x^k - x^*, d\rangle < 0.$$

This subsection shows that $-M(x^k - x^*)$ is a descent direction of the unknown distance function $\frac{1}{2}\|x - x^*\|^2$ at the point x^k.

Lemma 3.1. *Let* $\bar{x}^k$ *be a proximal point generated by* (3) *from given* $x^k \in C$. *Then, for any* $x^* \in S^*$ *we have*

$$\langle M(x^k - \bar{x}^k), x^k - x^*\rangle \geq \langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle. \tag{6}$$

Proof. Since $\bar{x}^k \in C$ is a solution of (3), we have

$$\langle w^k + M(\bar{x}^k - x^k), y - \bar{x}^k\rangle \geq 0 \quad \forall y \in C.$$

Substituting $x^* \in C$ into this, we obtain

$$\langle w^k + M(\bar{x}^k - x^k), x^* - \bar{x}^k\rangle \geq 0. \tag{7}$$

On the other hand, $w^* \in F(x^*)$ and $\bar{x}^k \in C$ imply

$$\langle w^*, \bar{x}^k - x^*\rangle \geq 0. \tag{8}$$

Combining (7) and (8) we have

$$\langle M(\bar{x}^k - x^k), x^* - \bar{x}^k\rangle \geq \langle w^k - w^*, \bar{x}^k - x^*\rangle.$$

Using the monotonicity of F, $\langle w^k - w^*, \bar{x}^k - x^*\rangle \geq 0$, it deduces

$$\langle M(\bar{x}^k - x^k), x^* - \bar{x}^k \rangle \geq 0.$$

Hence

$$\langle M(x^k - \bar{x}^k), x^k - x^* \rangle \geq \langle M(x^k - \bar{x}^k), x^k - \bar{x}^k \rangle,$$

which proves (6). ■

Since the matrix M is positive definite, from (2) we have

$$\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k \rangle \geq \tau \|x^k - \bar{x}^k\|^2.$$

Then, Lemma 3.1 shows that $-M(x^k - \bar{x}^k)$ is a descent direction of $\frac{1}{2}\|x - \bar{x}^k\|^2$ at the point $x^k \notin S^*$.

Remark 3.2. If the matrix M in (3) is symmetric, it follows from (6) that

$$\begin{aligned}\|\bar{x}^k - x^*\|_M^2 =& \|(x^k - x^*) - (x^k - \bar{x}^k)\|_M^2 \\ =& \|x^k - x^*\|_M^2 - 2\big\langle M(x^k - \bar{x}^k), x^k - x^* \big\rangle + \|x^k - \bar{x}^k\|_M^2 \\ \leq& \|x^k - x^*\|_M^2 - \|x^k - \bar{x}^k\|_M^2.\end{aligned}$$

Therefore, we can directly set $x^{k+1} := \bar{x}^k$ as the new iterate and the convergence follows from the above inequality. However, in the case that M is asymmetric, we cannot establish the convergence when we directly set $x^{k+1} := \bar{x}^k$ as the new iterate. In our proposed method, the new iterate x^{k+1} is updated by (4), the computational load for updating x^{k+1} in Step 2 is small.

In order to explain why we have the optimal step α_k^* as defined in (5), instead of the updating formula (4), we define the stepsize dependent new iterate by

$$x^{k+1} = x^k - \alpha M(x^k - \overline{x}^k). \tag{9}$$

In this way,

$$\theta(\alpha) := \|x^k - x^*\|^2 - \|x^{k+1}(\alpha) - x^*\|^2 \tag{10}$$

is a profit function in the k-th iteration by using the updating form (9). Since x^* is the solution point and thus is unknown, we cannot maximize $\theta(\alpha)$ directly. The following theorem introduces a tight lower bound of $\theta(\alpha)$, namely $q(\alpha)$, which does not include the unknown solution x^*.

Theorem 3.3. *For any $x^* \in S^*$ and $\alpha \geq 0$, we have*

$$\theta(\alpha) \geq q(\alpha), \tag{11}$$

where

$$q(\alpha) = 2\alpha \langle M(x^k - \bar{x}^k), x^k - \bar{x}^k \rangle - \alpha^2 \|M(x^k - \bar{x}^k)\|^2. \tag{12}$$

Proof. Combining (9), (10), (11) and (12), we get

$$\begin{aligned}\theta(\alpha) =&\|x^k - x^*\|^2 - \|(x^k - x^*) - \alpha M(x^k - \bar{x}^k)\|^2\\ =&2\alpha\langle M(x^k - \bar{x}^k), x^k - x^*\rangle - \alpha^2\|M(x^k - \bar{x}^k)\|^2\\ \le&2\alpha\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle - \alpha^2\|M(x^k - \bar{x}^k)\|^2\\ =&q(\alpha).\end{aligned}$$

The theorem is proved. ■

Noting that $q(\alpha)$ is a quadratic function of α and thus it reaches its maximum at

$$\alpha_k^* = \frac{\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle}{\|M(x^k - \bar{x}^k)\|^2},$$

this is just the same defined in (5). Because an inequality is used in proof of (11), in practical computation, taking a relaxed factor $\gamma \ge 1$ is good for fast convergence. Note that for any $\alpha_k = \gamma\alpha_k^*$, it follows from (11), (12) and (5) that

$$\theta(\gamma\alpha_k^*) \ge q(\gamma\alpha_k^*) = \gamma(2-\gamma)\alpha_k^*\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle. \tag{13}$$

In order to guarantee that the right-hand side of (13) is positive, we take $\gamma \in [1, 2)$.

The following theorem points out that a relation between the sequence $\{x^k\}$ generated by the proposed method and any solution of problem (MVI).

Theorem 3.4. *For any $(x^*, w^*) \in S^*$, the sequence $\{x^k\}$ generated by the proposed method satisfies*

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - c_0\|x^k - \bar{x}^k\|^2, \tag{14}$$

where $c_0 > 0$ is a constant.

Proof. First, it follows from (10) and (13) that

$$\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \ge \gamma(2-\gamma)\alpha_k^*\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle. \tag{15}$$

Thus

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - \gamma(2-\gamma)\alpha_k^*\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle.$$

Since

$$\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle \ge \tau\|x^k - \bar{x}^k\|^2, \tag{16}$$

we have

$$\begin{aligned}\alpha_k^* =&\frac{\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle}{\|M(x^k - \bar{x}^k)\|^2}\\ \ge&\frac{\tau}{\|M^TM\|}.\end{aligned}$$

Combining this with (16), we get

$$\alpha_k^* \langle M(x^k - \bar{x}^k), x^k - \bar{x}^k \rangle \geq \frac{\tau^2}{\|M^T M\|} \|x^k - \bar{x}^k\|^2. \tag{17}$$

Applying (17) to (15) and setting

$$c_0 = \frac{\gamma(2-\gamma)\tau^2}{\|M^T M\|},$$

we get (14). ■

The following theorem shows the convergence of the sequence $\{(x^k, w^k)\}$.

Theorem 3.5. *The sequences* $\{x^k\}$ *and* $\{w^k\}$ *generated by the proposed method are convergent to some* x^∞ *and* w^∞ *which are a solution point of* (MVI).

Proof. It follows from (14) that the sequence $\{\|x^k - x^*\|\}$ is nonincreasing. Since it is bounded below by 0, it must be convergent. Thus, the sequence $\{x^k\}$ is bounded. From (14), this also implies that

$$\lim_{k \to \infty} \|x^k - \bar{x}^k\| = 0,$$

and hence $\{\bar{x}^k\}$ is bounded. Then, by the upper semicontinuity of F, we have $\{w^k\}$ is also bounded. By Weierstrass theorem, there exist subsequences $\{\bar{x}^{k_j}\}$ of $\{\bar{x}^k\}$ and $\{w^{k_j}\}$ of $\{w^k\}$ such that $\bar{x}^{k_j} \to x^\infty$ and $w^{k_j} \to w^\infty$ as $j \to \infty$. For each $\bar{x}^{k_j} \in C, w^{k_j} \in F(\bar{x}^{k_j})$, we have

$$\langle w^{k_j} + M(\bar{x}^{k_j} - x^{k_j}), x - \bar{x}^{k_j} \rangle \geq 0 \quad \forall x \in C.$$

From the closedness of F and

$$\lim_{j \to \infty} \|x^{k_j} - \bar{x}^{k_j}\| = 0,$$

we get

$$x^\infty \in C, w^\infty \in F(x^\infty): \quad \langle w^\infty, x - x^\infty \rangle \geq 0 \quad \forall x \in C,$$

and thus (x^∞, w^∞) is a solution. Note that inequality (14) is true for all solution points of (MVI), hence

$$\|x^{k+1} - x^\infty\|^2 \leq \|x^k - x^\infty\|^2 \quad \forall k \geq 0,$$

and the sequence $\{x^k\}$ converges to x^∞. ■

4. Applications to the Banach Contraction Algorithm for (MVI)

In this section, we will combine Algorithm 1 with the Banach contraction method for solving the variational inequalities (MVI), where the monotone cost operator is Lipschitz continuous on C. The auxiliary problems (VIP_k) are strongly monotone variational inequalities with constant $\|M\|$. Then these can be solved efficiently by the Banach contraction algorithm. For every $x \in C, k = 0, 1, 2, \ldots$, we denote $F_k(x) := F(x) + M(x - x^k)$.

Algorithm 2. *Step 0.* Given an $n \times n$ positive definite matrix $M, \varepsilon > 0, x^0 \in C, w^0 \in F(x^0)$. Set $k = 0$. Choose $\beta > \frac{L^2}{2\|M\|}$.
Step 1. Iteration $j, j = 0, 1, 2, \ldots$. Choose $x^{k,0} = x^0, w^{k,0} \in F_k(x^{k,0})$ and solve the strongly convex program

$$x^{k,j} := \text{argmin}\left\{\frac{1}{2}\beta\|x - x^{k,j}\|^2 + \langle w^{k,j}, x - x^{k,j}\rangle |\ x \in C\right\}$$

to obtain its unique solution $x^{k,j+1}$. If $x^{k,j+1} = x^{k,j}$, then $\bar{x}^k := x^{k,j}, w^k = w^{k,j}$. Otherwise, choose $w^{k,j+1} \in F(x^{k,j+1})$, increase j by 1, and go to Iteration j.

If $\bar{x}^k = x^k$, then stop.
Otherwise, go to Step 2.
Step 2. Compute

$$x^{k+1} = x^k - \alpha_k M(x^k - \bar{x}^k),$$

with

$$\alpha_k = \gamma\frac{\langle M(x^k - \bar{x}^k), x^k - \bar{x}^k\rangle}{\|M(x^k - \bar{x}^k)\|^2}, \gamma \in [1, 2).$$

Increase k by 1 and go back to Step 1.

The convergence of the sequence $(x^{k,j})_{j=1}^{\infty}$ of the strongly monotone variational inequalities (VIP_k) is defined by the following proposition.

Proposition 4.1. (see Theorem 3.1 in [9]) *Suppose that F is L-Lipschitz continuous on C. If Algorithm 2 terminates at Iteration j in Step 1, then (x^k, w^k) is a solution to (VIP_k). Moreover, for $(x^{k,*}, w^{k,*})$ is an any solution to (VIP_k), we have*

$$\|x^{k,j} - x^{k,*}\| \le \frac{\delta^j}{1 - \delta^j}\|x^{k,1} - x^{k,0}\| \quad \forall j = 1, 2, \ldots$$

and every cluster point $w^{k,*}$ of the sequence $\{w^{k,j}\}$ satisfies $w^{k,*} \in F(x^{k,*})$, where $\delta := \sqrt{1 - \frac{2\beta}{\|M\|} + \frac{L^2}{\|M\|^2}}$.

Proposition 4.1 shows that Algorithm 2 terminates at Step 1. It means that $x^{k,j}$ is a solution to (VIP_k). By Theorem 3.3, we have the following convergence results of Algorithm 2.

Theorem 4.2. *The sequences $\{x^k\}$ and $\{w^k\}$ generated by the proposed method are convergent to some x^{∞} and w^{∞} which are a solution point of (MVI).*

Now we illustrate our Algorithms with the oligopolistic market equilibrium model considered in [17]. Assume that there are n firms supplying a homogeneous product and that the price p depends on its quantity $\sigma_x = x_1 + x_1 + \cdots + x_n$, i.e.,

$p = p(\sigma_x)$. Let $h_i(x_i)$ denote the total cost to the firm i for supplying x_i units of the product. Then, the profit of the firm i is $x_i p(\sigma_x) - h_i(x_i)$. Naturally, each firm seeks to maximize its own profit by choosing the corresponding production level. Suppose that the strategy set C is a polyhedral convex subset in $\mathbb{R}^n$ given by

$$C := \{x \in \mathbb{R}^n |\ 13 \leq \sum_{i=1}^{n} x_i \leq 25, 1 \leq x_i \leq 5 \quad i = 1, 2, \ldots, n\}. \tag{18}$$

Thus, the oligopolistic market equilibrium problem can be formulated as a Nash equilibrium noncooperative game, where the ith player has the strategy set C and the utility function

$$f_i(x_1, \ldots, x_n) = x_i p(\sum_{i=1}^{n} x_i) - h_i(x_i) \quad i = 1, 2, \ldots, n.$$

As usual, a point $x^* = (x_1^*, \ldots, x_n^*) \in C$ is said to be an equilibrium point for this problem if

$$f_i(x_1^*, \ldots, x_{i-1}^*, y_i, x_{i+1}^*, \ldots, x_n^*) \leq f_i(x_1^*, \ldots, x_n^*) \quad \forall i = 1, 2, \ldots, n.$$

Proposition 4.3. (see [17]) *A point x^* is an equilibrium point for the oligopolistic market problem if and only if it is a solution to (MVI), where C is the polyhedral given by* (18) *and*

$$F(x) = H(x) - p(\sigma_x)e - p'(\sigma_x)x,$$

where

$$H(x) = \big(h_1'(x_1), \ldots, h_n'(x_n)\big)^T, e = (1, \ldots, 1)^T, \sigma_x = \langle x, e\rangle.$$

Proposition 4.4. (see [17]) *Let $p : C \to \mathbb{R}_+$ be convex, twice continuously differentiable, nonincreasing and the function $\mu_\tau : \mathbb{R}_+ \to \mathbb{R}_+$, defined by $\mu_\tau(\sigma_x) = \sigma_x p(\sigma_x + \tau)$ be concave for every $\tau \geq 0$. Also, let the function $h_i : \mathbb{R}_+ \to \mathbb{R}, i = 1, \ldots, n$, be convex and twice continuously differentiable. Then, the cost mapping $F(x) = H(x) - p(\sigma_x)e - p'(\sigma_x)x$ is monotone on C.*

It is very easy to see that the cost mapping F is Lipschitz continuous on C with Lipschitz constant $L < 1$. In this example, we choose (randomly generalized)

$n := 7,$

$H(x) := (2x_1 + 1, 3x_2 + 4, 4x_3 + 2, 1.5x_4 + 3, 4x_5 + 1, x_6 - 2, 3x_7 + 1)^T,$

$p(t) := \frac{2}{3t} \quad t \in (0, +\infty),$

$x^0 := (1.9, 1, 1, 1, 1, 5, 1)^T \in C,$

The tolerance $\epsilon = 10^{-6}, \gamma = \frac{3}{2},$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1.5 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1.6 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}_{7\times 7}.$$

Then, the eigenvalues of the matrix M are: $2, 3, 1.5, 1.6, 3.4142, 0.5858, 1$, the norm of M is $\|M\| = 6.6248$ and $\beta > \frac{L^2}{2\|M\|} \approx 0.755$. If we choose $\beta = 1$, then $\delta \approx 0.7209$. In this case, we obtained the following iterates.

Iter (k)	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k
0	1.9	1	1	1	1	5	1
1	2.0475	1.0417	1.0246	1.6452	1.1338	5.0954	1.4095
2	2.0364	0.9797	0.9285	1.5003	1.0200	4.9796	1.3394
3	2.0558	1.0043	1.0409	1.5263	1.0573	5.0158	1.3717
4	2.0898	1.0170	0.9878	1.5089	1.0774	5.0317	1.4046
5	2.0813	0.9914	0.9858	1.4641	1.0368	4.9905	1.3784
6	2.0864	0.9997	1.0128	1.4741	1.0490	5.0025	1.3887
7	2.0903	1.0034	1.0072	1.4761	1.0543	5.0074	1.3948
8	2.0906	0.9983	0.9919	1.4641	1.0462	4.9989	1.3917
9	2.0920	1.0005	1.0036	1.4664	1.0493	5.0019	1.3946
10	2.0943	1.0017	0.9991	1.4648	1.0509	5.0029	1.3979
11	2.0933	0.9993	0.9982	1.4608	1.0472	4.9992	1.3955
12	2.0938	1.0000	1.0011	1.4617	1.0483	5.0003	1.3964
13	2.0940	1.0003	1.0003	1.4619	1.0487	5.0007	1.3970
14	2.0939	0.9998	0.9996	1.4608	1.0479	4.9998	1.3965
15	2.0940	1.0000	1.0003	1.4610	1.0482	5.0001	1.3968

The approximate solution obtained after 15 iterations is

$$x^{15} = (2.0940, 1.0000, 1.0003, 1.4610, 1.0482, 5.0001, 1.3968)^T.$$

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