Vietnam Journal of Mathematics **38**:4(2010) 435-451

Vietnam Journal of MATHEMATICS

A Nonlocal Boundary Value Problem for Fractional Differential Inclusions of Arbitrary Order Involving Convex and Non-convex Valued Maps

Bashir Ahmad and Naseer Shahzad

Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Received August 31, 2010
Revised November 6, 2010

Abstract. This paper investigates the existence of solutions for a nonlocal boundary value problem of fractional differential inclusions of arbitrary order involving convex and non-convex valued maps. Our results are based on the nonlinear alternative of Leray Schauder type and a fixed point theorem for generalized contractions.

2000 Mathematics Subject Classification: 34A12, 34A40.

Key words: Fractional differential inclusions, arbitrary order, nonlocal boundary conditions, existence, nonlinear alternative of Leray Schauder type, fixed point theorems.

1. Introduction

Fractional calculus (differentiation and integration of arbitrary order) is proved to be an important tool in the modelling of dynamical systems associated with phenomena such as fractal and chaos. In fact, this branch of calculus has found its applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity and damping, wave propagation, percolation, identification, fitting of experimental data, etc. [18, 29, 30, 31].

Recently, differential equations of fractional order have been addressed by several researchers with the sphere of study ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. For some recent work on fractional differential equations, see [2, 10, 14, 20, 24, 25, 26] and the references therein.

Differential inclusions arise in the mathematical modelling of certain problems in economics, optimal control, etc. and are widely studied by many authors, see [9, 23, 27, 32] and the references therein. For some recent development on differential inclusions of fractional order, we refer the reader to the references [1, 3, 11, 15, 16, 28].

In this paper, for $m \in \mathbb{N}, m \geq 2$, and $q \in (m-1, m]$, we consider the following fractional differential inclusions of order q with nonlocal boundary conditions

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)),\ t \in [0,1], \\ x(0) = \beta x(\eta),\ x'(0) = 0,\ x''(0) = 0,\ ..., x^{(m-2)}(0) = 0,\ x(1) = \alpha x(\eta), \\ 0 < \eta < 1,\ (\alpha - \beta)\eta^{m-1} \neq 1 - \beta,\ \beta,\ \alpha \in \mathbb{R}, \end{cases} \tag{1}$$

where ${}^cD^q$ denote the Caputo fractional derivative of order q, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact valued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

The concept of nonlocal multi-point boundary conditions is quite important in various physical problems of applied nature when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the censors located at intermediate points. Some recent results on nonlocal fractional boundary value problems can be found in [4, 5, 6].

2. Preliminaries

Let $C([0,1])$ denote the Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

Now we recall some basic definitions on multi-valued maps [12, 17].

For a normed space $(X, \|.\|)$, let $P(X) = \{Y \subseteq X : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{||y|| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that

$G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$. A multivalued map $G : [0,1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if
(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$;
(iii) for each $q > 0$, there exists $\varphi_q \in L^1([0,1], \mathbb{R}_+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_q(t) \text{ for all } |x| \leq q \text{ and for a. e. } t \in [0,1].$$

For each $y \in C([0,1], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0,1]\}.$$

Let E be a Banach space, X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1], \mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.2. Let Y be a separable metric space and let

$$\varPhi : Y \to \mathcal{P}(L^1([0,1], \mathbb{R}))$$

be a multivalued operator. We say $\varPhi$ has a property $(\mathcal{CD})$ if $\varPhi$ is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0;1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0;1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1], \mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nymetzki operator associated with F.

Definition 2.3. Let $F : [0;1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nymetzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let (X, d) be a metric space. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

where $d(a, B) = \inf_{b\in B} d(a, b)$. H_d is the (generalized) Pompeiu-Hausdorff functional. It is known that $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [19]). A function $l : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function (see [22]) if it is continuous strictly increasing and $\sum_{n=1}^{\infty} l^n(t) < \infty$, for each $t > 0$.

Definition 2.4. A multivalued operator $\Phi : X \to P_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(\Phi(x), \Phi(y)) \leq \gamma d(x, y) \text{ for each } x, y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$;

(c) a generalized contraction if and only if there is a strict comparison function $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H_d(\Phi(x), \Phi(y)) \leq l(d(x, y)) \text{ for each } x, y \in X;$$

The following lemmas will be used in the sequel.

Lemma 2.5. ([21]) *Let X be a Banach space. Let $F : [0;1] \times \mathbb{R} \to P_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$, then the operator*

$$\Theta \circ S_F : C([0,1], X) \to P_{cp,c}(C([0,1], X)), \ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

Lemma 2.6. ([7]) *Let Y be a separable metric space and let $\Phi : Y \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ be a multivalued operator satisfying the property $(\mathcal{CD})$. Then Φ has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1], \mathbb{R})$ such that $g(x) \in \Phi(x)$ for every $x \in Y$.*

Lemma 2.7. (Wegrzyk's fixed point theorem [22, 33]) *Let (X, d) be a complete metric space. If $\Phi : X \to P_{cl}(X)$ is a generalized contraction, then* Fix $\Phi \neq \emptyset$.

Lemma 2.8. (Nonlinear alternative of Leray-Schauder type [13]) *Let C be a convex set in a normed space and $U \subset C$ be open with $0 \in U$. Then each compact*

and upper semicontinuous mapping $\Phi : \overline{U} \to P(C)$ with compact convex values which is fixed point free on ∂U has at least one of the following two properties:
(a) *Φ has a fixed point,*
(b) *there exist $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda\Phi(x)$.*

Lemma 2.9. (Arzela-Ascoli Theorem [13]) *A subset of $C([0,1],\mathbb{R})$ is relatively compact if and only if it is bounded and equicontinuous.*

Let us recall some definitions on fractional calculus [31, 29, 18].

Definition 2.10. For a function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds,\ n-1 < q < n, n = [q]+1,\ q > 0,$$

where $[q]$ denotes the integer part of the real number q and Γ denotes the gamma function.

Definition 2.11. The Riemann-Liouville fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds,\ q > 0,$$

provided the right hand side is pointwisely defined on $(0,\infty)$.

Definition 2.12. The Riemann-Liouville fractional derivative of order q for a function g is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds,\ n = [q]+1,\ q > 0,$$

provided the right hand side is pointwisely defined on $(0,\infty)$.

To define the solution of (1), we consider the following lemma.

Lemma 2.13. *For a given $\sigma \in C[0,1]$, the unique solution of the boundary value problem*

$$\begin{cases} ^cD^q x(t) = \sigma(t),\ 0 < t < 1,\ q \in (m-1, m],\ m \in \mathbb{N},\ m \geq 2, \\ x(0) = \beta x(\eta),\ x'(0) = 0,\ x''(0) = 0, \dots, x^{(m-2)}(0) = 0,\ x(1) = \alpha x(\eta), \\ 0 < \eta < 1,\ (\alpha - \beta)\eta^{m-1} \neq 1 - \beta,\ \beta,\ \alpha \in \mathbb{R}, \end{cases} \quad (2)$$

is given by

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds$$
$$-\Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}\sigma(s)ds. \quad (3)$$

Proof. As argued in [20], the general solution of (2) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - c_0 - c_1 t - c_2 t^2 - \ldots - c_{m-1}t^{m-1}, \quad (4)$$

where $c_0, c_1, c_2, \ldots, c_{m-1} \in \mathbb{R}$ are arbitrary constants. In view of the relations ${}^cD^q\ I^q x(t) = x(t)$ and $I^q\ I^p x(t) = I^{q+p}x(t)$ for $q, p > 0, x \in L(0,1)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds - c_1 - 2c_2 t - \ldots - (m-1)c_{m-1}t^{m-2},$$
$$x''(t) = \int_0^t \frac{(t-s)^{q-3}}{\Gamma(q-2)}\sigma(s)ds - 2c_2 - \ldots - (m-1)(m-2)c_{m-1}t^{m-3}, \ldots.$$

Applying the boundary conditions for (2), we find that

$$c_0 = \Big(\frac{\beta}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}\sigma(s)ds$$
$$-\Big(\frac{\beta\eta^{m-1}}{\beta - 1 + (\alpha-\beta)}\eta^{m-1}\Big)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds,$$

$c_1 = 0, \ldots, c_{m-2} = 0$, and

$$c_{m-1} = \Big(\frac{\alpha-\beta}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}\sigma(s)ds$$
$$+\Big(\frac{\beta-1}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds,$$

Substituting the values of $c_0,\ c_1, \ldots, c_{m-1}$ in (4), we obtain (3). This completes the proof. ∎

Definition 2.14. A function $x \in C^m([0,1])$ is a solution of problem (1) if there exists a function $f \in L^1([0,1], \mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,1]$ and

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds - \Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s)ds.$$

3. Main Results

Theorem 3.1. *Assume that*

(H_1) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is* L^1-*Carathéodory, and has compact and convex values;*

(H_2) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,1], \mathbb{R}_+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(|x|) \text{ for each } (t,x) \in [0,1] \times \mathbb{R};$$

(H_3) *there exists a number* $M > 0$ *such that*

$$\frac{\Gamma(q)|\beta - 1 + (\alpha-\beta)\eta^{m-1}|M}{\Big(|\beta - 1 + (\alpha-\beta)\eta^{m-1}| + |\beta\eta^{m-1} + (1-\beta)| + |\alpha|\eta^{q-1}\Big)\psi(M)\|p\|_{L^1}} > 1.$$

Then the boundary value problem (1) *has at least one solution on* $[0,1]$.

Proof. Let an operator $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ be defined by

$$\Omega(x) = \Big\{h \in C([0,1],\mathbb{R}) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds - \Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s)ds, \ f \in S_{F,x}\Big\}. \tag{5}$$

We will show that Ω satisfies the assumptions of Lemma 2.8. The proof consists of several steps. As the first step, we show that $\Omega(x)$ is convex for each $x \in C([0,1],\mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,1]$, we have

$$h_i(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f_i(s)ds$$
$$-\Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f_i(s)ds, \ i = 1, 2.$$

Let $0 \leq \lambda \leq 1$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned}
&[\lambda h_1 + (1-\lambda)h_2](t) \\
&= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} [\lambda f_1(s) + (1-\lambda) f_2(s)]ds \\
&\quad + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} [\lambda f_1(s) + (1-\lambda) f_2(s)]ds \\
&\quad - \Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} [\lambda f_1(s) + (1-\lambda) f_2(s)]ds.
\end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\lambda h_1 + (1-\lambda)h_2 \in \Omega(x)$.

Next, we sow that Ω maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\|_\infty \leq r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds$$
$$-\Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s)ds$$

and in view of (H_1), we have

$$|h(t)| \quad \leq \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |f(s)|ds + \Big|\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big| \int_0^1 \frac{|1-s|^{q-1}}{\Gamma(q)} |f(s)|ds$$
$$+\Big|\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big| \int_0^\eta \frac{|\eta-s|^{q-1}}{\Gamma(q)} |f(s)|ds$$

$$\leq \Big(\frac{|\beta-1+(\alpha-\beta)\eta^{m-1}|+|\beta\eta^{m-1}+(1-\beta)|}{\Gamma(q)|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big)\int_0^1 \varphi_r(s)ds$$
$$+\frac{|\alpha|\eta^{q-1}}{\Gamma(q)|\beta-1+(\alpha-\beta)\eta^{m-1}|}\int_0^\eta \varphi_r(s)ds.$$

Thus,

$$\|h\|_\infty \leq \Big(\frac{|\beta-1+(\alpha-\beta)\eta^{m-1}|+|\beta\eta^{m-1}+(1-\beta)|}{\Gamma(q)|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big)\int_0^1 \varphi_r(s)ds$$
$$+\frac{|\alpha|\eta^{q-1}}{\Gamma(q)|\beta-1+(\alpha-\beta)\eta^{m-1}|}\int_0^\eta \varphi_r(s)ds.$$

Now we show that Ω maps bounded sets into equicontinuous sets in $C([0,1],\mathbb{R})$. Let $t',t''\in[0,1]$ with $t'<t''$ and $x\in B_r$, where B_r is a bounded set in $C([0,1],\mathbb{R})$. For each $h\in\Omega(x)$, we obtain

$$|h(t'')-h(t')|$$
$$=\Big|\int_0^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s)ds+\Big(\frac{\beta\eta^{m-1}+(1-\beta)(t'')^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds$$
$$-\Big(\frac{\beta+(\alpha-\beta)(t'')^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}f(s)ds$$
$$-\int_0^{t'}\frac{(t'-s)^{q-1}}{\Gamma(q)}f(s)ds-\Big(\frac{\beta\eta^{m-1}+(1-\beta)(t')^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds$$
$$+\Big(\frac{\beta+(\alpha-\beta)(t')^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}f(s)ds\Big|$$
$$\leq\Big|\int_0^{t'}\frac{[(t''-s)^{q-1}-(t'-s)^{q-1}]}{\Gamma(q)}f(s)ds\Big|+\Big|\int_{t'}^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s)ds\Big|$$
$$+\frac{((t'')^{m-1}-(t')^{m-1})}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big|\int_0^1(1-\beta)\frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds$$
$$-(\alpha-\beta)\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}f(s)ds\Big|.$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t'' - t' \to 0$. As Ω satisfies the above three assumptions, therefore it follows by Lemma 2.9 that Ω is completely continuous.

In our next step, we show that Ω has a closed graph. Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$h_n(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)ds + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f_n(s)ds$$
$$-\Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f_n(s)ds.$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$h_*(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f_*(s)ds$$
$$-\Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f_*(s)ds.$$

Let us consider a continuous linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$f \mapsto \Theta(f)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds$$
$$+\Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds$$
$$-\Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s)ds.$$

Observe that

$$\|h_n(t) - h_*(t)\|$$
$$= \Big\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds$$
$$+\Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds$$

$$-\Big(\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))ds\Big\| \to 0 \text{ as } n\to\infty.$$

Thus, it follows by Lemma 2.5 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds + \Big(\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f_*(s)ds$$
$$-\Big(\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f_*(s)ds,$$

for some $f_* \in S_{F,x_*}$.

As the last step, we discuss a priori bounds on solutions. Let x be a solution of (1). Then there exists $f \in L^1([0,1],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \Big(\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds$$
$$-\Big(\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s)ds.$$

In view of (H_2), for each $t \in [0,1]$, we obtain

$$|x(t)| \le \Big(\frac{|\beta-1+(\alpha-\beta)\eta^{m-1}|+|\beta\eta^{m-1}+(1-\beta)|}{\Gamma(q)|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big)\int_0^1 p(s)\psi(\|x\|_\infty)ds$$
$$+\frac{|\alpha|\eta^{q-1}}{\Gamma(q)|\beta-1+(\alpha-\beta)\eta^{m-1}|}\int_0^\eta p(s)\psi(\|x\|_\infty)ds$$
$$\le \Big(\frac{|\beta-1+(\alpha-\beta)\eta^{m-1}|+|\beta\eta^{m-1}+(1-\beta)|+|\alpha|\eta^{q-1}}{\Gamma(q)|\beta-1+(\alpha-\beta)\eta^{m-1}|}\Big)\times$$
$$\times\psi(\|x\|_\infty)\int_0^1 p(s)ds.$$

Consequently it follows that

$$\frac{\Gamma(q)|\beta-1+(\alpha-\beta)\eta^{m-1}|\|x\|_\infty}{\Big(|\beta-1+(\alpha-\beta)\eta^{m-1}|+|\beta\eta^{m-1}+(1-\beta)|+|\alpha|\eta^{q-1}\Big)\psi(\|x\|_\infty)\|p\|_{L^1}} \le 1.$$

By (H_3), there exists M such that $\|x\|_\infty \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\|_\infty < M\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by Lemma 2.8, we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (1). This completes the proof. ■

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [7] for lower semi-continuous maps with decomposable values.

Theorem 3.2. *Assume that* (H_2)-(H_3) *and the following conditions hold:*

(H_4) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(a) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*

(b) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,1]$*;*

(H_5) *for each* $\rho > 0$*, there exists* $\varphi_\rho \in L^1([0,1],\mathbb{R}_+)$ *such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq \varphi_\rho(t) \ \text{ for all } |x| \leq \rho \text{ and for a.e. } t \in [0,1].$$

Then the boundary value problem (1) *has at least one solution on* $[0,1]$.

Proof. It follows from (H_4) and (H_5) that F is of l.s.c. type. Then from Lemma 2.6, there exists a continuous function $f : C([0,T],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \ t \in [0,1], \\ x(0) = \beta x(\eta), \ x'(0) = 0, \ x''(0) = 0, \ldots, x^{(m-2)}(0) = 0, \ x(1) = \alpha x(\eta), \\ 0 < \eta < 1, \ (\alpha - \beta)\eta^{m-1} \neq 1 - \beta, \ \beta, \ \alpha \in \mathbb{R}, \end{cases} \tag{6}$$

Observe that if $x \in C^m([0,1])$ is a solution of (6), then x is a solution to the problem (1). In order to transform the problem (6) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\overline{\Omega}x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds$$

$$+\Big(\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(x(s))ds$$
$$-\Big(\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}f(x(s))ds.$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. ∎

Now we prove the existence of solutions for the problem (1) with a non-convex valued right hand side by applying Lemma 2.7.

Theorem 3.3. *Assume that the following conditions hold:*

(H_6) $F:[0,1]\times\mathbb{R}\to P_{cp}(\mathbb{R})$ *is such that* $F(.,x):[0,1]\to P_{cp}(\mathbb{R})$ *is measurable for each* $x\in\mathbb{R}$.

(H_7) $H_d(F(t,x),F(t,\bar{x}))\le\kappa(t)l(|x-\bar{x}|)$ *for almost all* $t\in[0,1]$ *and* $x,\bar{x}\in\mathbb{R}$ *with* $\kappa\in L^1([0,1],\mathbb{R}_+)$ *and* $d(0,F(t,0))\le\kappa(t)$ *for almost all* $t\in[0,1]$, *where* $l:\mathbb{R}_+\to\mathbb{R}_+$ *is strictly increasing.*

Then the boundary value problem (1) *has at least one solution on* $[0,1]$ *if* $\gamma l:\mathbb{R}_+\to\mathbb{R}_+$ *is a strict comparison function, where*

$$\gamma=\frac{\Big(|\beta-1+(\alpha-\beta)\eta^{m-1}|+|\beta\eta^{m-1}+(1-\beta)|+|\alpha|\eta^{q-1}\Big)\|\kappa\|_{L^1}}{\Gamma(q)|\beta-1+(\alpha-\beta)\eta^{m-1}|}.$$

Proof. Suppose that $\gamma l:\mathbb{R}_+\to\mathbb{R}_+$ is a strict comparison function. Observe that by the assumptions (H_6) and (H_7), $F(.,x(.))$ is measurable and has a measurable selection $v(.)$ (see [8, Theorem III.6]). Also $\kappa\in L^1([0,1],\mathbb{R})$ and

$$\begin{aligned}|v(t)|&\le d(0,F(t,0))+H_d(F(t,0),F(t,x(t)))\\&\le\kappa(t)+\kappa(t)l(|x(t)|)\\&\le(1+l(||x||_\infty))\kappa(t).\end{aligned}$$

Thus the set $S_{F,x}$ is nonempty for each $x\in C([0,1],\mathbb{R})$. Now we show that the operator Ω defined by (5) satisfies the assumptions of Lemma 2.7. To show that $\Omega(x)\in P_{cl}((C[0,1],\mathbb{R}))$ for each $x\in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\ge0}\in\Omega(x)$ be such that $u_n\to u$ $(n\to\infty)$ in $C([0,1],\mathbb{R})$. Then $u\in C([0,1],\mathbb{R})$ and there exists $v_n\in S_{F,x}$ such that, for each $t\in[0,1]$,

$$u_n(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_n(s)ds+\Big(\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_n(s)ds$$

$$-\Big(\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}v_n(s)ds.$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$,

$$u_n(t) \to u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}v(s)ds + \Big(\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}v(s)ds$$
$$-\Big(\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}v(s)ds.$$

Hence $u \in \Omega(x)$.

Next we show that

$$H_d(\Omega(x),\Omega(\bar{x})) \le \gamma l(\|x-\bar{x}\|_\infty) \text{ for each } x,\bar{x} \in C([0,1],\mathbb{R}).$$

Let $x,\bar{x} \in C([0,1],\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$h_1(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}v_1(s)ds + \Big(\frac{\beta\eta^{m-1}+(1-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}v_1(s)ds$$
$$-\Big(\frac{\beta+(\alpha-\beta)t^{m-1}}{\beta-1+(\alpha-\beta)\eta^{m-1}}\Big)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}v_1(s)ds,$$

By (H_7), we have

$$H_d(F(t,x),F(t,\bar{x})) \le \kappa(t)l(|x(t)-\bar{x}(t)|).$$

So, there exists $w \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w| \le \kappa(t)l(|x(t)-\bar{x}(t)|),\ t \in [0,1].$$

Define $V : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{w \in \mathbb{R} : |v_1(t)-w| \le \kappa(t)l(|x(t)-\bar{x}(t)|)\}.$$

Since the nonempty closed valued operator $V(t) \cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [33]), there exists a function $v_2(t)$ which is a measurable selection for $V(t) \cap F(t,\bar{x}(t))$. So $v_2(t) \in F(t,\bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t)-v_2(t)| \le \kappa(t)l(|x(t)-\bar{x}(t)|)$.

For each $t \in [0,1]$, let us define

$$h_2(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)ds + \Big(\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} v_2(s)ds$$
$$-\Big(\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big) \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} v_2(s)ds.$$

Thus

$$\begin{aligned}
&|h_1(t) - h_2(t)| \\
&\le \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|ds \\
&\quad + \Big|\frac{\beta\eta^{m-1} + (1-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big| \int_0^1 \frac{|1-s|^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|ds \\
&\quad + \Big|\frac{\beta + (\alpha-\beta)t^{m-1}}{\beta - 1 + (\alpha-\beta)\eta^{m-1}}\Big| \int_0^\eta \frac{|\eta-s|^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|ds \\
&\le \frac{\Big(|\beta - 1 + (\alpha-\beta)\eta^{m-1}| + |\beta\eta^{m-1} + (1-\beta)| + |\alpha|\eta^{q-1}\Big)}{\Gamma(q)|\beta - 1 + (\alpha-\beta)\eta^{m-1}|} \times \\
&\qquad \times \int_0^1 \kappa(s) l(\|x - \overline{x}\|_\infty)ds
\end{aligned}$$

Hence

$$\begin{aligned}
&\|h_1 - h_2\|_\infty \\
&\le \frac{\Big(|\beta - 1 + (\alpha-\beta)\eta^{m-1}| + |\beta\eta^{m-1} + (1-\beta)| + |\alpha|\eta^{q-1}\Big)\|\kappa\|_{L^1}}{\Gamma(q)|\beta - 1 + (\alpha-\beta)\eta^{m-1}|} l(\|x - \overline{x}\|_\infty).
\end{aligned}$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned}
&H_d(\Omega(x), \Omega(\bar{x})) \le \gamma l(\|x - \bar{x}\|_\infty) \\
&= \frac{\Big(|\beta - 1 + (\alpha-\beta)\eta^{m-1}| + |\beta\eta^{m-1} + (1-\beta)| + |\alpha|\eta^{q-1}\Big)\|\kappa\|_{L^1}}{\Gamma(q)|\beta - 1 + (\alpha-\beta)\eta^{m-1}|} l(\|x - \overline{x}\|_\infty)
\end{aligned}$$

for each $x, \bar{x} \in C([0,1], \mathbb{R})$.

Since Ω is a generalized contraction, it follows by Lemma 2.7 that Ω has a fixed point x which is a solution of (1). This completes the proof. ■

We notice that Theorem 3.3 holds for several values of the function l, for example, $l(t) = \frac{\ln(1+t)}{\gamma}$, where $\gamma \in (0,1)$; $l(t) = t$, etc.

Acknowledgements. The authors are grateful to the anonymous referee for his/her valuable comments.

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