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Behavior of the Sequence of Norms of Primitives of a Function in Lorentz Spaces*

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Abstract. Let $f \in N_\Psi(\mathbb{R})$ and $I^n f \in N_\Psi(\mathbb{R})$, for all $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} \|I^n f\|^{1/n}_{N_\Psi(\mathbb{R})} = \sigma^{-1},$$

where $\sigma := \inf\{|\xi| : \xi \in \text{supp}\widehat{f}\}$, $\|.\|_{N_\Psi(\mathbb{R})}$ is the norm in Lorentz space $N_\Psi(\mathbb{R})$, and for $g \in S'(\mathbb{R})$, the tempered generalized function Ig is a primitive of g if $D(Ig) = g$, that is

$$\langle Ig, \varphi' \rangle = -\langle g, \varphi \rangle, \quad \forall \varphi \in S(\mathbb{R})$$

and $S(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions.
In other words, in this paper we characterize behavior of the sequence of $N_\Psi(\mathbb{R})$-norms of primitives of a function by its spectrum (the support of its Fourier transform).

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1. Introduction

It was proven in [2] the following result: *Let* $1 \le p \le \infty$ *and* $f^{(n)} \in L_p(\mathbb{R})$, $n = 0, 1, 2, \ldots$. *Then there always exists the limit* $\lim_{n\to\infty} \|f^{(n)}\|_p^{\frac{1}{n}}$ *and*

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$$\lim_{n\to\infty} \|f^{(n)}\|_p^{\frac{1}{n}} = \sigma_f = \sup\{|\xi| : \ \xi \in \text{supp}\hat{f}\},$$

where $\hat{f}$ is the Fourier transform of f.

This theorem shows that behavior of the sequence $\|f^{(n)}\|_p^{\frac{1}{n}}$ is completely characterized by the spectrum of f. This result was studied by many authors such as V. K. Tuan, N. B. Andersen, J. J. Betancor, and J. D. Betancor,... (see [1-14, 16, 17, 19, 20]).

It is natural to ask what will happen when we replace derivatives by integrals. With $p = 2$, V. K. Tuan proved the following result in [21]: *Let $f \in L_2(\mathbb{R})$ and*

$$\sigma = \inf\{|\xi| : \ \xi \in \text{supp}\hat{f}\} > 0.$$

Then there exists $I^n f$, $I^n f \in L_2(\mathbb{R})$ for all n, and

$$\lim_{n\to\infty} \|I^{(n)} f\|_2^{\frac{1}{n}} = \sigma^{-1},$$

where by $\text{supp}\hat{f}$ *or the spectrum of f we denote the smallest closed set outside which $\hat{f}$ vanishes a.e.*

H. H. Bang and V. N. Huy proved this result for L_p-spaces for all $1 \leq p \leq \infty$ (see [15]).

In this paper, we solve the problem for Lorentz spaces $N_\Psi(\mathbb{R})$. For this purpose, we need a notion of primitive of a generalized function. Let $f \in S'(\mathbb{R})$. The tempered generalized function If is called the primitive of f if $D(If) = f$, that is

$$\langle If, \varphi' \rangle = -\langle f, \varphi \rangle, \quad \forall \varphi \in S(\mathbb{R}).$$

Note that the notation of primitive of a generalized function in $D^{'}(a, b)$, $a, b \in \mathbb{R}$ can be found in [22], here we define it for tempered generalized functions in $S^{'}(\mathbb{R})$.

The Fourier transform of a tempered generalized function $f \in S'(\mathbb{R})$ is defined by the formula

$$\langle Ff, \varphi \rangle = \langle f, F\varphi \rangle, \quad \forall \varphi \in S(\mathbb{R}),$$

where $F\varphi$ or $\hat{\varphi}(\xi) = \dfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) dx$, is the Fourier transform of φ.

2. Main Result

Let $\Psi : [0, \infty) \to [0, \infty]$ be a non-zero concave function, which is non-decreasing and $\Psi(0) = 0$. Denote by $N_\Psi(\mathbb{R})$ the set of all measurable functions f such that

$$\|f\|_{N_\Psi} = \int_0^\infty \Psi\big(\lambda_f(y)\big) dy < \infty,$$

where $\lambda_f(y) = \text{mes}\{x : |f(x)| > y\}$, $(y \geq 0)$, and by $M_\Psi(\mathbb{R})$ the set of all measurable functions g such that

$$\|g\|_{M_\Psi} = \sup\left\{\frac{1}{\Psi(\text{mes } \Delta)} \int\limits_{\Delta} |g(x)|dx : \ \Delta \subset \mathbb{R}, \ 0 < \text{mes } \Delta < \infty\right\} < \infty.$$

Then Lorentz spaces $N_\Psi(\mathbb{R})$ and $M_\Psi(\mathbb{R})$ are Banach spaces.

We state now our main result:

Theorem 2.1. *Suppose $f \in N_\Psi(\mathbb{R})$ and $I^n f \in N_\Psi(\mathbb{R})$, for all $n = 1, 2, \ldots$. Then*

$$\lim_{n\to\infty} \|I^n f\|_{N_\Psi(\mathbb{R})}^{1/n} = \sigma^{-1},$$

where

$$\sigma := \inf\{|\xi| : \xi \in \ \text{supp}\widehat{f}\}.$$

To prove the above theorem, we need the following results:

Lemma 2.2. [15] *Let $\delta > 0$ and $h \in C^\infty(\mathbb{R})$ satisfy*

$$\begin{cases} |h(\xi)| + |h'(\xi)| + |h''(\xi)| < C < \infty, \quad \forall \xi \in \mathbb{R} \\ h(\xi) = 0, \ \forall \xi \in (-\delta, \delta). \end{cases}$$

Then

$$\overline{\lim_{n\to\infty}} \left\| F \frac{h(\xi)}{(i\xi)^n} \right\|_1^{1/n} \leq \frac{1}{\delta}.$$

Lemma 2.3. [18] *Let $f \in N_\Psi(\mathbb{R})$ and $g \in M_\Psi(\mathbb{R})$, we have*

$$\left| \int\limits_{\mathbb{R}} f(x)g(x)dx \right| \leq \|f\|_{N_\Psi(\mathbb{R})} \|g\|_{M_\Psi(\mathbb{R})}.$$

Lemma 2.4. [18] *Let $f \in N_\Psi(\mathbb{R})$, then*

$$\|f\|_{N_\Psi(\mathbb{R})} = \sup_{\{g \in M_\Psi(\mathbb{R}) : \|g\|_{M_\Psi(\mathbb{R})} \leq 1\}} \left| \int\limits_{\mathbb{R}} f(x)g(x)dx \right|.$$

or

$$\|f\|_{N_\Psi(\mathbb{R})} = \sup_{\{g \in M_\Psi(\mathbb{R}) : \|g\|_{M_\Psi(\mathbb{R})} \leq 1\}} |\langle f, g \rangle|.$$

Let us now prove our theorem.

Proof of Theorem 2.1. First we show $S \subset M_\Psi(\mathbb{R})$. For any $\varphi \in S$ we have

$$\|f\|_{N_\Psi(\mathbb{R})} = \sup\left\{\frac{1}{\Psi(\text{mes}(E))} \int\limits_{E} |\varphi(x)|dx : E \subset \mathbb{R}, 0 < \text{mes}(E) < \infty\right\}.$$

If $1 < \text{mes}(E) < \infty$ then

$$\frac{1}{\Psi(\text{mes}(E))} \int_E |\varphi(x)|dx \leq \frac{\|\varphi\|_1}{\Psi(\text{mes}(E))} \leq \frac{\|\varphi\|_1}{\Psi(1)}$$

because $\Psi(x)$ is a non-decreasing function on $\mathbb{R}^+$.
If $0 < \text{mes}(E) < 1$ then

$$\frac{1}{\Psi(\text{mes}(E))} \int_E |\varphi(x)|dx \leq \|\varphi\|_\infty \frac{\text{mes}(E)}{\Psi(\text{mes}(E))} \leq \|\varphi\|_\infty \frac{1}{\Psi(1)}.$$

because $\dfrac{x}{\Psi(x)}$ is an increasing function.
Hence,

$$\|f\|_{N_\Psi(\mathbb{R})} \leq \frac{\max\{\|\varphi\|_\infty, \|\varphi\|_1\}}{\Psi(1)} < \infty.$$

So, $S \subset M_\Psi(\mathbb{R})$.
Next we prove

$$\|f\|_{N_\Psi(\mathbb{R})} = \sup_{\{g \in S: \|g\|_{M_\Psi(\mathbb{R})} \leq 1\}} |\langle f, g \rangle|.$$

Indeed, by Lemma 2.4 we have

$$\|f\|_{N_\Psi(\mathbb{R})} = \sup_{\{g \in M_\Psi(\mathbb{R}): \|g\|_{M_\Psi(\mathbb{R})} \leq 1\}} \left| \int_{\mathbb{R}} f(x)g(x)dx \right|.$$

Therefore, for any $\varepsilon > 0$ there exists a function $g \in M_\Psi(\mathbb{R}) : \|g\|_{M_\Psi(\mathbb{R})} \leq 1$ such that

$$\left| \int_{\mathbb{R}} f(x)g(x)dx \right| > \|f\|_{N_\Psi(\mathbb{R})} - \varepsilon.$$

We choose a number $M > 0$ such that

$$\left| \int_{|x| \leq M} f(x)g(x)dx \right| > \|f\|_{N_\Psi(\mathbb{R})} - 2\varepsilon.$$

Put

$$h = \chi_{[-M,M]} g,$$

$$\omega(x) = \begin{cases} C_1 e^{\frac{1}{x^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

where C_1 is defined by $\|\omega\|_{N_\Psi(\mathbb{R})} = 1$, and

$$\psi_\lambda = h * \omega_\lambda, \quad \lambda > 0,$$

with $\omega_\lambda(x) := \frac{1}{\lambda}\omega\left(\frac{x}{\lambda}\right)$.
Then $\psi_\lambda \in S, \|\psi_\lambda\|_{M_\Psi(\mathbb{R})} \leq \|h\|_{M_\Psi(\mathbb{R})}.\|\psi_\lambda\|_1$ and

$$\lim_{\lambda\to 0}\int\limits_{\mathbb{R}} f(x)\psi_\lambda(x)dx = \int\limits_{\mathbb{R}} f(x)h(x)dx.$$

Therefore,

$$\left|\int\limits_{\mathbb{R}} f(x)\psi_\lambda(x)dx\right| > \|f\|_{N_\Psi(\mathbb{R})} - 3\varepsilon$$

for some $\lambda > 0$ and $\psi_\lambda \in S : \|\psi_\lambda\|_{M_\Psi(\mathbb{R})} \leq 1$.
Because $\varepsilon > 0$ is arbitrarily chosen, we get

$$\|f\|_{N_\Psi(\mathbb{R})} = \sup_{\{g\in S:\|g\|_{M_\Psi(\mathbb{R})}\leq 1\}} |\langle f, g\rangle|.$$

Now we split the proof into two cases.

Case 1. $\sigma > 0$. Let us first prove

$$\operatorname{supp}\widehat{I^n f} = \operatorname{supp}\hat{f}, \quad n = 1, 2, \ldots, \tag{1}$$

where $\hat{f} = Ff$ is the Fourier transform of f. In fact, since $D^n I^n f = f$ we have

$$\hat{f} = \widehat{D^n I^n f} = (ix)^n \widehat{I^n f}.$$

Therefore,

$$\operatorname{supp}\widehat{I^n f} \subset \operatorname{supp}\hat{f} \cup \{0\}.$$

So, to prove (1), it is enough to show $0 \notin \operatorname{supp}\widehat{I^n f}$. We choose a number $0 < a < \sigma$ and a function $h \in C_0^\infty(-\sigma, \sigma)$ such that $h(x) = 1$ in $(-a, a)$. Then

$$\operatorname{supp} h\,\widehat{I^n f} \subset \{0\}.$$

Suppose $\operatorname{supp} h\,\widehat{I^n f} = \{0\}$. Then $h\,\widehat{I^n f}$ is a generalized function with point support. So, there is a number $N_n \in \mathbb{N}$ such that

$$h\,\widehat{I^n f} = \sum_{j=0}^{N_n} c_j^{N_n} D^j \delta,$$

where δ is the Dirac function defined by $< \delta, \varphi >= \varphi(0)$ for all $\varphi \in S$.
Hence, $F^{-1}h * I^n f = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{N_n} c_j^{N_n}(-ix)^j$. Since $I^n f \in N_\Psi(\mathbb{R})$ and $F^{-1}h \in M_\Psi(\mathbb{R})$ we have $F^{-1}h * I^n f \in L_\infty(\mathbb{R})$ and then

$$F^{-1}h * I^n f(x) = \frac{1}{\sqrt{2\pi}} c_0^{N_n}, n = 1, 2, \ldots.$$

Note that

$$F^{-1}h*I^n f(x) = F^{-1}h*(I^{n+1}f)'(x) = (F^{-1}h*I^{n+1}f(x))' = \left(\frac{1}{\sqrt{2\pi}}\, c_0^{N_{n+1}}\right)' \equiv 0.$$

So, $h\,\widehat{I^n f} = 0$. Now we assume the contrary that $0 \in \operatorname{supp}\widehat{I^n f}$. Then there always exists a function $\varphi \in C_0^\infty(-a, a)$ such that $\langle \widehat{I^n f}, \varphi\rangle \neq 0$. Since $h(x) = 1$ in $(-a, a)$ we get

$$0 \neq \langle \widehat{I^n f}, \varphi\rangle \;=\; \langle \widehat{I^n f}, h\,\varphi\rangle \;=\; \langle h\,\widehat{I^n f}, \varphi\rangle = 0,$$

which is impossible. Thus we have (1) or

$$\widehat{I^n f} \subset \mathbb{R} \setminus (-\sigma, \sigma).$$

Further, for any $0 < \varepsilon < \sigma$, we choose a function $h \in C^\infty(\mathbb{R})$ satisfying

$$\begin{cases} h(x) = 1 & \forall x \notin (-(\sigma - \frac{\varepsilon}{2}),\, \sigma - \frac{\varepsilon}{2}), \\ h(x) = 0 & \forall x \in (-(\sigma - \varepsilon),\, \sigma - \varepsilon). \end{cases}$$

Let $\varphi \in S(\mathbb{R})$. We put $g(x) = \varphi(x) - (F(F^{-1}(\varphi).h))(x)$ and $\psi = F(F^{-1}(\varphi).h)$. Then

$$\langle I^n f,\, \varphi\rangle \;=\; \langle I^n f,\, \psi\rangle. \tag{2}$$

Because $\hat{\psi} = 0$ in $(-(\sigma - \varepsilon),\, \sigma - \varepsilon)$, it is clear that

$$\psi_n(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{ix\xi}\, \frac{\hat{\psi}(\xi)}{(i\xi)^n}\, d\xi.$$

Then $\psi_n(x) \in S(\mathbb{R})$ and $D^n\psi_n = \psi$. Therefore,

$$|\langle f, \psi_n\rangle| = |\langle D^n(I^n f), \psi_n\rangle| = |\langle I^n f, D^n\psi_n\rangle| = |\langle I^n f, \psi\rangle|,$$

which together with (2) imply

$$|\langle I^n f, \varphi\rangle| = |\langle f, \psi_n\rangle|. \tag{3}$$

Since $\psi = F(F^{-1}(\varphi).h)$ we get $\hat{\psi}(\xi) = F^{-1}(\varphi)(-\xi).h(-\xi)$. We obtain for $n \geq 3$

$$\psi_n = F^{-1}(F(\psi_n)) = F^{-1}\left(\frac{\hat{\psi}(\xi)}{(i\xi)^n}\right) = F^{-1}\left((F^{-1}(\varphi))(-\xi).\frac{h(-\xi)}{(i\xi)^n}\right)$$
$$= F\left((F^{-1}(\varphi))(\xi).\frac{h(\xi)}{(-i\xi)^n}\right) = \frac{1}{\sqrt{2\pi}}\,\varphi * F\left(\frac{h(\xi)}{(-i\xi)^n}\right).$$

which together with (3) imply

$$|\langle I^n f, \varphi\rangle| = |\langle f, \psi_n\rangle| = \frac{1}{\sqrt{2\pi}} \left|\left\langle f, \varphi * F\left(\frac{h(\xi)}{(i\xi)^n}\right)\right\rangle\right|.$$

By virtue of Lemma 2.2, we have

$$\overline{\lim_{n\to\infty}} \left\| F\frac{h(\xi)}{(i\xi)^n}\right\|_1^{1/n} \le \frac{1}{\sigma-\varepsilon}.$$

On the other hand, it follows from $\varphi \in M_\Psi(\mathbb{R})$ and $F\dfrac{h(\xi)}{(i\xi)^n} \in L_1(\mathbb{R})$ that

$$\varphi * F\frac{h(\xi)}{(i\xi)^n} \in M_\Psi(\mathbb{R}).$$

If $\|\varphi\|_{M_\Psi(\mathbb{R})} \le 1$ then

$$\left\|\varphi * F\frac{h(\xi)}{(i\xi)^n}\right\|_{M_\Psi(\mathbb{R})} \le \|\varphi\|_{M_\Psi(\mathbb{R})}\cdot\left\|F\frac{h(\xi)}{(i\xi)^n}\right\|_1 \le \left\|F\frac{h(\xi)}{(i\xi)^n}\right\|_1.$$

Therefore,

$$\begin{aligned}\|I^n f\|_{N_\Psi(\mathbb{R})} &= \sup_{\{\varphi\in S:\, \|\varphi\|_{M_\Psi(\mathbb{R})}\le 1\}} |\langle I^n f, \varphi\rangle| = \frac{1}{\sqrt{2\pi}} \sup_{\{\varphi\in S:\, \|\varphi\|_{M_\Psi(\mathbb{R})}\le 1\}} \left|\left\langle f, \varphi * F\frac{h(\xi)}{(i\xi)^n}\right\rangle\right| \\ &\le \frac{1}{\sqrt{2\pi}} \sup_{\{\|t\|_{M_\Psi(\mathbb{R})}\le \|F\frac{h(\xi)}{(i\xi)^n}\|_1\}} |\langle f, t\rangle| = \frac{1}{\sqrt{2\pi}}\|f\|_{N_\Psi(\mathbb{R})}\cdot\left\|F\frac{h(\xi)}{(i\xi)^n}\right\|_1.\end{aligned}$$

So, we obtain

$$\overline{\lim_{n\to\infty}} \|I^n f\|_{N_\Psi(\mathbb{R})}^{1/n} \le \frac{1}{\sigma-\varepsilon}$$

and then

$$\overline{\lim_{n\to\infty}} \|I^n f\|_{N_\Psi(\mathbb{R})}^{1/n} \le \frac{1}{\sigma} \tag{4}$$

by letting $\varepsilon \to 0$.

On the other hand, without loss of generality we can assume that

$$\sigma = \inf\{\xi : 0 < \xi \in \mathrm{supp}\hat{f}\}.$$

Hence there exists a function $\varphi(x) \in C_0^\infty(\sigma-\varepsilon, \sigma+\varepsilon)$ such that $\langle \hat{f}, \varphi\rangle \ne 0$. Then $0 \ne \langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$. Therefore,

$$|\langle f, \hat{\varphi}\rangle| = |\langle D^n(I^n f), \hat{\varphi}\rangle| = |\langle I^n f, D^n\hat{\varphi}\rangle| \le \|I^n f\|_{N_\psi(\mathbb{R})}\cdot \|D^n\hat{\varphi}\|_{M_\Psi(\mathbb{R})}.$$

So, since $\varphi(x) = 0$ for all $x \notin (-(\sigma+\varepsilon), \sigma+\varepsilon)$, using the following result in [9]:

$$\lim_{n\to\infty} \|D^n \hat{\varphi}\|_{M_\psi(\mathbb{R})}^{1/n} \leq \sigma + \varepsilon,$$

and letting $\varepsilon \to 0$ we have

$$\varliminf_{n\to\infty} \|I^n f\|_{N_\psi(\mathbb{R})}^{1/n} \geq \frac{1}{\sigma}. \tag{5}$$

Combining (4) and (5), we arrive at

$$\lim_{n\to\infty} \|I^n f\|_{N_\Psi(\mathbb{R})}^{1/n} = \sigma^{-1}.$$

Case 2. $\sigma = 0$. Since $\sigma = 0, 0 \in \mathrm{supp}\hat{f}$. Hence, for any $\varepsilon > 0$ there is a function $\varphi(x) \in C_0^\infty(-\varepsilon, \varepsilon)$ such that $\langle \hat{f}, \varphi \rangle \neq 0$. Then arguing as just above, we obtain

$$\varliminf_{n\to\infty} \|I^n f\|_{N_\psi(\mathbb{R})}^{1/n} \geq \lim_{n\to\infty} \|D^n \hat{\varphi}\|_{M_\psi(\mathbb{R})}^{-1/n} \geq \frac{1}{\varepsilon}$$

and then

$$\lim_{n\to\infty} \|I^n f\|_{N_\psi(\mathbb{R})}^{1/n} = \infty.$$

So, we always have

$$\lim_{n\to\infty} \|I^n f\|_{N_\psi(\mathbb{R})}^{1/n} = \sigma^{-1}.$$

The theorem is proved. ∎

References

1. N. B. Andersen, Real Paley-Wiener theorems for the inverse Fourier transform on a Riemannian symmetric space, *Pacific J. Math.* **213** (2004), 1–13.
2. H. H. Bang, A property of infinitely differentiable functions, *Proc. Amer. Math. Soc.* **108** (1) (1990), 73–76.
3. H. H. Bang and M. Morimoto, On the Bernstein-Nikolsky inequality, *Tokyo J. Math.* **14** (1991), 231–238.
4. H. H. Bang, Remarks on a property of infinitely differentiable functions, *Bull. Polish Acad. Sci.* **40** (1993), 197–206.
5. H. H. Bang and M. Morimoto, The sequence of Luxemburg norms of derivatives, *Tokyo J. Math.* **17** (1994), 141–147.
6. H. H. Bang, On the Bernstein-Nikolsky inequality II, *Tokyo J. Math.* **18** (1995), 123–131.
7. H. H. Bang, An algebra of pseudodifferential operators, *Mat. Sbornik* **186** (1995), 3–14.
8. H. H. Bang, Asymptotic behavior of the sequence of norms of derivatives, *J. Math. Sci. Univ. Tokyo* **2** (1995), 611–620.
9. H. H. Bang, A property of entire functions of exponential type, *Analysis* **15** (1995), 17–23.
10. H. H. Bang, Functions with bounded spectrum, *Trans. Amer. Math. Soc.* **347** (1995), 1067–1080.
11. H. H. Bang, The existence of a point spectral radius of pseudodifferential operators, *Doklady Akad. Nauk* **348** (1996), 740–742.

12. H. H. Bang, A study of the properties of functions depending on the geometry of their spectrum, *Doklady Akad. Nauk* **355** (1997), 740–743.
13. H. H. Bang, The study of the properties of functions belonging to an Orlicz space depending on the geometry of their spectra (Russian), *Izv. Russ. Akad. Nauk Ser. Mat.* **61** (1997), 163–198; translation in Izv. Math. **61**, 399–434.
14. H. H. Bang, Investigation of the properties of functions in the space N_Φ depending on the geometry of their spectrum, *Doklady Akad. Nauk* **374** (2000), 590–593.
15. H. H. Bang and V. N. Huy, Behavior of the sequence of norms of primitives of a function, *J. Approximation Theory* **162** (2010), 1178–1186.
16. J. J. Betancor, J. D. Betancor and J. M. R. Méndez, Paley-Wiener type theorems for Chébli- Trimèche transforms *Publ. Math. Debrecen* **60** (2002), 347–358.
17. H. M. Le, On a property of infinitely differentiable functions, *Acta Math. Vietnam.* **24** (3) (199 9), 263–270.
18. M. S. Steigerwalt and A. J. White, Some function spaces related to L_p,, *Proc. London. Math. Soc.* **22** (1971), 137–163.
19. M. T. Thu, *Some Inequalities for Derivatives in Orlicz and Lorentz spaces*, Ph.D. Thesis, Institute of Mathematics, Vietnamese Academy of Science and Technology, 2006.
20. V. K. Tuan and A. Zayed, Generalization of a theorem of Boas to a class of integral transforms, *Result. Math.* **38** (2000), 362–376.
21. V. K. Tuan, Spectrum of signals, *J. Fourier Anal. Appl.* **7** (2001), 319–323.
22. V. S. Vladimirov, *Methods of the Theory of Generalized Functions*, Taylor & Francis, London, New York, 2002.