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Solvability of a System of Dual Integral Equations involving Fourier Transforms

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Abstract. The aim of the present work is to consider the solvability and the solution method of a system of dual integral equations involving Fourier transforms occurring in mixed boundary value problems for an elastic strip. The uniqueness and existence theorems are proved. A method for reducing the system of dual integral equations to infinite systems of linear algebraic equations is proposed.

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1. Introduction

Dual integral equations arise when integral transforms are used to solve mixed boundary value problems of mathematical physics and mechanics. Formal techniques for solving such equations have been extensively developed, but the solvability of problems have been relatively fewer considered [3, 10, 13]. The solvability of dual integral equations involving Fourier transforms and dual series equations involving orthogonal expansions of generalized functions were considered in [4-7].

The solvability for systems of dual equations obtained from some mixed boundary value problems for harmonic equations in a strip was considered in [8]. The aim of the present work is to consider the solvability and the solution method of a system of dual integral equations involving Fourier transforms encountered in a mixed boundary value problem for an elastic strip. The uniqueness and existence theorems are proved. A method for reducing this systems of dual equations to an infinite system of linear algebraic equations is also proposed.

There is a considerable literature devoted to mixed problems and contact problems of the indentation of punch (stamp) in an elastic strip without friction and adhesion or with friction or adhesion between the punch and strip (see for example, [9-12]). These problems can be reduced to solving dual integral equations or systems of dual integral equations involving Fourier transforms.

2. Formulation of the Problem

The equilibrium equations for an isotropic elastic medium indentically satisfied if displacements $u(x,y), v(x,y)$ and pressures $\sigma_y(x,y), \tau_{xy}(x,y)$ are expressed in terms of the harmonic functions $\Phi(x,y)$ and $\Psi(x,y)$ by the formulas [14]

$$\begin{aligned}
&2\mu u = -\Phi_x - y\Psi_x,\\
&2\mu v = (3-4\nu)\Psi - \Phi_y - y\Psi_y,\\
&\sigma_y = 2(1-\nu)\Psi_y + \Phi_{xx} + y\Psi_{xx},\\
&\tau_{xy} = (1-2\nu)\Psi_x - \Phi_{xy} - y\Psi_{xy},
\end{aligned}$$

where μ and ν are the shear modulus and Poisson ratio, respectively ($\mu > 0,\ 0 < \nu < 1/2$).

Consider a mixed boundary value problem of an elastic strip. Mathematical formulation of the problem states as follows: let us consider harmonic equations

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \quad \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0, \quad (-\infty < x < \infty,\ 0 < y < h) \tag{1}$$

subject to the boundary conditions

$$\tau_{xy}(x,h) = \tau_0(x), \quad \sigma_y(x,h) = \sigma_0(x), \quad -\infty < x < \infty, \tag{2}$$

$$\begin{cases} \tau_{xy}(x,0) = \sigma_y(x,0) = 0, & x \in (a,b),\\ u(x,0) = v(x,0) = 0, & x \in \mathbb{R}\setminus(a,b). \end{cases} \tag{3}$$

We shall solve problem (1)-(3) by the method of the Fourier transformation and reduce it to a system of dual equations involving the inverse Fourier transforms. For a suitable function $f(x), x \in \mathbb{R}$ (for example, $f(x) \in L^1(\mathbb{R})$), direct and inverse Fourier transforms are defined by the formulas

$$\hat{f}(\xi) = F[f](\xi) = \int_{-\infty}^{\infty} f(x)e^{ix\xi}dx, \tag{4}$$

$$\breve{f}(\xi) = F^{-1}[f](\xi) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} f(x) e^{-ix\xi} dx. \tag{5}$$

The Fourier transforms of tempered generalized functions can be found, for example, in [15, 16].

By applying the Fourier transform with respect to x to the harmonic equations (1), we obtain

$$\frac{d^2\widehat{\Phi}(\xi, y)}{dy^2} - \xi^2 \widehat{\Phi}(\xi, y) = 0, \quad \frac{d^2\widehat{\Psi}(\xi, y)}{dy^2} - \xi^2 \widehat{\Psi}(\xi, y) = 0, \tag{6}$$

$$(-\infty < \xi < \infty, \quad 0 < y < h),$$

where $\widehat{\Phi}(\xi, y) = F_x[\Phi(x, y)](\xi)$ and $\widehat{\Psi}(\xi, y) = F_x[\Psi(x, y)](\xi)$ are the Fourier transforms with respect to x of the function $\Phi(x, y)$ and $\Psi(x, y)$ understood in the sense of generalized functions [15]. The general solutions of the differential equations (6) are

$$\widehat{\Phi}(\xi, y) = A(\xi) \cosh(|\xi| y) + B(\xi) \sinh(|\xi| y),$$
$$\widehat{\Psi}(\xi, y) = C(\xi) \cosh(|\xi| y) + D(\xi) \sinh(|\xi| y),$$

where $A(\xi)$, $B(\xi)$,$C(\xi)$ and $D(\xi)$ are arbitrary functions. We have

$$\widehat{\Phi}_y(\xi, y) = |\xi| A(\xi) \sinh(|\xi| y) + |\xi| B(\xi) \cosh(|\xi| y),$$
$$\widehat{\Psi}_y(\xi, y) = |\xi| C(\xi) \sinh(|\xi| y) + |\xi| D(\xi) \cosh(|\xi| y),$$
$$2\mu \widehat{u}(\xi, y) = i\xi \widehat{\Phi} + i\xi y \widehat{\Psi},$$
$$2\mu \widehat{v}(\xi, y) = (3 - 4\nu) \widehat{\Psi} - \widehat{\Phi}_y - y \widehat{\Psi}_y,$$
$$\widehat{\sigma}_y(\xi, y) = 2(1 - \nu) \widehat{\Psi}_y - \xi^2 \widehat{\Phi} - y \xi^2 \widehat{\Psi},$$
$$\widehat{\tau}_{xy}(\xi, y) = (1 - 2\nu)(-i\xi) \widehat{\Psi} + i\xi \widehat{\Phi}_y + i\xi y \widehat{\Psi}_y.$$

We introduce the auxiliary functions

$$\widehat{u}_1(\xi) = 2\mu \widehat{u}(\xi, 0), \quad \widehat{u}_2(\xi) = 2\mu \widehat{v}(\xi, 0).$$

The problem (1)-(3) is reduced to the system of dual integral equations

$$\begin{cases} F^{-1}[|\xi| \mathbf{A}_0(\xi) \widehat{\mathbf{u}}(\xi)](x) = \mathbf{f}(x), \quad x \in \Omega := (a, b), \\ F^{-1}[\hat{\mathbf{u}}(\xi)](x) = \mathbf{0}, \quad x \in \Omega' = \mathbb{R} \setminus \Omega, \end{cases} \tag{7}$$

where

$$u_1(x) = 2\mu u(x, 0), \quad u_2(x) = 2\mu v(x, 0),$$
$$\hat{\mathbf{u}}(\xi) = F[\mathbf{u}(x)](\xi), \quad \mathbf{u}(x) = (u_1(x), u_2(x))^T,$$
$$\mathbf{f}(x) = (f_1(x), f_2(x))^T,$$

$$f_1(x) = F^{-1}\Big[-\frac{\widehat{\tau}_0(\xi)2(1-\nu)\{-2(1-\nu)\cosh(|\xi|h)+|\xi|h\sinh(|\xi|h)\}}{|\xi|^2h^2+4(1-\nu)^2+(3-4\nu).\sinh^2(|\xi|h)}$$
$$-\frac{\widehat{\sigma}_0(\xi)\cdot i\xi\{|\xi|h\cosh(|\xi|h)-(1-2\nu)\sinh(|\xi|h)\}}{|\xi|\{|\xi|^2h^2+4(1-\nu)^2+(3-4\nu).\sinh^2(|\xi|h)\}}\Big](x), \tag{8}$$
$$f_2(x) = F^{-1}\Big[\frac{\widehat{\tau}_0(\xi)2(1-\nu)|\xi|\{(1-2\nu)\sinh(|\xi|h)+|\xi|h\cosh(|\xi|h)\}}{i\xi\{|\xi|^2h^2+4(1-\nu)^2+(3-4\nu).\sinh^2(|\xi|h)\}} \tag{9}$$
$$+\frac{\widehat{\sigma}_0(\xi)2(1-\nu)\{|\xi|h\sinh(|\xi|h)+2(1-\nu)\cosh(|\xi|h)\}}{|\xi|^2h^2+4(1-\nu)^2+(3-4\nu)\sinh^2(|\xi|h)}\Big](x), \tag{10}$$

$$\widehat{\sigma}_0(\xi) = F[\sigma_0](\xi), \quad \widehat{\tau}_0(\xi) = F[\tau_0](\xi),$$
$$\mathbf{A}_0(\xi) = \begin{pmatrix} a_{11}(\xi) & i\cdot\mathrm{sign}(\xi)a_{12}(\xi) \\ -i\cdot\mathrm{sign}(\xi)a_{21}(\xi) & a_{22}(\xi)\end{pmatrix},$$

with

$$a_{11}(\xi) = \frac{2(1-\nu)[\cosh(|\xi|h)\sinh(|\xi|h)+|\xi|h]}{4(1-\nu)^2+|\xi|^2h^2+(3-4\nu)\sinh^2(|\xi|h)},$$
$$a_{21}(\xi) = a_{12}(\xi) = \frac{(1-2\nu)\sinh^2(|\xi|h)+|\xi|^2h^2}{4(1-\nu)^2+|\xi|^2h^2+(3-4\nu)\sinh^2(|\xi|h)},$$
$$a_{22}(\xi) = \frac{2(1-\nu)[\cosh(|\xi|h)\sinh(|\xi|h)-|\xi|h]}{4(1-\nu)^2+|\xi|^2h^2+(3-4\nu)\sinh^2(|\xi|h)}.$$

3. Solvability of Systems of Dual Equations

3.1. Functional Spaces

Let $\mathbb{R}$ be a real axis, $\mathcal{S} = \mathcal{S}(\mathbb{R})$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ be the Schwartz spaces of basic and generalized functions, respectively [15, 16]. Denote by F and F^{-1} the Fourier transform and inverse Fourier transform defined on $\mathcal{S}'$. It is known that these operators are automorphisms on $\mathcal{S}'$. For a suitable ordinary function $f(x)$ (for example, $f \in L^1(\mathbb{R})$), the direct and inverse Fourier transforms defined by formulas (4) and (5) respectively. The symbol $< f, \varphi >$ denotes the value of the generalized function $f \in \mathcal{S}'$ on the basic function $\varphi \in \mathcal{S}$, besides, $(f, \varphi) := < f, \overline{\varphi} >$.

Definition 3.1. Let $H^s := H^s(\mathbb{R})$ $(s \in \mathbb{R})$ be the Sobolev-Slobodeskii space defined as the closure of the set $C_o^\infty(\mathbb{R})$ of infinitely differentiable functions with compact support with respect to the norm [15, 16]

$$||u||_s := \left[\int_{-\infty}^{\infty}(1+\xi^2)^s|\hat{u}(\xi)|^2d\xi\right]^{1/2} < \infty, \quad \hat{u} = F[u]. \tag{11}$$

The space H^s is a Hilbert space with the following scalar product

$$(u, v)_s := \int_{-\infty}^{\infty} (1 + \xi^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi. \tag{12}$$

Let $\Omega = (a, b)$ be a certain interval in $\mathbb{R}$. The subspace of $H^s(\mathbb{R})$ consisting of functions $u(x)$ with support in $\overline{\Omega}$ is denoted by $H^s_o(\Omega)$ [16], while the space of functions $v(x) = pu(x)$, where $u \in H^s(\mathbb{R})$ and p is the restriction operator to Ω is denoted by $H^s(\Omega)$. The norm in $H^s(\Omega)$ is defined by

$$||v||_{H_s(\Omega)} = \inf_l ||lv||_s,$$

where the infimum is taken over all possible extensions $lv \in H^s(\mathbb{R})$.

Let X be a linear topological space. We denote the direct product of two elements X by X^2. A topology in X^2 is given by the usual topology of the direct product. We shall use bold letters for denoting vector-values and matrices. Denote by $\mathbf{u}$ a vector of the form (u_1, u_2), and $\mathcal{S}^2 = \mathcal{S} \times \mathcal{S}$, $\quad (S')^2 = \mathcal{S}' \times \mathcal{S}'$. For the vectors $\mathbf{u} \in (\mathcal{S}')^2$, $\boldsymbol{\varphi} \in \mathcal{S}^2$ we set

$$< \mathbf{u}, \boldsymbol{\varphi} >= \sum_{j=1}^{2} < u_j, \varphi_j > .$$

The Fourier transform and inverse Fourier transform of a vector $\mathbf{u} \in (S')^2$ are the vectors $\hat{\mathbf{u}} = F^{\pm 1}[\mathbf{u}] = (F^{\pm 1}[u_1], F^{\pm 1}[u_2])$, defined by the equations [15]:

$$< F[\mathbf{u}], \boldsymbol{\varphi} >=< \mathbf{u}, F[\boldsymbol{\varphi}] >, \; < F^{-1}[\mathbf{u}], \boldsymbol{\varphi} >= \frac{1}{2\pi} < \mathbf{u}, F[\boldsymbol{\varphi}](-x) >, \; \boldsymbol{\varphi} \in \mathcal{S}^2. \tag{13}$$

Let H^{s_j}, $H^{s_j}_o(\Omega)$, $H^{s_j}(\Omega)$ be the Sobolev spaces, where $j = 1, 2$; Ω is a certain set of intervals in $\mathbb{R}$. We put $\vec{s} = (s_1, s_2)$ and

$$\mathbb{H}^{\vec{s}} = H^{s_1} \times H^{s_2}, \; \mathbb{H}^{\vec{s}}_o(\Omega) = H^{s_1}_o(\Omega) \times H^{s_2}_o(\Omega), \; \mathbb{H}^{\vec{s}}(\Omega) = H^{s_1}(\Omega) \times H^{s_2}(\Omega).$$

A scalar product and a norm in $\mathbb{H}^{\vec{s}}$ and $\mathbb{H}^{\vec{s}}_o(\Omega)$ are given by the formulas

$$(\mathbf{u}, \mathbf{v})_{\vec{s}} = \sum_{j=1}^{2} (u_j, v_j)_{s_j}, \; ||\mathbf{u}||_{\vec{s}} = \Big(\sum_{j=1}^{2} ||u_j||^2_{s_j} \Big)^{1/2},$$

where $||u_j||_{s_j}$ and $(u_j, v_j)_{s_j}$ are given by the formulas (11) and (12), respectively. A norm in $\mathbb{H}^{\vec{s}}(\Omega)$ is defined by the equality

$$||u||_{\mathbb{H}^{\vec{s}}(\Omega)} := \Big(\sum_{j=1}^{2} \inf_{l_j} ||l_j u_j||^2_{s_j} \Big)^{1/2},$$

where l_j are extension operators of the $u_j \in H^{s_j}(\Omega)$ from Ω to $\mathbb{R}$.

Theorem 3.2. *Let* $\Omega \subset \mathbb{R}, \mathbf{u} = (u_1, u_2) \in \mathbb{H}^{\vec{s}}(\Omega), \mathbf{f} \in \mathbb{H}^{-\vec{s}}(\Omega)$ *and* $\mathbf{lf} = (l_1 f_1, l_2 f_2)$ *be an extension of* $\mathbf{f}$ *from* Ω *to* $\mathbb{R}$ *belonging to* $\mathbb{H}^{-\vec{s}}(\mathbb{R})$*. Then the integrals*

$$[\mathbf{f}, \mathbf{u}] := \sum_{j=1}^{2} \int_{-\infty}^{\infty} \widehat{l_j f_j}(t) \overline{\widehat{u_j(t)}} dt \tag{14}$$

do not depend on the choice of the extension $\mathbf{lf}$*. Therefore, this formula defines a linear continuous functional on* $\mathbb{H}_o^{\vec{s}}(\Omega)$*. Conversely, for every linear continuous functional* $\Phi(\mathbf{u})$ *on* $\mathbb{H}_o^{\vec{s}}(\Omega)$ *there exists an element* $\mathbf{f} \in \mathbb{H}^{-\vec{s}}(\Omega)$ *such that* $\Phi(\mathbf{u}) = [\mathbf{u}, \mathbf{f}]$ *and* $||\Phi|| = ||\mathbf{f}||_{\mathbb{H}^{-\vec{s}}(\Omega)}$.

Proof. The proof is based on the fact, that the set $(C_o^\infty(\Omega))^2$ is dense in $\mathbb{H}_o^{\vec{s}}(\Omega), \vec{s} = (s_1, s_2)$, and on the Riesz theorem. ■

3.2. Pseudo-differential Operators

Consider pseudo-differential operators of the form

$$(A\mathbf{u})(x) := F^{-1}[\mathbf{A}(t)\widehat{\mathbf{u}}(t)](x),$$

where $\mathbf{A}(t) = ||a_{ij}(t)||_{2\times 2}$ is a square matrix of order two, $\mathbf{u} = (u_1, u_2)^T$ is a vector, transposed to the line vector (u_1, u_2), and $\widehat{\mathbf{u}}(t) := F[\mathbf{u}] = (F[u_1], F[u_2])^T$. We introduce the following classes.

Definition 3.3. Let $\alpha \in \mathbb{R}$. We say that the function $a(t)$ belongs to the class $\sigma^\alpha(\mathbb{R})$, if $|a(t)| \leqslant C_1(1 + |t|)^\alpha$, for all $t \in \mathbb{R}$, and belongs to the class $\sigma_+^\alpha(\mathbb{R})$, if $C_2(1 + |t|)^\alpha \leqslant a(t) \leqslant C_1(1 + |t|)^\alpha$, for all $t \in \mathbb{R}$, where C_1 and C_2 are certain positive constants.

Lemma 3.4. [5]. *Let* $a(t) > 0$ *be such that* $(1+|t|)^{-\alpha}a(t)$ *is a bounded continuous function on* $\mathbb{R}$*. Suppose moreover that there are positive limits of the function* $(1 + |t|)^{-\alpha}a(t)$ *when* $t \to \pm\infty$*. Then* $a(t) \in \sigma_+^\alpha(\mathbb{R})$.

Definition 3.5. Let $\mathbf{A}(t) = ||a_{ij}(t)||_{2\times 2},\ t \in \mathbb{R}$ be a square matrix of second order, where $a_{ij}(t)$ are continuous functions on $\mathbb{R}, \alpha_j \in \mathbb{R}, (j = 1, 2),\ \vec{\alpha} = (\alpha_1, \alpha_2)$. Denote by $\boldsymbol{\Sigma}^{\vec{\alpha}}(\mathbb{R})$ the class of square matrices $\mathbf{A}(t) = ||a_{ij}(t)||_{2\times 2}$, such that

$$a_{ii}(t) \in \sigma^{\alpha_i}(\mathbb{R}),\ \ a_{ij}(t) \in \sigma^{\alpha_{ij}}(\mathbb{R}),\ \ \alpha_{ij} \leqslant \frac{1}{2}(\alpha_i + \alpha_j).$$

We shall say that the matrix $\mathbf{A}(t)$ belongs to the class $\boldsymbol{\Sigma}_+^{\vec{\alpha}}(\mathbb{R})$, if $\mathbf{A}(t) \in \boldsymbol{\Sigma}^{\vec{\alpha}}(\mathbb{R})$ and it is Hermitian, i.e. $\overline{(\mathbf{A}(t))^T} = \mathbf{A}(t)$, and satisfies the condition:

$$\overline{\mathbf{w}^T}\mathbf{A}\mathbf{w} \geq C_1 \sum_{j=1}^{2} (1+|t|)^{\alpha_j} |w_j|^2, \ \forall \mathbf{w} = (w_1, w_2)^T \in \mathbb{C}^2,$$

where C_1 is a positive constant. Finally, we say that the matrix $\mathbf{A}(t) \in \boldsymbol{\Sigma}^{\vec{\alpha}}(\mathbb{R})$ belongs to the class $\boldsymbol{\Sigma}_o^{\vec{\alpha}}(\mathbb{R})$, if it is positive-definite for almost everywhere $t \in \mathbb{R}$.

Lemma 3.6. *Let the matrix $\mathbf{A}(t) = \mathbf{A}_+(t)$ belong to the class $\boldsymbol{\Sigma}_+^{\vec{\alpha}}(\mathbb{R})$. Then the scalar product and norm in $\mathbb{H}^{\vec{\alpha}/2}(\mathbb{R})$ can be defined by the formulas*

$$(\mathbf{u}, \mathbf{v})_{\mathbf{A}_+, \vec{\alpha}/2} = \int_{-\infty}^{\infty} \overline{F[\mathbf{v}^T](t)} \mathbf{A}_+(t) F[\mathbf{u}](t) dt, \tag{15}$$

$$||\mathbf{u}||_{\mathbf{A}_+, \vec{\alpha}/2} = \Big(\int_{-\infty}^{\infty} \overline{F[\mathbf{u}^T](t)} \mathbf{A}_+(t) F[\mathbf{u}](t) dt \Big)^{1/2}, \tag{16}$$

respectively.

Lemma 3.7. *Let $\mathbf{A}(t) \in \boldsymbol{\Sigma}^{\alpha}(\mathbb{R})$. Then the Fourier integral operator $A\mathbf{u}$ defined by the formula $F^{-1}[\mathbf{A}(t)\hat{\mathbf{u}}(t)](x)$ is bounded from $\mathbb{H}^{\vec{\alpha}/2}(\mathbb{R})$ into $\mathbb{H}^{-\vec{\alpha}/2}(\mathbb{R})$.*

Lemma 3.8. *Let Ω be a bounded subset of intervals in $\mathbb{R}$. Then the imbedding $\mathbb{H}^{\vec{s}}(\Omega)$ into $\mathbb{H}^{\vec{s}-\vec{\varepsilon}}(\Omega)$ is compact, where $\vec{\varepsilon} = (\varepsilon, \varepsilon) > \mathbf{0}$ if and only if $\varepsilon > 0$.*

Proof. The proof based on the fact that the imbedding $H^{s_j}(\Omega)$ into $H^{s_j-\varepsilon}(\Omega)$, $\varepsilon > 0$ is completely continuous if Ω is bounded in $\mathbb{R}$ (see, [16]). ■

3.3. Solvability of the System of Dual Equations (7)

System (7) can be rewritten in the form

$$\begin{cases} pF^{-1}[|\xi|\mathbf{A}_0(\xi)\hat{\mathbf{u}}(\xi)](x) = \mathbf{f}(x), & x \in \Omega, \\ p'F^{-1}[\hat{\mathbf{u}}(\xi)](x) = \mathbf{0}, & x \in \Omega' := \mathbb{R} \setminus \Omega, \end{cases} \tag{17}$$

where $\mathbf{f}(x) = (f_1(x), f_2(x))^T$, $\hat{\mathbf{u}}(\xi) = F[\mathbf{u}] = (\widehat{u}_1(\xi), \widehat{u}_2(\xi))^T$, the operator F^{-1} is understood in the generalized sense (13), and

$$\mathbf{A}_0(\xi) = \begin{pmatrix} a_{11}(\xi) & i \cdot \mathrm{sign}(\xi).a_{12}(\xi) \\ -i \cdot \mathrm{sign}(\xi).a_{21}(\xi) & a_{22}(\xi) \end{pmatrix},$$

$$a_{11}(\xi) = \frac{2(1-\nu)[\cosh(|\xi|h)\sinh(|\xi|h) + |\xi|h]}{4(1-\nu)^2 + |\xi|^2h^2 + (3-4\nu)\sinh^2(|\xi|h)},$$

$$a_{21}(\xi) = a_{12}(\xi) = \frac{(1-2\nu)\sinh^2(|\xi|h) + |\xi|^2h^2}{4(1-\nu)^2 + |\xi|^2h^2 + (3-4\nu)\sinh^2(|\xi|h)},$$

$$a_{22}(\xi) = \frac{2(1-\nu)[\cosh(|\xi|h)\sinh(|\xi|h) - |\xi|h]}{4(1-\nu)^2 + |\xi|^2h^2 + (3-4\nu)\sinh^2(|\xi|h)},$$

p and p' denote restriction operators to Ω and Ω', respectively.

It is clear that $|\xi|\mathbf{A}_0(\xi) \in \boldsymbol{\Sigma}^{\vec{\alpha}}, \vec{\alpha} = (1,1)$, besides, $\lim_{|\xi|\to\infty} a_{ij}(\xi) = \gamma_{ij}$, where

$$\gamma_{11} = \gamma_{22} = \frac{2(1-\nu)}{3-4\nu}, \quad \gamma_{12} = \gamma_{21} = \frac{1-2\nu}{3-4\nu}.$$

One can show that

$$a_{ij}(\xi) - \gamma_{ij} = 0(|\xi|^{-\infty}), \quad |\xi| \longrightarrow \infty.$$

Lemma 3.9. *The matrix $\mathbf{A}_0(\xi)$ is positve-definite for all $\xi \neq 0$.*

Proof. We have to prove that

$$a_{11}a_{22} - a_{12}a_{21} > 0, \ \forall \xi \neq 0.$$

It is equivalent to

$$\triangle = 4(1-\nu)^2[\cosh^2(|\xi|h).\sinh^2(|\xi|h) - |\xi|^2h^2] - [(1-2\nu)\sinh^2(|\xi|h) + |\xi|^2h^2]^2 > 0$$

for all $\xi \neq 0$.

Indeed, putting $t = |\xi|h$, we have

$$\begin{aligned}\triangle &= (4-8\nu+4\nu^2)(\sinh^4 t + \sinh^2 t - t^2) - (1-4\nu+4\nu^2)\sinh^4 t \\ &\quad - 2(1-2\nu)t^2\sinh^2 t - t^4 \\ &= (3-4\nu)\sinh^4 t + 4(1-\nu)^2(\sinh^2 t - t^2) - (2-4\nu)t^2\sinh^2 t - t^4 \\ &= (2-4\nu)(\sinh^4 t - t^2\sinh^2 t) + (\sinh^4 t - t^4) + 4(1-\nu)^2(\sinh^2 t - t^2) \\ &= (\sinh^2 t - t^2)[(2-4\nu)\sinh^2 t + \sinh^2 t + t^2 + 4(1-\nu)^2].\end{aligned}$$

It is clear that $\triangle > 0$ for all $t > 0$. The lemma is proved. ∎

According to Lemma 3.9, $\mathbf{A}(\xi) := |\xi|\mathbf{A}_0(\xi) \in \boldsymbol{\Sigma}_0^{\vec{\alpha}}, \ \vec{\alpha} = (1,1)$.

Theorem 3.10. (Uniqueness) *The system of dual equations* (17) *has at most one solution in the space* $\mathbb{H}_0^{\vec{\alpha}/2}(\Omega)$.

Proof. Let $\mathbf{u} \in \mathbb{H}_0^{\vec{\alpha}/2}(\Omega)$ be a solution of the homogeneous system of system (17). Using the formulas (14)-(16) and Lemma 3.7 we can show that

$$[A\mathbf{u}, \mathbf{u}] = \int_{-\infty}^{\infty} \overline{\widehat{\mathbf{u}}^{\mathbf{T}}(\xi)}\mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi)d\xi = 0,$$

from which follows $\mathbf{u} \equiv 0$. ∎

Denote

$$(A\mathbf{u})(x) = pF^{-1}[\mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi)](x) \tag{18}$$

and rewrite (17) in the form

$$(A\mathbf{u})(x) = \mathbf{f}(x), \quad x \in \Omega. \tag{19}$$

Our purpose now is to establish the existence of solution of system (19) in the space $\mathbb{H}_o^{\vec{\alpha}/2}(\Omega)$, $\vec{\alpha} = (1,1)^T$.

We introduce the matrices

$$\mathbf{A}_+(\xi) = |\xi| \coth(|\xi|h) \begin{pmatrix} \alpha & i\beta.\mathrm{sign}\xi \\ -i\beta.\mathrm{sign}\xi & \alpha \end{pmatrix},$$

$$\begin{aligned} \mathbf{B}(\xi) &= |\xi|\mathbf{A}_0(\xi) - \mathbf{A}_+(\xi) \\ &= |\xi| \begin{pmatrix} a_{11}(\xi) - \alpha\coth(|\xi|h) & i\cdot\mathrm{sign}\xi(a_{12}(\xi) - \beta\coth(|\xi|h)) \\ -i\cdot\mathrm{sign}\xi(a_{21}(\xi) - \beta\coth(|\xi|h)) & a_{22}(\xi) - \alpha\coth(|\xi|h) \end{pmatrix}, \end{aligned}$$

where

$$\alpha = \frac{2(1-\nu)}{3-4\nu}, \ \ \beta = \frac{1-2\nu}{3-4\nu}, \ \ \alpha - \beta = \frac{1}{3-4\nu}.$$

Theorem 3.11. *We have* $\mathbf{A}_+(\xi) \in \boldsymbol{\Sigma}_+^{\vec{\alpha}}$, $\vec{\alpha} = (1,1)$.

Proof. Let

$$\widehat{u}_1 = a_1 + ib_1, \quad \widehat{u}_2 = a_2 + ib_2, \quad a_1, b_1, a_2, b_2 \in \mathbb{R}.$$

We have

$$|\hat{u}_1|^2 = a_1^2 + b_1^2, \ |\hat{u}_2|^2 = a_2^2 + b_2^2.$$

It is not difficult to show that

$$\begin{aligned} \overline{\hat{\mathbf{u}}}^T \mathbf{A}_+ \hat{\mathbf{u}} &= \xi\coth(\xi h)[\alpha(a_1^2 + b_1^2 + a_2^2 + b_2^2) - 2\beta\mathrm{sign}(\xi)(a_1b_2 - a_2b_1)] \\ &\geq \xi\coth(\xi h)[\alpha(a_1^2 + b_1^2 + a_2^2 + b_2^2) - 2\beta\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}] \\ &\geq (\alpha - \beta)\xi\coth(\xi h)(a_1^2 + b_1^2 + a_2^2 + b_2^2) \\ &\Leftrightarrow \beta\xi\coth(\xi h)[|\hat{u}_1|^2 + |\hat{u}_2|^2 - 2|\hat{u}_1||\hat{u}_2|] \geq 0. \end{aligned}$$

Thus,

$$\overline{\hat{\mathbf{u}}}^T \mathbf{A}_+ \hat{\mathbf{u}} \geq (\alpha - \beta)\xi\coth(\xi h)(|\hat{u}_1|^2 + |\hat{u}_2|^2) = \frac{\xi\coth(\xi)h}{3-4\nu}(|\hat{u}_1|^2 + |\hat{u}_2|^2). \tag{20}$$

Using Lemma 3.4, we can show that $\xi\coth(\xi h) \in \sigma_+^1(\mathbb{R})$, that means there exists a positive constant C such that

$$\xi\coth(\xi h) \geq C(1 + |\xi|), \ \forall\xi \in \mathbb{R}. \tag{21}$$

From (20) and (21) it follows that $\mathbf{A}_+(\xi) \in \boldsymbol{\Sigma}_+^{\vec{\alpha}},\ \vec{\alpha} = (1,1)$. The theorem is proved. ■

It is not difficult to show that

$$\mathbf{B}(\xi) \in \boldsymbol{\Sigma}^{-\vec{\beta}},\ \vec{\beta} = (\beta, \beta),\ \beta >> 1.$$

We have

Theorem 3.12. (Existence) *Let $\tau_0(x)$ and $\sigma_0(x)$ be such that the function $\mathbf{f}(x)$ defined by (8)-(10) belongs to $\mathbb{H}^{-\vec{\alpha}/2}(\Omega),\ \vec{\alpha} = (1,1)$. Then the system of dual equations (17) has a unique solution $\mathbf{u} = F^{-1}[\widehat{\mathbf{u}}] \in \mathbb{H}_o^{\vec{\alpha}/2}(\Omega)$, i.e.*

$$u(x,0) \in H_o^{1/2}(a,b), \quad v(x,0) \in H_o^{1/2}(a,b),$$

where $u(x,0)$ and $v(x,0)$ are horizontal and vertical displacements on the axis $y = 0$, respectively.

Proof. Represent the operator $\mathbf{A}$ defined by formula (18) in the form $\mathbf{A} = \mathbf{A}_+ + \mathbf{B}$, with

$$A_+\mathbf{u} = pF^{-1}[\mathbf{A}_+\hat{\mathbf{u}}], \quad B\mathbf{u} = pF^{-1}[\mathbf{B}\hat{\mathbf{u}}], \quad \mathbf{u} = F[\mathbf{u}], \tag{22}$$

and then consider the system of equations

$$A_+\mathbf{u}(x) = \mathbf{g}(x), \quad \mathbf{u}(x) \in \mathbb{H}_o^{\vec{\alpha}/2}(\Omega) \tag{23}$$

with $\mathbf{g}(x) \in \mathbb{H}^{-\vec{\alpha}/2}(\Omega)$ being a given vector-function. From (14) and (15) we have

$$[\mathbf{A}_+\mathbf{u}, \mathbf{v}] = \int_{-\infty}^{\infty} \overline{F[\mathbf{v}^T](t)}\mathbf{A}_+(t)F[\mathbf{u}](t)dt = (\mathbf{u}, \mathbf{v})_{\mathbf{A}_+,\vec{\alpha}/2}$$

for arbitrary vector-functions $\mathbf{u}$ and $\mathbf{v}$ belonging to $\mathbb{H}_o^{\vec{\alpha}/2}(\Omega)$. Therefore, if $\mathbf{u} \in \mathbb{H}_o^{\vec{\alpha}/2}(\Omega)$ satisfies (23) then

$$(\mathbf{u}, \mathbf{v})_{\mathbf{A}_+,\vec{\alpha}/\mathbf{2}} = [\mathbf{g}, \mathbf{v}], \ \ \forall \mathbf{v} \in \mathbb{H}_o^{\vec{\alpha}/2}(\Omega). \tag{24}$$

Since $[\mathbf{g}, \mathbf{v}]$ is a continuous linear functional on the Hilbert space $\mathbb{H}_o^{\vec{\alpha}/2}(\Omega)$, by virtue of Riesz theorem, there exists a unique element $\mathbf{u}_o \in \mathbb{H}_o^{\vec{\alpha}/2}(\Omega)$ such that

$$[\mathbf{g}, \mathbf{v}] = (\mathbf{u_o}, \mathbf{v})_{\mathbf{A}_+,\vec{\alpha}/2}, \quad \mathbf{v} \in \mathbb{H}_o^{\vec{\alpha}/2}(\Omega). \tag{25}$$

From (24) and (25) it follows that $\mathbf{u} = \mathbf{u}_0$. Moreover, the estimation

$$||\mathbf{u_o}||_{\mathbf{A}_+,\vec{\alpha}/2} = ||A^{-1}\mathbf{g}||_{\mathbf{A}_+,\vec{\alpha}/2} \leqslant C||\mathbf{g}||_{\mathbb{H}^{-\vec{\alpha}/2}(\Omega)}$$

holds for a positive constant C. Hence the operator A^{-1} is bounded. Next, representing system (17) in the form

$$A_+\mathbf{u} + B\mathbf{u} = \mathbf{f}.$$

we obtain

$$\mathbf{u} + A_+^{-1}B\mathbf{u} = A_+^{-1}\mathbf{f}. \tag{26}$$

In virtue of Lemma 3.8, the operator $B\mathbf{u}$ defined by (22) is completely continuous from $\mathbb{H}_o^{\vec{\alpha}/2}(\Omega)$ into $\mathbb{H}^{-\vec{\alpha}/2}(\Omega)$. Thus the operator $A_+^{-1}B$ is completely continuous. It follows that system (26) is Fredholm. Due to the uniqueness of its solution (Theorem 3.10) it follows that this system has a unique solution $\mathbf{u} \in \mathbb{H}_o^{\vec{\alpha}/2}(\Omega)$. ■

4. Reduction to Infinite Systems of Algebraic Equations

In this section we propose a method for reducing the system of dual integral equations (7) to an infinite system of linear algebraic equations of second kind.

4.1. Some Preliminary Considerations

Definition 4.1. Let $\rho(x) = \sqrt{(x-a)(b-x)}(a < x < b)$. We denote by $L^2_{\rho^{\pm 1}}(a,b)$ the Hilbert spaces of functions with respect to the scalar products and the norms

$$(u,v)_{L^2_{\rho^{\pm 1}}} = \int_a^b \rho^{\pm 1}(x)u(x)\overline{v(x)}dx, \quad ||u||_{L^2_{\rho^{\pm 1}}} = \sqrt{(u,u)_{L^2_{\rho^{\pm 1}}}} < +\infty.$$

We have the following result [7].

Lemma 4.2. *Let $\varphi \in L^2_\rho(a,b)$. Denote by φ_0 the zero-extension of the function φ on $\mathbb{R}$. Then $\varphi_0 \in H_o^{-1/2}(a,b)$.*

In the spaces $L^2_{\rho^{\pm 1}}(a,b)$ we consider the singular integral operator

$$S_\Omega[\varphi](x) = \frac{1}{\pi i}\int_a^b \frac{\varphi(t)}{x-t}dt, \quad x \in \Omega := (a,b),$$

where the integral is taken in the sense of Cauchy principal value. The following theorem is due to Khvedelidze [1].

Theorem 4.3. *The operator S_Ω is bounded in the spaces $L^2_{\rho^{\pm 1}}(a,b)$:*

$$||S_\Omega[\varphi]||_{L^2_{\rho^{\pm 1}}(a,b)} \leqslant C||\varphi||_{L^2_{\rho^{\pm 1}}(a,b)}.$$

We shall need some relations for Chebyshev polynomials. Let $T_k(x)$ and $U_k(x)$ be the Chebyshev polynomials of first and second kind, respectively. We have the following relations [10]:

$$T_n(\cos\theta) = \cos n\theta, \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

$$\int_a^b \frac{T_k[\eta(x)]T_j[\eta(x)]}{\rho(x)}dx = \alpha_k\delta_{kj},$$

$$\int_a^b U_k[\eta(x)]U_j[\eta(x)]\rho(x)dx = \beta\delta_{kj},$$

$$\int_a^b \frac{T_k[\eta(y)]dy}{(x-y)\rho(y)} = \frac{-2\pi}{b-a}U_{k-1}[\eta(x)],\ k = 0, 1, \dots$$

$$\int_a^b \frac{\rho(y)U_{k-1}[\eta(y)]dy}{x-y} = \frac{\pi(b-a)}{2}T_k[\eta(x)], \quad k = 1, 2, \dots,$$

where δ_{kj} is the Kronecker symbol and

$$\alpha_k = \begin{cases} \pi, & k = 0, \\ \dfrac{\pi}{2}, & k = 1, 2, \dots \end{cases}, \ \beta = \frac{\pi(b-a)^2}{8},\ \eta(x) = \frac{2x-(a+b)}{b-a}.$$

Consider the following system of linear algebraic equations [2]:

$$x_i = \sum_{k=1}^{\infty} c_{i,k}x_k + b_i \quad (i = 1, 2, ...), \tag{27}$$

where the numbers x_i are to be determined.

Definition 4.4. [2] The infinite system (27) is called regular if

$$\sum_{k=1}^{\infty} |c_{i,k}| < 1 \quad (i = 1, 2, \dots) \tag{28}$$

and completely regular if

$$\sum_{k=1}^{\infty} |c_{i,k}| \leqslant 1 - \theta < 1 \quad (i = 1, 2, \dots). \tag{29}$$

If the inequalities (28) (respectively, (29)) hold only for $i = N+1, N+2, \dots,$ then system (27) is called quasi-regular (respectively, quasi-completely regular).

The theory and applications of regular infinite systems can be found in [2].

4.2. Reduction of the System of Dual Equations (7) to an Infinite System of Algebraic Equations

Now we turn to system (7):

$$\begin{cases} F^{-1}[|\xi|a_{11}(\xi)\widehat{u}_1(\xi) + i \cdot |\xi|\text{sign}(\xi)a_{12}(\xi)\widehat{u}_2(\xi)](x) & = f_1(x), \\ F^{-1}[(-i) \cdot |\xi|\text{sign}(\xi)a_{21}(\xi)\widehat{u}_1(\xi) + |\xi|a_{22}(\xi)\widehat{u}_2(\xi)](x) & = f_2(x), \quad x \in (a,b), \end{cases} \tag{30}$$

where

$$a_{11} = \frac{2(1-\nu)[\cosh(|\xi|h)\sinh(|\xi|h) + |\xi|h]}{4(1-\nu)^2 + |\xi|^2h^2 + (3-4\nu)\sinh^2(|\xi|h)},$$

$$a_{21} = a_{12} = \frac{(1-2\nu)\sinh^2(|\xi|h) + |\xi|^2h^2}{4(1-\nu)^2 + |\xi|^2h^2 + (3-4\nu)\sinh^2(|\xi|h)},$$

$$a_{22} = \frac{2(1-\nu)[\cosh(|\xi|h)\sinh(|\xi|h) - |\xi|h]}{4(1-\nu)^2 + |\xi|^2h^2 + (3-4\nu)\sinh^2(|\xi|h)}.$$

We find the functions $u_1(x) = F^{-1}[\widehat{u}_1](x)$ and $u_2(x) = F^{-1}[\widehat{u}_2](x)$ in the form

$$u_m(x) = \frac{1}{2}\int_a^b v_m(t)\text{sign}(x-t)dt, \tag{31}$$

where $v_m \in L^2_\rho(a,b) \subset H_o^{-1/2}(a,b)$ and

$$\int_a^b v_m(x)dx = 0, \quad (m = 1,2). \tag{32}$$

Taking the Fourier transform of (31) we get

$$\widehat{u}_m(\xi) = \frac{1}{(-i\xi)}\int_a^b v_m(t)e^{i\xi t}dt = \frac{1}{(-i\xi)}\widehat{v}_m(\xi), \quad (m = 1,2). \tag{33}$$

Substituting equations (33) into (30), after some transforms we get the system

$$\begin{cases} F^{-1}\left[i \cdot \text{sign}(\xi)a_{11}(\xi)\widehat{v}_1(\xi) - a_{12}(\xi)\widehat{v}_2(\xi)\right](x) & = f_1(x), \quad x \in (a,b), \\ F^{-1}\left[a_{21}(\xi)\widehat{v}_1(\xi) + i \cdot \text{sign}(\xi)a_{22}(\xi)\widehat{v}_2(\xi)\right](x) & = f_2(x), \quad x \in (a,b). \end{cases} \tag{34}$$

Further, using the formula

$$F^{-1}[\text{sign}(\xi)F[v]](x) = \frac{1}{\pi i}\int_a^b \frac{v(t)dt}{x-t}, \quad v \in L^2_{\rho^{\pm 1}}(a,b)$$

we can transform system (34) to the following system of singular integral equations:

$$\begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_a^b \frac{v_1(t)dt}{x-t} + \int_a^b v_1(t)k_{11}(x-t)dt - \int_a^b v_2(t)k_{12}(x-t)dt - \beta v_2(x) \\ \qquad\qquad = f_1(x), \quad x \in (a,b), \\ \dfrac{\alpha}{\pi}\displaystyle\int_a^b \frac{v_2(t)dt}{x-t} + \int_a^b v_1(t)k_{21}(x-t)dt + \int_a^b v_2(t)k_{22}(x-t)dt + \beta v_1(x) \\ \qquad\qquad = f_2(x), \quad x \in (a,b), \end{cases} \tag{35}$$

where

$$k_{11}(x) = \frac{1}{\pi}\int_0^\infty (a_{11}(\xi) - \alpha)\sin \xi x d\xi,$$

$$k_{22}(x) = \frac{1}{\pi}\int_0^\infty (a_{22}(\xi) - \alpha)\sin \xi x d\xi,$$

$$k_{12}(x) = k_{21}(x) = \frac{1}{\pi}\int_0^\infty (a_{12}(\xi) - \beta)\cos \xi x d\xi.$$

In (35) replacing the functions $v_m(t)$ by $\dfrac{\psi_m(t)}{\rho(t)}$, where $\psi_m(t) \in L^2_{\rho^{-1}}(a,b)$, we have the following system of integral equations

$$\begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_a^b \frac{\psi_1(t)dt}{\rho(t)(x-t)} + \int_a^b \frac{\psi_1(t)}{\rho(t)}k_{11}(x-t)dt - \int_a^b \frac{\psi_2(t)}{\rho(t)}k_{12}(x-t)dt \\ \qquad\qquad -\beta \cdot \dfrac{\psi_2(x)}{\rho(x)} = f_1(x), \quad x \in (a,b), \\ \dfrac{\alpha}{\pi}\displaystyle\int_a^b \frac{\psi_2(t)dt}{\rho(t)(x-t)} + \int_a^b \frac{\psi_1(t)}{\rho(t)}k_{21}(x-t)dt + \int_a^b \frac{\psi_2(t)}{\rho(t)}k_{22}(x-t)dt \\ \qquad\qquad +\beta \cdot \dfrac{\psi_1(x)}{\rho(x)} = f_2(x), \quad x \in (a,b). \end{cases} \tag{36}$$

Further, we expand the functions $\psi_1(t)$ and $\psi_2(t)$ to series

$$\psi_m(t) = \sum_{j=1}^{\infty} A_j^{(m)} T_j[\eta(t)], \quad (m = 1, 2), \tag{37}$$

where $A_j^{(m)}$ are unknown constants, besides, $\{A_j^{(m)}\}_{j=1}^{\infty} \in l_2 (m = 1, 2)$. It is not difficult to verify that the functions $v_m(t) = \rho^{-1}(t)\psi_m(t)$ $(m = 1, 2)$ satisfy conditions (32). Substituting (37) into (36), in virtue of Theorem 4.3, changing the order of integrations and summations, after some transforms, we have the following system

$$\begin{cases} \dfrac{-\alpha(b-a)\pi}{4} A_{n+1}^{(1)} + \displaystyle\sum_{j=1}^{\infty} (A_j^{(1)} C_{nj}^{(11)} - A_j^{(2)} C_{nj}^{(12)}) = F_n^{(1)}, \\ \dfrac{-\alpha(b-a)\pi}{4} A_{n+1}^{(2)} + \displaystyle\sum_{j=1}^{\infty} (A_j^{(1)} C_{nj}^{(21)} + A_j^{(2)} C_{nj}^{(22)}) = F_n^{(2)}, \\ \quad n = 0, 1, 2, \ldots, \end{cases} \tag{38}$$

where

$$C_{nj}^{(11)} = \int_a^b \rho(x) U_n[\eta(x)] dx \int_a^b \frac{T_j[\eta(t)]}{\rho(t)} k_{11}(x-t) dt,$$

$$C_{nj}^{(22)} = \int_a^b \rho(x) U_n[\eta(x)] dx \int_a^b \frac{T_j[\eta(t)]}{\rho(t)} k_{22}(x-t) dt,$$

$$C_{nj}^{(12)} = \begin{cases} \widetilde{C}_{nj}^{(12)} + \dfrac{\beta(b-a)(n+1)}{(n+1)^2 - j^2}, \ (n = 0, 2, 4, \ldots; j = 2, 4, 6, \ldots \\ \qquad\qquad\qquad\qquad \text{or } n = 1, 3, 5, \ldots;\ j = 1, 3, 5, \ldots), \\ \widetilde{C}_{nj}^{(12)}, \quad (n = 0, 2, 4, \ldots;\ j = 1, 3, 5, \ldots \text{or } n = 1, 3, 5, \ldots;\ j = 2, 4, 6, \ldots), \end{cases}$$

$$C_{nj}^{(21)} = \begin{cases} \widetilde{C}_{nj}^{(21)} + \dfrac{\beta(b-a)(n+1)}{(n+1)^2 - j^2}, \ (n = 0, 2, 4, \ldots;\ j = 2, 4, 6, \ldots, \\ \qquad\qquad\qquad\qquad \text{or } n = 1, 3, 5, \ldots;\ j = 1, 3, 5, \ldots), \\ \widetilde{C}_{nj}^{(21)}, \quad (n = 0, 2, 4, \ldots;\ j = 1, 3, 5, \ldots \ \textit{or} \ n = 1, 3, 5, \ldots;\ j = 2, 4, 6, \ldots), \end{cases}$$

$$\widetilde{C}_{nj}^{(12)} = \int_a^b \rho(x) U_n[\eta(x)] dx \int_a^b \frac{T_j[\eta(t)]}{\rho(t)} k_{12}(x-t) dt,$$

$$\widetilde{C}_{nj}^{(21)} = \int_a^b \rho(x) U_n[\eta(x)] dx \int_a^b \frac{T_j[\eta(t)]}{\rho(t)} k_{21}(x-t) dt,$$

$$F_n^{(1)} = \int\limits_a^b \rho(x)U_n[\eta(x)]f_1(x)dx,$$

$$F_n^{(2)} = \int\limits_a^b \rho(x)U_n[\eta(x)]f_2(x)dx.$$

One can prove the following theorem.

Theorem 4.5. *The system of singular integral equations* (35) *and the system of linear algebraic equations* (38) *are equivalent.*

We introduce notations

$$X_{2k-1} = A_k^{(1)},\ \ X_{2k} = A_k^{(2)}, \quad (k = 1, 2, 3, \ldots) \tag{39}$$

$$E_{2l+1} = -\frac{4}{\alpha(b-a)\pi}F_l^{(1)},\ \ E_{2l+2} = -\frac{4}{\alpha(b-a)\pi}F_l^{(2)},\ (l = 0, 1, 2, ...), \tag{40}$$

$$C_{2j+1,2n-1} = -\frac{4}{\alpha(b-a)\pi}C_{jn}^{(11)}, \quad C_{2j+1,2n} = +\frac{4}{\alpha(b-a)\pi}C_{jn}^{(12)}, \tag{41}$$

$$C_{2j+2,2n-1} = -\frac{4}{\alpha(b-a)\pi}C_{jn}^{(21)}, \quad C_{2j+2,2n} = -\frac{4}{\alpha(b-a)\pi}C_{jn}^{(22)}. \tag{42}$$

(43)

Then system (38) can be written in the form

$$X_n + \sum_{j=1}^{\infty} C_{nj}X_j = E_n \ \ (n = 1, 2, \ldots). \tag{44}$$

The following lemmas hold.

Lemma 4.6. *The following inequalities hold*

$$|C_{nj}| \leqslant \frac{L}{nj^2} \ \ (n \geq 1, \ \ j \geq 2), \tag{45}$$

where L is a certain positive constant.

Lemma 4.7. *If derivatives $f_m^{(k)}(x), m = 1, 2$ are continuous functions on $[a, b]$, then the following inequalities hold*

$$|E_n| \leqslant \frac{L}{n^k} \ \ (n = 1, 2, \ldots; k = 0, 1, \ldots).$$

Theorem 4.8. *Let $f_1(x)$ and $f_2(x)$ be such that the set $\{E_n\}_{n=1}^{\infty}$, defined by* (40) *belongs to l_2. Then the infinite system of linear algebraic equations* (44) *possesses a unique solution $\{X_n\}_{n=1}^{\infty} \in l_2$. This infinite system is quasi-completely regular.*

Proof. Denote by L the infinite coefficient matrix in the left-hand side of (44). According to the estimations (45) the double series formed of the squares of the components of L is convergent, so the infinite matrix L defines a completely continuous operator mapping the Hilbert space l_2 into itself. Therefore, the infinite system (44) is Fredholm in l_2. The uniqueness of the solution of this system follows from the uniqueness of that of the system of dual equations (7). Hence, it follows that the infinite system (44) has a unique solution in l_2. For a sufficiently large number $n = N$, we have

$$\sum_{j=1}^{\infty} |C_{nj}| \leqslant \frac{L}{n} \sum_{j=1}^{\infty} \frac{1}{j^2} \leqslant 1 - \theta < 1 \quad (n = N+1, N+2, \ldots),$$

therefore the infinite system (44) is quasi-completely regular [2]. ∎

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