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On the Stability and Boundedness of Solutions in a Class of Nonlinear Differential Equations of Fourth Order with Constant Delay

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Abstract. In this paper, we discuss the stability and boundedness of solutions of the nonlinear delay differential equation of fourth-order:

$$\begin{aligned}&x^{(4)}(t) + f_1(x''(t))x'''(t) + f_2(x''(t-r)) + g(x'(t-r)) + h(x(t-r))\\&= p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t), x''(t-r), x'''(t)),\end{aligned}$$

when $p = 0$ and $p \neq 0$, respectively, where $r > 0$ is a constant delay. In proving our main results, we use the Lyapunov functional approach.

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1. Introduction

To the best of our knowledge, first, in 1973, Sinha [5] discussed the stability of solutions of a certain nonlinear delay differential equation of fourth order, in the literature. Namely, Sinha [5] considered the following fourth-order nonlinear delay differential equation:

$$x^{(4)}(t)+f(x''(t))x'''(t)+f_2(x'(t),x''(t))x''(t)+g(x'(t-r))+h(x(t-r)) = 0, \quad (1)$$

where r is a positive constant, and by defining a Lyapunov functional, he proved the asymptotic stability of null solution of equation (1) (see [5, Theorem 2]).

Later, in 1989, Okoronkwo [4] took into consideration to the nonlinear delay differential equation of fourth-order:

$$\begin{aligned} & x^{(4)}(t) + f(x''(t))x'''(t) + \alpha_2 x''(t) + \beta_2 x''(t-r) \\ & + g(x'(t-r)) + \alpha_4 x(t) + \beta_4 x(t-r) \\ & = p(t). \end{aligned}$$

He established some sufficient conditions, which guarantee the stability and boundedness of solutions of the mentioned equation, when $p = 0$ and $p \neq 0$, respectively (see [4, Theorems 3.1, 3.2]).

After that, in 2000, Tejumola and Tchegnani [6] considered the following nonlinear delay differential equation of fourth-order:

$$\begin{aligned} & x^{(4)}(t) + \varphi(t, x(t), x'(t), x''(t), x'''(t))x'''(t) + \psi(t, x'(t-r), x''(t-r)) \\ & + \chi(t, x(t-r), x'(t-r)) + h(x(t-r)) \\ & = p_2(t, x(t), x'(t), x''(t), x'''(t), x(t-r), x'(t-r), x''(t-r)). \end{aligned} \tag{2}$$

The authors proved two results [6, Theorems 2.3, 2.4] related to the stability, uniformly boundedness and uniformly ultimately boundedness of solutions of equation (2), when $p = 0$ and $p \neq 0$, respectively.

The author [9] also established a similar result on the boundedness of solutions of the delay differential equation of fourth-order:

$$\begin{aligned} & x^{(4)}(t) + f_1(x''(t))x'''(t) + f_2(x'(t), x''(t))x''(t) + g(x'(t-r)) + h(x(t-r)) \\ & = p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t), x''(t-r), x'''(t)). \end{aligned} \tag{3}$$

On the other hand, we refer the reader to our papers [7, 8] and the references therein for the rest of contributions to the topic related to nonlinear delay differential equations of fourth-order.

In this paper, we consider the nonlinear delay differential equation of fourth-order:

$$\begin{aligned} & x^{(4)}(t) + f_1(x''(t))x'''(t) + f_2(x''(t-r)) + g(x'(t-r)) + h(x(t-r)) \\ & = p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t), x''(t-r), x'''(t)), \end{aligned} \tag{4}$$

where $t \in [0, \infty)$, $x \in (-\infty, \infty)$, and $r > 0$ is a constant delay. The real valued functions f_1, f_2 , g, h and p are supposed to be continuous in their arguments and satisfy Lipschitz condition; in deed, the existence and uniqueness of solutions of equation (4) are guaranteed. Moreover, we assume that f_2, g and h are continuously differentiable functions with $f_2(0) = g(0) = h(0) = 0$.

We transform equation (4) to the system as follows

$$x' = y, \ y' = z, \ z' = u,$$

$$
\begin{aligned}
u' = &- f_1(z)u - f_2(z) - g(y) - h(x) \\
&+ \int_{t-r}^{t} f_2'(z(s))u(s)ds + \int_{t-r}^{t} g'(y(s))z(s)ds + \int_{t-r}^{t} h'(x(s))y(s)ds \\
&+ p(t, x, x(t-r), y, y(t-r), z, z(t-r), u), \qquad (5)
\end{aligned}
$$

where $x(t)$, $y(t)$, $z(t)$ and $u(t)$ are, respectively, abbreviated as x, y, z and u, and throughout the paper.

The motivation for the present paper comes especially from the paper of Sinha [5], which is related to the stability of the null solution of equation (1) (see [5, Theorem 2]). It is clear that the term $f_2(x'(t), x''(t))x''(t)$, which is included in equation (1) and equation (3), has no delay, however, our equation, equation (4), includes the term $f_2(x''(t-r))$, which has the constant delay r, $r > 0$. This fact is a clear improvement of the works of Sinha [5, Theorem 2] and Tunç [9] for the special case $f_2(x'(t), x''(t))x''(t) = f_2(x'(t))x''(t)$ in equation (1) and equation (3). Here, our aim is to improve the results of Sinha [5] and Tunç [9] to the stability and boundedness of solutions of equation (4). It should be noted that in proving the main results of this paper; we take advantage of the Lyapunov functional approach as used in the papers of Okoronkwo [4], Sinha [5], Tejumola and Tchegnani [6] and Tunç [7, 8, 9] (see also Lyapunov's second method [3]). Next, the assumptions will be established and the Lyapunov functional will be used here are completely different from that in Okoronkwo [4], Tejumola and Tchegnani [6] and Tunç [7, 8].

2. Preliminaries

We give some basic information for the general non-autonomous and autonomous delay differential systems (see Burton [2] and Yoshizawa [10]). Consider the general non-autonomous delay differential system

$$\dot{x} = F(t, x_t), x_t(\theta) = x(t+\theta), -r \le \theta \le 0, t \ge 0, \qquad (6)$$

where $F : [0, \infty) \times C_H \to \Re^n$ is a continuous mapping, $F(t, 0) = 0$, and we suppose that F takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \|.\|)$ is the Banach space of continuous function $\phi : [-r, 0] \to \Re^n$ with supremum norm, $r > 0$; C_H is the open H -ball in C ; $C_H := \{\phi \in C([-r, 0], \Re^n) : \|\phi\| < H\}$.

Definition 2.1. (See [10]) A function $x(t_0, \phi)$ is said to be a solution of (6) with the initial condition $\phi \in C_H$ at $t = t_0$, $t_0 \ge 0$, if there is a constant $A > 0$ such that $x(t_0, \phi)$ is a function from $[t_0 - r,\ t_0 + A]$ into $\Re^n$ with the properties:

(i) $x_t(t_0, \phi) \in C_H$ for $t_0 \le t < t_0 + A$,

(ii) $x_{t_0}(t_0, \phi) = \phi$,

(iii) $x(t_0, \phi)$ satisfies (6) for $t_0 \le t < t_0 + A$.

Standard existence theory, see Burton [2], shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ satisfying on $[t_0,\ t_0 + \alpha)$ system (6) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.\ |$ will denote a convenient norm in $\Re^n$ with $|x| = \max_{1\leq i\leq n} |x_i|$. Now, let us assume that $C(t) = \{\phi : [t - \alpha, t] \to \Re^n \mid \phi \text{ is continuous}\}$ and ϕ_t denotes the ϕ in the particular $C(t)$, and that $\|\phi_t\| = \max_{t-\alpha\leq s\leq t} |\phi(t)|$.

Definition 2.2. (See [2]) Let $V(t, \phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of V along solutions of (6) will be denoted by $\dot{V}$ and is defined by

$$\dot{V}(t, \phi) = \limsup_{h\to 0} \frac{V(t + h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))}{h},$$

where $x(t_0, \phi)$ is the solution of (6) with $x_{t_0}(t_0, \phi) = \phi$.

Consider the general autonomous delay differential system:

$$\dot{x} = G(x_t), x_t = x(t + \theta), -r \leq \theta \leq 0, t \geq 0,$$

where $g : C_H \to \Re^n$ is a continuous mapping, $g(0) = 0$, and we suppose that G maps closed bounded sets into bounded sets of $\Re^n$. Here $(C,\ \|.\ \|)$ is the Banach space of continuous function $\phi : [-r,\ 0] \to \Re^n$ with supremum norm, $r > 0$; C_H is the open H -ball in C; $C_H := \{\phi \in C([-r, 0],\ \Re^n) : \|\phi\| < H\}$.

Lemma 2.3. (See[5]) *Suppose $G(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \leq s < \infty$, $u(s) \to \infty$ as $u \to \infty$ with $u(0) = 0$. If for all $\phi \in C$, $u(|\phi(0)|) \leq V(\phi)$, $V(\phi) \geq 0$, $\dot{V}(\phi) \leq 0$, then the solution $x = 0$ of $\dot{x} = G(x_t)$ is stable.*

If we define $Z = \{\phi \in C_H : \dot{V}(\phi) = 0\}$, then the solution $x = 0$ of $\dot{x} = G(x_t)$ is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3. Main Results

For the case $p(t, x, x(t-r), y, y(t-r), z, z(t-r), u) = 0$, our first result is given by the following theorem.

Theorem 3.1. *In addition to the basic assumptions imposed on the functions f_1, f_2, g and h, we assume that there exist positive constants a, b, β, λ, δ, δ_0 and δ_1 such that the following conditions hold:*

$$b \geq f_1(z) \geq f_1^0 \geq a + 2\lambda,$$

$$\frac{f_2(z)}{z} \geq f_2^0 > 0, (z \neq 0), \delta_0 \geq f_2'(z) > 0,$$

$$\frac{g(y)}{y} \geq g^0 > 0, (y \neq 0), \delta_1 \geq g'(y) \geq g^0,$$

$$\sqrt{h^0\delta} \geq \frac{h(x)}{x} \geq h^0 > 0, (x \neq 0), \delta \geq h'(0) \geq h'(x) \geq h^0,$$

$$\frac{1}{z}\int_0^z f_1(s)ds - f_1(z) < \frac{\beta}{a^2 g^0 h^0}, (z \neq 0),$$

and

$$a f_2^0 g^0 h^0 - \delta a b h^0 - \delta_1^2 \delta = \beta > 0,$$

$$\frac{\beta}{(g^0)^2 (2a g^0 h^0)^{\frac{1}{2}}} > \left(\frac{\delta_1}{g^0} - \frac{h^0}{\delta}\right).$$

Then the null solution of equation (4) *is asymptotic stable, provided that*

$$r < \min\left\{\frac{2g^0\delta_2}{\delta(\delta_0 + \delta_1 + \delta) + 2g^0\lambda_1}, \frac{2\gamma}{\delta_0 + \delta_1 + \delta + 2\lambda_2}, \frac{4\lambda}{\delta_0 + \delta_1 + \delta}\right\}.$$

Here, the superscript 0 *designates the evaluation of the given function at the origin.*

Remark 3.2. Consider the linear constant coefficient differential equation of fourth order without delay:

$$x^{(4)} + a_1 x''' + a_2 x'' + a_3 x' + a_4 x = 0.$$

It is well known that a necessary and sufficient condition that all solutions of this equation tend to the null solution, $x = 0$, as $t \to \infty$ is the Routh-Hurwitz criterion:

$$a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0.$$

It can be seen that the conditions of Theorem 3.1 imposed on the given nonlinear functions f_1, f_2 , g and h are based on the usual generalized Routh-Hurwitz conditions.

Moreover, in view of the above discussion, one can see that in the relative works of Sinha [5] and Tunç [9] the function f_2 does not include any delay argument. In this paper, however, we proceed to this case, which is important from the theoretical point of view. Since we use the Lyapunov functional approach based on the Routh-Hurwitz criterion, our conditions are compatible with these conditions.

Proof of Theorem 3.1. To verify Theorem 3.1, we define the following Lyapunov functional:

$$2V(x_t, y_t, z_t, u_t) = \frac{2\delta}{g^0}\int_0^x h(\xi)d\xi + \left[\frac{f_2^0\delta}{g^0} - \frac{h'(0)}{a}\right]y^2 + \left[\frac{f_2^0}{a} - \frac{\delta}{g^0}\right]z^2 + \frac{1}{a}u^2 + 2uz$$

$$+\frac{2\delta}{g^0}yu+\frac{2\delta}{g^0}y\int_0^z f_1(\tau)d\tau+2h(x)y+\frac{2}{a}h(x)z+\frac{2}{a}g(y)z$$

$$+2\int_0^y g(\eta)d\eta+\frac{2}{a}\int_0^z \tau\{f_2(\tau)-f_2^0\}d\tau+\frac{2\delta}{g^0}\int_0^z \tau\{f_2(\tau)-f_2^0\}d\tau$$

$$+2\int_0^z \tau\ f_1(\tau)d\tau+2\lambda_1\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds+2\lambda_2\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds$$

$$+2\lambda_3\int_{-r}^0\int_{t+s}^t u^2(\theta)d\theta ds, \tag{7}$$

where λ_1, λ_2 and λ_3 are some positive constants which will be determined later in the proof.

Let

$$F(z)=\int_0^z f_1(\tau)d\tau.$$

Then, the functional $2V(x_t,y_t,z_t,u_t)$ is rearranged as follows

$$2V(x_t,y_t,z_t,u_t)=\frac{z}{F(z)}\left[u+F(z)+\frac{\delta}{g^0}\frac{F(z)}{z}y\right]^2+\left[\frac{1}{a}-\frac{z}{F(z)}\right]u^2$$

$$+\frac{2}{a}\int_0^z \tau\{f_2(\tau)-f_2^0\}d\tau+\frac{2\delta}{g^0}\int_0^z \tau\{f_2(\tau)-f_2^0\}d\tau$$

$$+\frac{2\delta}{g^0}\int_0^x \xi\left[\frac{h(\xi)}{\xi}-h^0\right]d\xi+\frac{h^0\delta}{g^0}\left[x+\frac{g^0}{h^0\delta}\frac{h(x)y}{x}+\frac{g^0}{ah^0\delta}\frac{h(x)z}{x}\right]^2$$

$$+2\int_0^y\left[\frac{g(\eta)}{\eta}-\frac{g^0}{h^0\delta}\left(\frac{h(x)}{x}\right)^2\right]\eta d\eta+\left[\frac{f_2^0\delta}{g^0}-\frac{\delta^2}{(g^0)^2}\frac{F(z)}{z}-\frac{h'(0)}{a}\right]y^2$$

$$+\left[\frac{f_2^0}{a}-\frac{\delta}{g^0}-\frac{g^0}{\delta h^0a^2}\left(\frac{h(x)}{x}\right)^2\right]z^2+2\int_0^z \tau\ f_1(\tau)d\tau-zF(z)$$

$$+\frac{2}{a}\left[\frac{g(y)}{y}-\frac{g^0}{\delta\ h^0}\left(\frac{h(x)}{x}\right)^2\right]yz+2\lambda_1\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds$$

$$+2\lambda_2\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds+2\lambda_3\int_{-r}^0\int_{t+s}^t u^2(\theta)d\theta ds.$$

The assumption $f_1(z)\geq f_1^0\geq a+2\lambda$ implies that

$$\left[\frac{1}{a}-\frac{z}{F(z)}\right]u^2 \geq \left(\frac{1}{a}-\frac{1}{a+2\lambda}\right)u^2.$$

Similarly, the assumptions $\sqrt{h^0\delta} \geq \frac{h(x)}{x} > 0$ $(x \neq 0)$ and $\frac{g(y)}{y} \geq g^0 > 0$ $(y \neq 0)$ yield

$$2\int_0^y \left[\frac{g(\eta)}{\eta}-\frac{g^0}{h^0\delta}\left(\frac{h(x)}{x}\right)^2\right]\eta d\eta \geq 2\int_0^y \left[\frac{g(\eta)}{\eta}-g^0\right]\eta d\eta \geq 0.$$

By using the assumptions $0 < \frac{F(z)}{z} < b$ and $h^0h'(0) \leq \delta^2$, it follows that

$$\begin{aligned}\left[\frac{f_2^0\delta}{g^0}-\frac{\delta^2}{(g^0)^2}\frac{F(z)}{z}-\frac{h'(0)}{a}\right]y^2 &\geq \left[\frac{f_2^0\delta}{g^0}-\frac{\delta^2 b}{(g^0)^2}-\frac{\delta}{a}\right]y^2\\ &= \frac{\delta}{a(g^0)^2h^0}[af_2^0g^0h^0-\delta abh^0-\delta(g^0)^2]y^2\\ &\geq \frac{\delta}{a(g^0)^2h^0}[af_2^0g^0h^0-\delta abh^0-\delta\delta_1^2]y^2\\ &= \left(\frac{\delta\beta}{a(g^0)^2h^0}\right)y^2\end{aligned}$$

and

$$\begin{aligned}\left[\frac{f_2^0}{a}-\frac{\delta}{g^0}-\frac{g^0}{\delta h^0a^2}\left(\frac{h(x)}{x}\right)^2\right]z^2 &\geq \left[\frac{f_2^0}{a}-\frac{\delta}{g^0}-\frac{g^0}{a^2}\right]z^2\\ &= \frac{1}{a^2g^0h^0}[af_2^0g^0h^0-a^2\delta\, h^0-(g^0)^2h^0]z^2\\ &\geq \frac{1}{a^2g^0h^0}[af_2^0g^0h^0-a^2\delta\, h^0-\delta\delta_1^2]z^2\\ &= \left(\frac{\beta}{a^2g^0h^0}\right)z^2.\end{aligned}$$

Performing partial integration gives

$$\int_0^z \tau\, f_1(\tau)d\tau = z\int_0^z f_1(\tau)d\tau - \int_0^z\left(\int_0^\tau f_1(\tau)d\tau\right)d\tau = z\int_0^z f_1(\tau)d\tau - \int_0^z F(\tau)d\tau.$$

By using the assumption

$$\frac{1}{z}\int_0^z f_1(s)ds - f_1(z) < \frac{\beta}{a^2g^0h^0},$$

we get

$$2\int_0^z \tau\, f_1(\tau)d\tau - zF(z) = \int_0^z \tau\, f_1(\tau)d\tau - \int_0^z F(\tau)d\tau$$
$$= -\int_0^z \left(\frac{F(\tau)}{\tau} - f_1(\tau)\right)\tau d\tau$$
$$> -\int_0^z \frac{\beta}{a^2 g^0 h^0}\tau d\tau = -\left(\frac{\beta}{2a^2 g^0 h^0}\right) z^2.$$

The assumption $f_2(z) \geq f_2^0 > 0$ implies that

$$\frac{2}{a}\int_0^z \tau\{f_2(\tau) - f_2^0\}d\tau + \frac{2\delta}{g^0}\int_0^z \tau\{f_2(\tau) - f_2^0\}d\tau \geq 0.$$

Thus, subject to the foregoing discussion, we arrive at

$$2V(x_t, y_t, z_t, u_t) \geq \frac{z}{F(z)}\left[u + F(z) + \frac{\delta}{g^0}\frac{F(z)}{z}y\right]^2$$
$$+ \frac{h^0\delta}{g^0}\left[x + \frac{g^0}{h^0\delta}\frac{h(x)y}{x} + \frac{g^0}{ah^0\delta}\frac{h(x)z}{x}\right]^2$$
$$+ \left(\frac{\delta\beta}{a(g^0)^2h^0}\right)y^2 + \left(\frac{\beta}{2a^2g^0h^0}\right)z^2 + \left(\frac{1}{a} - \frac{1}{a+2\lambda}\right)u^2$$
$$+ \frac{2}{a}\left[\frac{g(y)}{y} - \frac{g^0}{\delta\, h^0}\left(\frac{h(x)}{x}\right)^2\right]yz + 2\lambda_1\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds$$
$$+ 2\lambda_2\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds + 2\lambda_3\int_{-r}^0\int_{t+s}^t u^2(\theta)d\theta ds. \tag{8}$$

Let

$$\phi(x, y) = \left[\frac{g(y)}{y} - \frac{g^0}{\delta\, h^0}\left(\frac{h(x)}{x}\right)^2\right].$$

Hence,

$$\phi(x, y) \leq \left(\delta_1 - \frac{g^0h^0}{\delta}\right) = g^0\left(\frac{\delta_1}{g^0} - \frac{h^0}{\delta}\right)$$

by $\frac{g(y)}{y} \geq g^0 > 0$, $(y \neq 0)$ and $\frac{h(x)}{x} \geq h^0 > 0$, $(x \neq 0)$.

We now consider the terms

$$W = \left(\frac{\beta}{2a^2g^0h^0}\right)z^2 + \frac{2}{a}\phi(x, y)yz + \left(\frac{\delta\beta}{a(g^0)^2h^0}\right)y^2,$$

which are included in (8).

In view of the assumptions of Theorem 3.1, we have

$$\begin{aligned}
W &= \frac{\beta}{a}\left[\frac{y}{g^0}+\frac{g^0}{\beta}\phi(x,y)z\right]^2+\left[\frac{\beta}{a(g^0)^2}\left(\frac{\delta}{h^0}-1\right)\right]y^2 \\
&\quad+\left(\frac{\beta}{2a^2g^0h^0}\right)z^2-\frac{(g^0)^2}{a\beta}\phi^2(x,y)z^2 \\
&\geq \left[\frac{\beta}{a(g^0)^2}\left(\frac{\delta}{h^0}-1\right)\right]y^2+\left(\frac{\beta}{2a^2g^0h^0}\right)z^2-\frac{(g^0)^4}{a\beta}\left(\frac{\delta_1}{g^0}-\frac{h^0}{\delta}\right)^2z^2 \\
&= \left[\frac{\beta}{a(g^0)^2}\left(\frac{\delta}{h^0}-1\right)\right]y^2+\left[\frac{\beta}{2a^2g^0h^0}-\frac{(g^0)^4}{a\beta}\left(\frac{\delta_1}{g^0}-\frac{h^0}{\delta}\right)^2\right]z^2 \\
&= \left[\frac{\beta}{a(g^0)^2}\left(\frac{\delta}{h^0}-1\right)\right]y^2+\delta_3z^2,
\end{aligned}$$

where

$$\delta_3=\frac{\beta}{2a^2g^0h^0}-\frac{(g^0)^4}{a\beta}\left(\frac{\delta_1}{g^0}-\frac{h^0}{\delta}\right)^2>0.$$

As a result, it follows from (8) that

$$\begin{aligned}
2V &\geq \frac{z}{F(z)}\left[u+F(z)+\frac{\delta}{g^0}\frac{F(z)}{z}y\right]^2+\frac{h^0\delta}{g^0}\left[x+\frac{g^0}{h^0\delta}\frac{h(x)y}{x}+\frac{g^0}{ah^0\delta}\frac{h(x)z}{x}\right]^2 \\
&\quad+\left[\frac{\beta}{a(g^0)^2}\left(\frac{\delta}{h^0}-1\right)\right]y^2+\delta_3z^2+\left(\frac{1}{a}-\frac{1}{a+2\lambda}\right)u^2 \\
&\quad+2\lambda_1\int_{-r}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds+2\lambda_2\int_{-r}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds+2\lambda_3\int_{-r}^{0}\int_{t+s}^{t}u^2(\theta)d\theta ds.
\end{aligned}$$

In view of the assumptions of Theorem 3.1, one can easily obtain for some positive constants D_i , $(i=1,\ 2,\ 3,\ 4)$, that

$$\begin{aligned}
2V &\geq D_1x^2+D_2y^2+D_3z^2+D_4u^2+2\lambda_1\int_{-r}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds \\
&\quad+2\lambda_2\int_{-r}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds+2\lambda_3\int_{-r}^{0}\int_{t+s}^{t}u^2(\theta)d\theta ds,
\end{aligned}$$

and hence

$$V\geq D_5(x^2+y^2+z^2+u^2), \tag{9}$$

where $D_5=2^{-1}\min\{D_1,D_2,D_3,D_4\}$.

Thus, the existence of a continuous function $u(s) \geq 0$ with $u(|\phi(0)|) \geq 0$, which satisfies the inequality $u(|\phi(0)|) \leq V(\phi)$, can be easily verified, since the integrals $\int\limits_{-r}^{0} \int\limits_{t+s}^{t} y^2(\theta)d\theta ds$, $\int\limits_{-r}^{0} \int\limits_{t+s}^{t} z^2(\theta)d\theta ds$ and $\int\limits_{-r}^{0} \int\limits_{t+s}^{t} u^2(\theta)d\theta ds$ are non-negative.

By a straightforward calculation from (7) and (5), we have

$$\begin{aligned}
\frac{d}{dt}V(x_t, y_t, z_t, u_t) = &- \left(\frac{\delta}{g^0}\frac{g(y)}{y} - h'(x)\right) y^2 - \left(f_2(z) - \frac{1}{a}g'(y) - \frac{\delta}{g^0}\frac{F(z)}{z}\right) z^2 \\
&- \frac{1}{a}(f_1(z) - a)u^2 - \frac{1}{a}\left[h'(0) - h'(x)\right] yz \\
&+ \left(\frac{\delta}{g^0}y + z + \frac{1}{a}u\right) \int\limits_{t-r}^{t} f_2'(z(s))u(s)ds \\
&+ \left(\frac{\delta}{g^0}y + z + \frac{1}{a}u\right) \int\limits_{t-r}^{t} g'(y(s))z(s)ds \\
&+ \left(\frac{\delta}{g^0}y + z + \frac{1}{a}u\right) \int\limits_{t-r}^{t} h'(x(s))y(s)ds + \lambda_1 y^2 r + \lambda_2 z^2 r \\
&+ \lambda_3 u^2 r - \lambda_1 \int\limits_{t-r}^{t} y^2(s)ds - \lambda_2 \int\limits_{t-r}^{t} z^2(s)ds - \lambda_3 \int\limits_{t-r}^{t} u^2(s)ds.
\end{aligned}$$

By using the assumptions of the theorem and the mean value theorem (for derivative), we have the estimates:

$$\begin{aligned}
-\left(\frac{\delta}{g^0}\frac{g(y)}{y} - h'(x)\right) y^2 &\leq - \left[\delta - h'(0)\right] y^2 = -\delta_2 y^2, \\
\delta_2 &= \delta - h'(0) > 0, \\
-\left(f_2(z) - \frac{1}{a}g'(y) - \frac{\delta}{g^0}\frac{F(z)}{z}\right) z^2 &\leq - \left\{f_2^0 - \frac{1}{a}g'(y) - \frac{\delta b}{g^0} - \frac{\left[h'(0) - h'(x)\right]}{4a^2}\right\} z^2 \\
&\leq -\frac{1}{ag^0h^0}\left[af_2^0 g^0 h^0 - ab\delta h^0 - \delta_1^2 \delta\right] z^2 + \frac{\gamma_1}{4a^2} z^2 \\
&= -\left(\frac{\beta}{ag^0h^0} - \frac{\gamma_1}{4a^2}\right) z^2 = -\gamma\, z^2, (\gamma > 0),
\end{aligned}$$

since

$$h'(0) > 0,\ h'(0) \geq h'(x) \geq h'(0) - \gamma_1,\ \gamma_1 > 0,$$
$$\gamma = \left(\frac{\beta}{ag^0h^0} - \frac{\gamma_1}{4a^2}\right), \quad \gamma_1 < \frac{4a\beta}{g^0h^0}, \quad -\frac{1}{a}(f_1(z) - a)u^2 \leq -\frac{2\lambda}{a}u^2.$$

Now, making use of the assumptions $0 < f_2'(z) \leq \delta_0$, $0 < g'(y) \leq \delta_1$, $0 < h'(x) \leq \delta$ and the elementary inequality $2\,|st| \leq s^2 + t^2$, we arrive at the estimates:

$$\left(\frac{\delta}{g^0}y+z+\frac{1}{a}u\right)\int\limits_{t-r}^{t} f_2'(z(s))u(s)ds$$

$$\leq \frac{\delta\delta_0}{2g^0}ry^2+\frac{\delta_0}{2}rz^2+\frac{\delta_0}{2a}ru^2+\frac{\delta_0}{2}\left(1+\frac{\delta}{g^0}+\frac{1}{a}\right)\int\limits_{t-r}^{t} u^2(s)ds,$$

$$\left(\frac{\delta}{g^0}y+z+\frac{1}{a}u\right)\int\limits_{t-r}^{t} g'(y(s))z(s)ds$$

$$\leq \frac{\delta\ \delta_1}{2g^0}ry^2+\frac{\delta_1}{2}rz^2+\frac{\delta_1}{2a}ru^2+\frac{\delta_1}{2}\left(1+\frac{\delta}{g^0}+\frac{1}{a}\right)\int\limits_{t-r}^{t} z^2(s)ds,$$

$$\left(\frac{\delta}{g^0}y+z+\frac{1}{a}u\right)\int\limits_{t-r}^{t} h'(x(s))y(s)ds$$

$$\leq \frac{\delta^2}{2g^0}ry^2+\frac{\delta}{2}rz^2+\frac{\delta}{2a}ru^2+\frac{\delta}{2}\left(1+\frac{\delta}{g^0}+\frac{1}{a}\right)\int\limits_{t-r}^{t} y^2(s)ds.$$

Subject to the above discussion, it follows that

$$\begin{aligned}\frac{dV}{dt}\leq &-\left[\delta_2-\left(\frac{\delta\ \delta_1+\delta\delta_0+\delta^2}{2g^0}+\lambda_1\right)r\right]y^2-\left[\gamma-\left(\frac{\delta_0+\delta_1+\delta}{2}+\lambda_2\right)r\right]z^2\\ &-\left[\frac{2\lambda}{a}-\left(\frac{\delta_0+\delta_1+\delta}{2a}\right)r\right]u^2+\left[\frac{\delta}{2}\left(1+\frac{\delta}{g^0}+\frac{1}{a}\right)-\lambda_1\right]\int\limits_{t-r}^{t} y^2(s)ds\\ &+\left[\frac{\delta_1}{2}\left(1+\frac{\delta}{g^0}+\frac{1}{a}\right)-\lambda_2\right]\int\limits_{t-r}^{t} z^2(s)ds\\ &+\left[\frac{\delta_0}{2}\left(1+\frac{\delta}{g^0}+\frac{1}{a}\right)-\lambda_3\right]\int\limits_{t-r}^{t} u^2(s)ds.\end{aligned}$$

Let

$$\lambda_1=\frac{\delta}{2}\left(1+\frac{\delta}{g^0}+\frac{1}{a}\right),\quad \lambda_2=\frac{\delta_1}{2}\left(1+\frac{\delta}{g^0}+\frac{1}{a}\right),\quad \lambda_3=\frac{\delta_0}{2}\left(1+\frac{\delta}{g^0}+\frac{1}{a}\right).$$

Then, we have

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,u_t)\leq &-\left[\delta_2-\left(\frac{\delta\ \delta_1+\delta\delta_0+\delta^2}{2g^0}+\lambda_1\right)r\right]y^2\\ &-\left[\gamma-\left(\frac{\delta_0+\delta_1+\delta}{2}+\lambda_2\right)r\right]z^2\end{aligned}$$

$$-\left[\frac{2\lambda}{a}-\left(\frac{\delta_0+\delta_1+\delta}{2a}\right)r\right]u^2.$$

From the foregoing inequality, one can obtain

$$\frac{d}{dt}V(x_t,y_t,z_t,u_t)\le -k(y^2+z^2+u^2)$$

for some constant $k>0$ provided that

$$r<\min\left\{\frac{2g^0\delta_2}{\delta(\delta_0+\delta_1+\delta)+2g^0\lambda_1},\ \frac{2\gamma}{\delta_0+\delta_1+\delta+2\lambda_2},\ \frac{4\lambda}{\delta_0+\delta_1+\delta}\right\}.$$

At the end, it is also clear that the largest invariant set in Z is $Q=\{0\}$, where $Z=\{\phi\in C_H:\dot V(\phi)=0\}$. That is, the only solution of equation (4) for which $\frac{d}{dt}V(x_t,y_t,z_t,u_t)=0$ is the solution $x\equiv 0$. Thus, in view of the above discussion and Lemma, we conclude that the null solution of equation (4) is asymptotically stable. The proof of Theorem 3.1 is now complete. ■

Let us denote by $L^1(0,\infty)$ the space of Lebesgue integrable functions.

For the case, $p(t,x,x(t-r),y,y(t-r),z,z(t-r),u)\ne 0$, our last result is given by the following theorem.

Theorem 3.3. *Let us assume that all the assumptions of Theorem 3.1 hold. In addition, we assume that*

$$|p(t,x,x(t-r),y,y(t-r),z,z(t-r),u)|\le q(t),$$

where $q\in L^1(0,\infty)$*. Then, there exists a positive constant* K *such that the solution* $x(t)$ *of equation* (4) *satisfies*

$$|x(t)|\le\sqrt{K},|x'(t)|\le\sqrt{K},|x''(t)|\le\sqrt{K},|x'''(t)|\le\sqrt{K}$$

for all $t\ge t_0$*, where* $\phi\in C^3([t_0-r,t_0],\ \Re)$*, provided that*

$$r<\min\left\{\frac{2g^0\delta_2}{\delta(\delta_0+\delta_1+\delta)+2g^0\lambda_1},\ \frac{2\gamma}{\delta_0+\delta_1+\delta+2\lambda_2},\ \frac{4\lambda}{\delta_0+\delta_1+\delta}\right\}.$$

Proof. Subject to the assumptions of Theorem 3.3, the result of Theorem 3.1 can be revised as follows:

$$\frac{d}{dt}V(x_t,y_t,z_t,u_t)$$
$$\le -k(y^2+z^2+u^2)+\left(\frac{\delta}{g^0}y+z+\frac{1}{a}u\right)p(t,x,x(t-r),y,y(t-r),z,z(t-r),u).$$

Hence, we have

$$\frac{d}{dt}V(x_t,y_t,z_t,u_t)\le\left|\frac{\delta}{g^0}y+z+\frac{1}{a}u\right||p(t,x,x(t-r),y,y(t-r),z,z(t-r),u)|$$

$$\leq \left(\frac{\delta}{g^0} |y| + |z| + \frac{1}{a} |u| \right) q(t)$$
$$\leq D_6(|u| + |z| + |y|)\, q(t)$$
$$\leq D_6(3 + y^2 + z^2 + u^2)\, q(t), \tag{10}$$

where $D_6 = \max\left\{1,\ \frac{\delta}{g^0},\ \frac{1}{a}\right\}$.
It is also clear from (9) that

$$y^2 + z^2 + u^2 \leq x^2 + y^2 + z^2 + u^2 \leq D_5^{-1}\, V(x_t, y_t, z_t, u_t).$$

Using the last inequality, from (10) we get

$$\frac{d}{dt}V(x_t, y_t, z_t, u_t) \leq D_6\{3 + D_5^{-1}V(x_t, y_t, z_t, u_t)\}\, q(t)$$
$$= 3D_6 q(t) + D_6 D_5^{-1} V(x_t, y_t, z_t, u_t) q(t).$$

Integrating the above estimate from 0 to t , using assumption $q \in L^1(0, \infty)$, integrability nature of q and Gronwall-Reid-Bellman inequality (see Ahmad and Rao [1]), we obtain

$$V(x_t, y_t, z_t, u_t) \leq V(x_0, y_0, z_0, u_0) + 3D_6 A + D_6 D_5^{-1} \int_0^t V(x_s, y_s, z_s, u_s)\, q(s)ds$$
$$\leq \{V(x_0, y_0, z_0, u_0) + 3D_6 A\} \exp\left(D_6 D_5^{-1} \int_0^t q(s)ds \right)$$
$$\leq \{V(x_0, y_0, z_0, u_0) + 3D_6 A\} \exp(D_6 D_5^{-1} A) = K_1 < \infty, \tag{11}$$

where $K_1 > 0$ is a constant, $K_1 = \{V(x_0, y_0, z_0, u_0) + 3D_6 A\} \exp(D_6 D_5^{-1} A) < \infty$, and $A = \int_0^\infty q(s)ds$. As a result, the inequalities (9) and (11) together imply that

$$x^2 + y^2 + z^2 + u^2 \leq D_5^{-1} V(x_t, y_t, z_t, u_t) \leq K,$$

where $K = K_1 D_5^{-1}$. Thus, one can conclude that

$$|x(t)| \leq \sqrt{K}, |y(t)| \leq \sqrt{K}, |z(t)| \leq \sqrt{K}, |u(t)| \leq \sqrt{K}$$

for all $t \geq t_0$.
Hence

$$|x(t)| \leq \sqrt{K}, |x'(t)| \leq \sqrt{K}, |x''(t)| \leq \sqrt{K}, |x'''(t)| \leq \sqrt{K}$$

for all $t \geq t_0$.

The proof of Theorem 3.3 is complete. ■

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