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On Contact CR-Lightlike Submanifolds of Indefinite Sasakian Manifolds

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Abstract. In the present paper we obtain a condition for an invariant lightlike submanifold of indefinite Sasakian space form to be of constant ϕ-sectional curvature. Then, we study contact CR-lightlike submanifolds of (ϵ)-Sasakian manifolds extensively and concluded with the study of totally contact umbilical contact CR-lightlike submanifolds.

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1. Introduction

Cauchy Riemann (CR) submanifolds of Kaehlerian manifolds with Riemannian metric were introduced by Bejancu in 1978, [2]. Then, contact CR-submanifolds of Sasakian manifolds with definite metric were introduced and studied by Yano-Kon in 1982, [9]. Recently, Duggal-Sahin [6] introduced the theory of contact CR-lightlike submanifolds of indefinite Sasakian manifolds and studied the integrability conditions of their distributions and investigated the geometry of leaves of the distributions involved in the induced contact CR-structure. They also studied geometric conditions for an irrotational contact CR-lightlike submanifold of an indefinite Sasakian manifold to be a contact CR-lightlike product. Contact geometry has a significant role in optics, phase spaces of a dynamical system and many more, see [1, 8]. So, we study geometry of contact submanifolds, in particular, contact CR-lightlike submanifolds of indefinite Sasakian manifolds.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [5] by Duggal-Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u, v) = 0,\ \forall v \in T_xM, x \in M\},$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{Rad}(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$\mathrm{Rad}(TM) : x \in M \longrightarrow \mathrm{Rad}(T_xM),$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called r-lightlike submanifold and $\mathrm{Rad}(TM)$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is,

$$TM = \mathrm{Rad}(TM) \perp S(TM), \tag{1}$$

$S(TM^{\perp})$ is a complementary vector subbundle to $\mathrm{Rad}(TM)$ in $TM^{\perp}$. Let $\mathrm{tr}(TM)$ and $\mathrm{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $\mathrm{Rad}(TM)$ in $S(TM^{\perp})^{\perp}$ respectively. Then, we have

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S(TM^{\perp}), \tag{2}$$

$$T\bar{M}\mid_M = TM \oplus \mathrm{tr}(TM) = (\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{3}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as

$$\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\},$$

where $\{\xi_1, \ldots, \xi_r\}, \{N_1, \ldots, N_r\}$ are local lightlike bases of $\Gamma(\mathrm{Rad}(TM)\mid_u)$, $\Gamma(\mathrm{ltr}(TM)\mid_u)$ and $\{W_{r+1}, \ldots, W_n\}, \{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp}) \mid_u)$ and $\Gamma(S(TM) \mid_u)$, respectively. For this quasi-orthonormal fields of frames, we have

Theorem 2.1. [5] *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, there exists a complementary vector bundle $\mathrm{ltr}(TM)$ of $\mathrm{Rad}(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(\mathrm{ltr}(TM) \mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp} \mid_{\mathrm{u}}$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{4}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, according to decomposition (3), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \forall X, Y \in \Gamma(TM), \tag{5}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \ \forall X \in \Gamma(TM), U \in \Gamma(\mathrm{tr}(TM)), \tag{6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_V is a linear operator on M, known as shape operator.

According to (2), considering the projection morphisms L and S of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively, (5) and (6) give

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{7}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{8}$$

where, we put $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore, we call them the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{9}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{10}$$

where $X \in \Gamma(TM), N \in \Gamma(\mathrm{ltr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

Using (2)-(3) and (7)-(10), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{11}$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0,$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0,$$

$$\bar{g}(A_N X, \bar{P} Y) = \bar{g}(N, \bar{\nabla}_X \bar{P} Y), \tag{12}$$

for any $\xi \in \Gamma(\mathrm{Rad}(TM))$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(\mathrm{ltr}(TM))$. $\bar{P}$ is a projection of TM on $S(TM)$.

Now, we consider decomposition (1), we can write

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{13}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*\perp}_X \xi, \tag{14}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$, where $\{\nabla^*_X \bar{P}Y, A^*_\xi X\}$ and $\{h^*(X, \bar{P}Y), \nabla^{*\perp}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}(TM))$, respectively. Here ∇^* and $\nabla^{*\perp}_X$ are linear connections on $S(TM)$ and $\mathrm{Rad}(TM)$ respectively. By using (7)-(8) and (13)-(14), we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y),$$

$$\bar{g}(h^*(X, \bar{P}Y), N) = \bar{g}(A_N X, \bar{P}Y). \tag{15}$$

3. Invariant Lightlike Submanifolds

Let $\bar{M}$ be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ, η, V), where ϕ is a tensor field of type $(1,1)$, η is a 1-form and V is a characteristic vector field on M, such that

$$\phi^2(X) = -X + \eta(X)V; \qquad \eta(V) = 1. \tag{16}$$

It follows that

$$\eta(\phi(X)) = 0; \qquad \phi(V) = 0; \qquad \mathrm{rank}\phi = 2n, \tag{17}$$

then $\bar{M}$ is called an almost contact manifold. If there exists a semi-Riemannian metric $\bar{g}$ satisfying

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon\eta(X)\eta(Y),$$

where $\epsilon \pm 1$, then $(\phi, \eta, V, \bar{g})$ is called an (ϵ)-almost contact metric structure and $\bar{M}$ is called an (ϵ)-almost contact manifold [4, 7]. Here

$$\eta(X) = \epsilon\bar{g}(X, V); \qquad \epsilon = \bar{g}(V, V). \tag{18}$$

If $d\phi(X, Y) = \bar{g}(X, \phi Y)$, then $\bar{M}$ is said to have (ϵ)-contact Riemannian structure $(\phi, V, \eta, \bar{g})$. If, moreover, this structure is normal then it is known an (ϵ)-Sasakian structure and $\bar{M}$ is known an (ϵ)-Sasakian manifold. Also, an (ϵ)-almost contact metric structure is (ϵ)-Sasakian, if and only if,

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(X, Y)V + \epsilon\eta(Y)X, \tag{19}$$

$$\bar{\nabla}_X V = \phi X. \tag{20}$$

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of $(\bar{M}, \bar{g})$. For any vector field X tangent to M, we put

$$\phi X = PX + FX, \tag{21}$$

where PX and FX are the tangential and transversal components of ϕX, respectively. Therefore, (17) implies that $PV = 0$ and $FV = 0$.

Let M be tangent to structure vector field V, then $V \in \Gamma(S(TM))$, [3]. It follows that M is invariant in $\bar{M}$ if $\phi X \in \Gamma(TM)$, that is, $\phi X = PX$, for all $X \in \Gamma(TM)$. For any $\lambda \in \Gamma(\mathrm{tr}(TM))$, we put

$$\phi\lambda = t\lambda + f\lambda,$$

where $t\lambda$ and $f\lambda$ are the tangential and transversal components of $\phi\lambda$, respectively. Clearly, M tangent to the structure vector field V is invariant in $\bar{M}$ if $\phi\lambda = f\lambda$. Therefore, if M is an invariant submanifold of an indefinite Sasakian manifold $\bar{M}$ then $F = 0$ and $t = 0$. For any vector fields $\lambda, \acute{\lambda} \in \Gamma(\mathrm{tr}(TM))$, we have $\bar{g}(\phi\lambda, \acute{\lambda}) = \bar{g}(f\lambda, \acute{\lambda})$, this implies $\bar{g}(f\lambda, \acute{\lambda})$ is skew-symmetric. Also, for any $X \in \Gamma(TM)$, we have

$$\bar{g}(FX, \lambda) + g(X, t\lambda) = 0.$$

Define covariant derivatives of P, t, F and f, respectively as

$$(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y), \tag{22}$$

$$\begin{aligned} (\nabla_X t)\lambda &= \nabla_X(t\lambda) - t(\nabla^{\perp}_X\lambda), \\ (\nabla_X F)Y &= \nabla^{\perp}_X(FY) - F(\nabla_X Y), \\ (\nabla_X f)\lambda &= \nabla^{\perp}_X(f\lambda) - f(\nabla^{\perp}_X\lambda). \end{aligned} \tag{23}$$

From (19), we have

$$\begin{aligned} -\bar{g}(X,Y)V + \epsilon\eta(Y)X &= \bar{\nabla}_X(\phi Y) - \phi(\bar{\nabla}_X Y) \\ &= \nabla_X(PY) + h(X, PY) - A_{FY}X + \nabla^{\perp}_X(FY) \\ &\quad - P(\nabla_X Y) - F(\nabla_X Y) - th(X,Y) - fh(X,Y). \end{aligned}$$

Use (22) and (23), then compare the tangential and transversal components, we get

$$(\nabla_X P)Y = -g(X,Y)V + \epsilon\eta(Y)X + A_{FY}X + th(X,Y), \tag{24}$$

$$(\nabla_X F)Y = -h(X, PY) + fh(X,Y). \tag{25}$$

Similarly, we can obtain

$$(\nabla_X t)\lambda = A_{f\lambda}X - PA_{\lambda}X, \tag{26}$$

$$(\nabla_X f)\lambda = -FA_{\lambda}X - h(X, t\lambda).$$

Lemma 3.1. *Let M be an invariant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then,*

$$h^l(X,V)=0, \qquad h^s(X,V)=0, \qquad A_N V=0, \qquad A_W V=0,$$
$$\phi h(X,Y)=h(\phi X,Y)=h(X,\phi Y). \tag{27}$$

Proof. For an invariant lightlike submanifold, (7) and (20) give

$$\nabla_X V=PX, \qquad h^l(X,V)=0, \qquad h^s(X,V)=0. \tag{28}$$

Let $N\in\Gamma(\mathrm{ltr}(TM))$, then $\bar{g}(N,\phi X)=\bar{g}(N,\bar{\nabla}_X V)$. Since M is tangent to the structure vector field V and $\bar{\nabla}$ is metric connection, we have $\bar{g}(\bar{\nabla}_X N,V)+\bar{g}(N,\bar{\nabla}_X V)=0$, therefore

$$\bar{g}(N,\phi X)=-\bar{g}(\bar{\nabla}_X N,V)=\bar{g}(A_N X,V). \tag{29}$$

Using (7), we also have

$$\bar{g}(N,\phi X)=\bar{g}(N,\nabla_X V)+\bar{g}(N,h^l(X,V)). \tag{30}$$

Therefore, from (29) and (30), we obtain

$$\bar{g}(A_N X,V)=\bar{g}(N,\nabla_X V)+\bar{g}(N,h^l(X,V)). \tag{31}$$

Using (28) in (31), we get

$$\bar{g}(A_N X,V)=\bar{g}(N,PX). \tag{32}$$

Replace X by V and use V being a non-null vector field then we get

$$A_N V=0.$$

Similarly, let $W\in\Gamma(S(TM^{\perp}))$, then we have

$$\bar{g}(W,\phi X)=\bar{g}(A_W X,V), \tag{33}$$

and

$$\bar{g}(W,\phi X)=\bar{g}(W,h^s(X,V)). \tag{34}$$

Therefore, from (33) and (34), we have

$$\bar{g}(A_W X,V)=\bar{g}(W,h^s(X,V)). \tag{35}$$

Using (28) in (35), we get

$$A_W X=0,$$

in particular, we have

$$A_W V=0. \tag{36}$$

Since for an invariant lightlike submanifold, $F = 0$, therefore (25) implies (27). ■

The Screen distribution $S(TM)$ is said to define the totally geodesic foliation in M, if and only if, $\nabla_X Y \in \Gamma(S(TM))$, for any $X, Y \in \Gamma(S(TM))$.

Lemma 3.2. *Let M be an invariant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that the screen distribution defines the totally geodesic foliation in M. Then,*

$$\phi A_N X = -A_N \phi X = A_{\phi N} X, \tag{37}$$

$$\phi A_W X = -A_W \phi X = A_{\phi W} X. \tag{38}$$

Proof. Replace X by ϕX in (31) and using the hypothesis, we get $\bar{g}(A_N \phi X, V) = -\bar{g}(A_{\phi N} X, V)$. Moreover, for invariant lightlike submanifolds $t = 0$, therefore (37) follows from (26) with above equation, by using the non-degeneracy of vector field V.

Similarly, we can obtain (38), replacing X by ϕX in (35). ■

Theorem 3.3. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an invariant lightlike submanifold of an indefinite Sasakian space form $\bar{M}(c)$. Then, M is of constant ϕ-sectional curvature c if M is totally geodesic.*

Proof. Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇, respectively, then by straightforward calculations [5], we have

$$\begin{aligned} \bar{R}(X,Y)Z = R(X,Y)Z &+ A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ &- A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) + (\nabla_X h^s)(Y,Z) \\ &- (\nabla_Y h^s)(X,Z) + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) \\ &+ D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)). \end{aligned} \tag{39}$$

The Gauss equation is

$$\begin{aligned} \bar{R}(X,Y)Z = R(X,Y)Z &+ A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X \\ &+ A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X, \end{aligned} \tag{40}$$

then using (16), (18), (27), (36), (37), and (38), we have

$$\begin{aligned} \bar{g}(\bar{R}(X,\phi X)\phi X, X) = g(R(X,\phi X)\phi X, X) &+ 2\bar{g}(A_{h^l(X,X)}X, X) \\ &- 2\epsilon\eta(A_{h^l(X,X)}X)\eta(X). \end{aligned} \tag{41}$$

Further using (32), we have

$$\begin{aligned} \bar{g}(\bar{R}(X,\phi X)\phi X, X) = g(R(X,\phi X)\phi X, X) &+ 2\bar{g}(A_{h^l(X,X)}X, X) \\ &- 2\bar{g}(h^l(X,X), PX)\eta(X). \end{aligned} \tag{42}$$

Hence, the proof follows from the above equation. ■

4. Contact CR-lightlike submanifolds

Definition 4.1. [6] Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. Then, M is a contact CR-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $\mathrm{Rad}(TM)$ is a distribution on M such that $\mathrm{Rad}(TM) \cap \phi(\mathrm{Rad}(TM)) = \{0\}$,

(ii) there exist vector bundles D_0 and D' over M such that

$$S(TM) = \{\phi(\mathrm{Rad}(TM)) \oplus D'\} \perp D_0 \perp \{V\},$$

$$\phi D_0 = D_0, \qquad \phi D' = L_1 \perp \mathrm{ltr}(TM),$$

where D_0 is nondegenerate and L_1 is a vector subbundle of $S(TM^{\perp})$. Therefore we have

$$TM = D \oplus \{V\} \oplus D',$$
$$D = \mathrm{Rad}(TM) \perp \phi(\mathrm{Rad}(TM)) \perp D_0. \tag{43}$$

A contact CR-lightlike submanifold is said to be proper if $D_0 \neq \{0\}$ and $L_1 \neq \{0\}$. If $D_0 = \{0\}$, then M is said to be a totally real lightlike submanifold.

Example 4.2. [6] Let M be a lightlike hypersurface of $\bar{M}$, then for $\xi \in \Gamma(\mathrm{Rad}(TM))$, we have $\bar{g}(\phi\xi, \xi) = 0$. Hence $\phi\xi \in \Gamma(TM)$. Thus, we get a rank-1 distribution $\phi(TM^{\perp})$ on M such that $\phi(TM^{\perp}) \cap TM^{\perp} = \{0\}$ and $\phi(TM^{\perp}) \in S(TM)$. Now, let $N \in \Gamma(\mathrm{ltr}(TM))$ such that $\bar{g}(\phi N, \xi) = -\bar{g}(N, \phi\xi) = 0$ and $\bar{g}(\phi N, N) = 0$. Thus, $\phi N \in \Gamma(S(TM))$. Let $D' = \phi(\mathrm{tr}(TM))$, we obtain $S(TM) = \{\phi(TM^{\perp}) \oplus D'\} \perp D_0$, where D_0 is a nondegenerate distribution and $\phi D' = \mathrm{tr}(TM)$. Hence, M is a contact CR-lightlike hypersurface.

Using (16), we have

$$P^2 = -I - tF + \eta \otimes V, \tag{44}$$
$$FP + fF = 0,$$
$$f^2 = -I - Ft, \tag{45}$$
$$Pt + tf = 0.$$

Lemma 4.3. *In a contact* CR*-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$, in order to a vector field X tangent to M belongs to $D \oplus \{V\}$, it is necessary and sufficient that $FX = 0$.*

Proof. The proof follows from (21). ∎

Theorem 4.4. *In a contact* CR*-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$, the distribution $D \oplus \{V\}$ has an almost contact metric structure (P, V, η, g) and hence the dimension of D is even.*

Proof. From (44), we have

$$P^2X = -X - tFX + \eta(X)V.$$

Let $X, Y \in \Gamma(D \oplus \{V\})$, then we obtain

$$P^2X = -X + \eta(X)V.$$

Since $PV = 0$ and $g(PX, PY) = \bar{g}(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y)$, then $D \oplus \{V\}$ has an almost contact metric structure (P, V, η, g). ■

Define the orthogonal complement subbundle to the vector subbundle L_1 in $S(TM^\perp)$ by $L_1^\perp$. Therefore

$$\mathrm{tr}(TM) = \phi D' \oplus L_1^\perp,$$

so put

$$\phi W = BW + CW, \quad \forall\, W \in \Gamma(S(TM^\perp)),$$

where $BW \in \Gamma(\phi L_1)$ and $CW \in \Gamma(L_1^\perp)$.

Theorem 4.5. *For a contact* CR*-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$, the subbundle $L_1^\perp$ has an almost complex structure f and hence the dimension of $L_1^\perp$ is even.*

Proof. For any $\lambda \in \Gamma(L_1^\perp))$, from (45) we have $f^2\lambda = -\lambda - Ft\lambda$, this implies $f^2\lambda = -\lambda$, which completes the proof. ■

Lemma 4.6. *In a contact* CR*-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$, $\nabla_X V \in \Gamma(S(TM))$ for any $X \in D_0$.*

Proof. Let $N \in \Gamma(\mathrm{ltr}(TM))$ then $\bar{g}(\nabla_X V, N) = \bar{g}(\bar{\nabla}_X V, N) = \bar{g}(\phi X, N) = 0$. Hence, from (4) the result follows. ■

Theorem 4.7. *In a contact* CR*-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$, the distribution D_0 has K-contact metric structure (P, V, η, g).*

Proof. Use (6) and (19) for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D')$, we have

$$\phi(\nabla_X Z + h(X, Z)) = -A_{\phi Z}X + \nabla_X^\perp \phi Z + g(X, Z)V. \tag{46}$$

Let $Y \in \Gamma(D_0 \perp \{V\})$, then we have

$$\bar{g}(\nabla_X Z, \phi Y) = \bar{g}(A_{\phi Z}X, Y) - \epsilon g(X, Z)\eta(Y).$$

Therefore, in particular, for $X \in \Gamma(D_0)$ and $Y = V$, we have

$$\bar{g}(A_{\phi Z}X, V) = 0. \tag{47}$$

Since $\phi D' = L_1 \perp \mathrm{ltr}(TM)$ therefore using (47) in (31) and (35) with Lemma 4.6, we have $\bar{g}(h^l(X,V), \phi Z) = 0$ and $\bar{g}(h^s(X,V), \phi Z) = 0$.

Let $U \in \Gamma(L_1^\perp)$ and $X \in \Gamma(D_0)$, then

$$\bar{g}(h^l(X,V), U) = 0,$$

and

$$\bar{g}(h^s(X,V), U) = \bar{g}(\bar{\nabla}_X V - \nabla_X V - h^l(X,V), U) \quad = \bar{g}(\phi X, U) = \bar{g}(X, U) = 0.$$

Thus, we have $h^l(X,V) = 0, h^s(X,V) = 0$ for any $X \in \Gamma(D_0$. Since for any $X \in \Gamma(D_0)$, we have $\bar{\nabla}_X V = \nabla_X V + h^l(X,V) + h^l(X,V)$, therefore, $\bar{\nabla}_X V = \nabla_X V$ or $PX = \nabla_X V$. Thus, with Theorem 4.4, the proof follows. ∎

Lemma 4.8. *For any* $\lambda \in \Gamma(\mathrm{tr}(TM))$*, if* $t\lambda = 0$ *then* $\lambda \in \Gamma(L_1^\perp$*.*

Proof. For any $\lambda \in \Gamma(\mathrm{tr}(TM))$, put $\phi\lambda = t\lambda + f\lambda$ and let $t\lambda = 0$. Then, for any $X \in \Gamma(D')$, we have $\bar{g}(\phi X, \lambda) = -\bar{g}(X, \phi\lambda) = -\bar{g}(X, f\lambda) = 0$. This implies that $\lambda \in \Gamma(L_1^\perp)$. ∎

Theorem 4.9. *In a contact* CR*-lightlike submanifold* M *of an indefinite Sasakian manifold* $\bar{M}$*, the almost contact structure* (P, V, η, g) *is Sasakian, if and only if,* $th(X,Y) = 0$ *for any* $X, Y \in \Gamma(D)$ *or, if and only if,* $h(X,Y) \in \Gamma(L_1^\perp)$*.*

Proof. By virtue of the above lemma and (24), the proof follows. ∎

In [6], Duggal and Sahin studied the integrability of distributions of contact CR-lightlike submanifolds of indefinite Sasakian manifolds and proved

Theorem 4.10. *Let* M *be a contact* CR*-lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$*. Then,* D *and* $D' \oplus D$ *are not integrable.*

Theorem 4.11. *Let* M *be a contact* CR*-lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$*. Then,* $D \oplus \{V\}$ *is integrable, if and only if,* $h(X, \phi Y) = h(\phi X, Y)$*, for any* $X, Y \in \Gamma(D \oplus \{V\})$*.*

Next, we discuss the integrability of the distribution D'.

Lemma 4.12. *For any* $X, Z \in \Gamma(D')$*,* $\nabla^\perp_X \phi Z - \nabla^\perp_Z \phi X \in \Gamma(\phi D')$*.*

Proof. Let $U \in \Gamma(L_1^\perp)$. Then

$$\begin{aligned}\bar{g}(\nabla^\perp_X \phi Z - \nabla^\perp_Z \phi X, U) &= \bar{g}(\bar{\nabla}_X \phi Z - \bar{\nabla}_Z \phi X, U)\\ &= \bar{g}((\bar{\nabla}_X \phi)Z + \phi\bar{\nabla}_X Z - (\bar{\nabla}_Z \phi)X - \phi\bar{\nabla}_Z X, U)\\ &= \bar{g}(\phi(\bar{\nabla}_X Z - \bar{\nabla}_Z X) + (\bar{\nabla}_X \phi)Z - (\bar{\nabla}_Z \phi)X, U). \quad (48)\end{aligned}$$

From (19), for any $X, Z \in \Gamma(D')$, we have $(\bar{\nabla}_X \phi)Z = -g(X,Z)V$ then using the symmetric property of the second fundamental form, the above equation gives

$$\bar{g}(\nabla^{\perp}_X\phi Z - \nabla^{\perp}_Z\phi X, U) = -\bar{g}(\bar{\nabla}_X Z - \bar{\nabla}_Z X, \phi U)$$
$$= -\bar{g}(\nabla_X Z - \nabla_Z X, \phi U) = 0.$$

Therefore, $\nabla^{\perp}_X\phi Z - \nabla^{\perp}_Z\phi X \in \Gamma(\phi D')$. ■

Theorem 4.13. *In a contact* CR*-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$, the distribution D' is integrable.*

Proof. Since ∇ is torsion free then for any $X, Z \in \Gamma(D')$, from (46), we have

$$\phi([X, Z]) = A_{\phi X}Z - A_{\phi Z}X + \nabla^{\perp}_X\phi Z - \nabla^{\perp}_Z\phi X. \tag{49}$$

For any $Y \in \Gamma(D')$, (46) gives $\bar{g}(A_{\phi Z}X^{\star}, Y) = \bar{g}(\bar{\nabla}_{X^{\star}}Z, \phi Y)$, for any $X^{\star} \in \Gamma(TM)$, $Z \in \Gamma(D')$. Then, particularly from (12), we obtain

$$\bar{g}(A_{\phi Z}X^{\star}, Y) = \bar{g}(A_{\phi Y}X^{\star}, Z). \tag{50}$$

For any $\bar{P}X \in \Gamma(S(TM))$, (15) gives

$$\bar{g}(h^*(\bar{P}X, \bar{P}Y), N) = \bar{g}(A_N\bar{P}X, \bar{P}Y),$$

since h^* is bilinear and symmetric, therefore

$$\bar{g}(A_N\bar{P}X, \bar{P}Y) = \bar{g}(\bar{P}X, A_N\bar{P}Y). \tag{51}$$

Choose, particularly $X^{\star} \in \Gamma(D_0)$, then from (50) and (51), we obtain

$$\bar{g}(X^{\star}, A_{\phi Z}Y) = \bar{g}(X^{\star}, A_{\phi Y}Z),$$

then the non-degeneracy of D_0 implies that

$$A_{\phi Z}Y = A_{\phi Y}Z, \tag{52}$$

for any $Y, Z \in \Gamma(D')$. Thus, from (49), (52) and Lemma 4.12, the proof follows. ■

5. Totally Contact Umbilical Contact CR-lightlike submanifolds

Definition 5.1. [10] If the second fundamental form h of a submanifold, tangent to the structure vector field V, of an indefinite Sasakian manifold $\bar{M}$ is of the form

$$h(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y, V) + \eta(Y)h(X, V),$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called totally contact umbilical and totally contact geodesic if $\alpha = 0$.

The above definition also holds for a lightlike submanifold. For a totally contact umbilical lightlike submanifold M, we have

$$h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_L + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \quad (53)$$
$$h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_s + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \quad (54)$$

where $\alpha_L \in \Gamma(\mathrm{ltr}(TM))$ and $\alpha_s \in \Gamma(S(TM^\perp))$.

Lemma 5.2. [6] *Let M be a totally contact umbilical proper contact* CR*-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, $\alpha_L = 0$.*

The distribution D' is said to define totally geodesic foliation in M, if and only if, $\nabla_X Y \in \Gamma(D')$, for any $X, Y \in \Gamma(D')$.

Theorem 5.3. *Let M be a totally contact umbilical proper contact* CR*-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that D' defines totally geodesic foliation in M. Then, M is totally contact geodesic or $\alpha_s \in \Gamma(L_1^\perp)$ or* $\dim D' = 1$.

Proof. For any $X, Y \in \Gamma(D')$, from (53) and (54), we have

$$h^l(X,Y) = 0, \quad h^s(X,Y) = g(X,Y)\alpha_s, \quad (55)$$

then (7) implies

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)\alpha_s. \quad (56)$$

Use (55) in (11), we get

$$A_W X = \bar{g}(\alpha_s, W)X, \quad W \in \Gamma(S(TM^\perp)),$$

then (10) implies

$$\bar{\nabla}_X W = -\bar{g}(\alpha_s, W)X + \nabla^s_X W + D^l(X,W). \quad (57)$$

Since $Y \in \Gamma(D')$ so particularly let $W = \phi Y \in \Gamma(L_1)$ then

$$\bar{\nabla}_X \phi Y = -\bar{g}(\alpha_s, \phi Y)X + \nabla^s_X \phi Y + D^l(X, \phi Y). \quad (58)$$

Since $(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi\bar{\nabla}_X Y$, then, from (19), we have

$$\bar{\nabla}_X \phi Y = \phi\bar{\nabla}_X Y - g(X,Y)V. \quad (59)$$

Use (56) and (59) in (58), we get

$$\phi\nabla_X Y + g(X,Y)\phi\alpha_s - g(X,Y)V = -\bar{g}(\alpha_s, \phi Y)X + \nabla^s_X \phi Y + D^l(X, \phi Y).$$

Taking inner product with X and then use the hypothesis, we get

$$\bar{g}(\alpha_s, \phi Y)||X||^2 = g(X,Y)\bar{g}(\alpha_s, \phi X). \quad (60)$$

Change the role of X and Y, we get

$$\bar{g}(\alpha_s, \phi X)||Y||^2 = g(X, Y)\bar{g}(\alpha_s, \phi Y),$$

using (60) in the above equation, we get

$$\bar{g}(\alpha_s, \phi Y) = \frac{g(X, Y)^2}{||X||^2||Y||^2}\bar{g}(\alpha_s, \phi Y). \tag{61}$$

Then, possible solutions of equation (61) are $\alpha_s = 0$ or $\alpha_s \perp \phi Y$ or $X||Y$, which completes the proof. ■

Lemma 5.4. *Let M be a totally contact umbilical contact* CR*-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, $\nabla_X \phi W \in \Gamma(D')$ if $\alpha_s \in \Gamma(L_1^{\perp})$, $W \in \Gamma(\phi D')$ and $X \in \Gamma(D_0)$.*

Proof. Taking into account of (43), it is clear that $\nabla_X \phi W \in \Gamma(D')$, if and only if, $\bar{g}(\nabla_X \phi W, N) = \bar{g}(\nabla_X \phi W, Y) = \bar{g}(\nabla_X \phi W, \phi N) = 0$, for any $N \in \Gamma(\mathrm{ltr}(TM)), Y \in \Gamma(D_0)$.

Let $X, Y \in \Gamma(D_0), W \in \Gamma(\phi D')$ then, from $(\bar{\nabla}_X \phi)W = \bar{\nabla}_X \phi W - \phi \bar{\nabla}_X W$ and (19), we get $\bar{\nabla}_X \phi W = \phi \bar{\nabla}_X W$. Then, $\bar{g}(\nabla_X \phi W, N) = \bar{g}(\bar{\nabla}_X \phi W, N) - \bar{g}(h^l(X, \phi W), N) = -\bar{g}(\bar{\nabla}_X W, \phi N)$, by using (53). This further gives

$$\bar{g}(\nabla_X \phi W, N) = g(A_W X, \phi N) = 0,$$

by the use of (11) and (54).

Next, since $Y \in \Gamma(D_0)$ so let $\phi Y = Y' \in \Gamma(D_0)$, then

$$\begin{aligned} g(\nabla_X \phi W, Y) &= \bar{g}(\phi \bar{\nabla}_X W, Y) = -\bar{g}(\bar{\nabla}_X W, \phi Y) \\ &= -\bar{g}(\bar{\nabla}_X W, Y') = \bar{g}(A_W X, Y') = \bar{g}(h^s(X, Y'), W), \end{aligned} \tag{62}$$

by using (11). Now, for $X, Y \in \Gamma(D_0)$, from (54), we have $h^s(X, Y) = g(X, Y)\alpha_s$, then (62) gives $g(\nabla_X \phi W, Y) = g(X, Y')\bar{g}(\alpha_s, W)$, since $\alpha_s \in \Gamma(L_1^{\perp})$. Therefore, $\bar{g}(\nabla_X \phi W, Y) = 0$.

Similarly, by using (11) and (54), finally, we can prove that $\bar{g}(\nabla_X \phi W, \phi N) = 0$. This completes the proof. ■

Theorem 5.5. *Let M be a totally contact umbilical contact* CR*-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ and the lightlike transversal vector bundle is parallel with respect to $\nabla^{\perp}$. Then, M is totally real lightlike submanifold if $\alpha_S \neq 0$ and $\alpha_s \in \Gamma(L_1^{\perp})$.*

Proof. Here, for $X, Y \in \Gamma(D_0), W \in \Gamma(\phi D')$, we have

$$\begin{aligned} \bar{\nabla}_X \phi W &= \phi \bar{\nabla}_X W \\ \nabla_X \phi W + h(X, \phi W) &= -\phi A_W X + \phi \nabla_X^{\perp} W \end{aligned}$$

$$\nabla_X \phi W + g(X, \phi W)\alpha_s = -\phi \bar{g}(\alpha_s, W)X + \phi \nabla^{\perp}_X W$$
$$\nabla_X \phi W - g(\phi X, W)\alpha_s = \phi \nabla^{\perp}_X W,$$

this gives

$$\nabla_X \phi W = \phi \nabla^{\perp}_X W, \tag{63}$$

then, by using the above lemma with (63), we get $\nabla^{\perp}_X W \in \Gamma(\phi D')$. By hypothesis, the lightlike transversal vector bundle $\mathrm{ltr}(TM)$ is parallel with respect to $\nabla^{\perp}$ then, by [5, Theorem 2.3, p. 159], $\nabla^{\perp}$ gives a metric connection on $\mathrm{tr}(TM)$. Then, $\bar{g}(W, \nabla^{\perp}_X \alpha_s) = \bar{g}(\nabla^{\perp}_X W, \alpha_s)$. But $\nabla^{\perp}_X W \in \Gamma(\phi D')$, then $\bar{g}(W, \nabla^{\perp}_X \alpha_s) = 0$. Hence for any $X \in \Gamma(D_0)$, we have $\nabla^{\perp}_X \alpha_s \in \Gamma(L_1^{\perp})$. Now, for $X \in \Gamma(D_0)$, we have

$$\bar{\nabla}_X \phi \alpha_s = \phi \bar{\nabla}_X \alpha_s. \tag{64}$$

Replace W by α_s in (57), we get

$$\bar{\nabla}_X \alpha_s = -\bar{g}(\alpha_s, \alpha_s)X + \nabla^{\perp}_X \alpha_s. \tag{65}$$

Use (65) in (64), we get

$$\nabla^{\perp}_X \phi \alpha_s = -\bar{g}(\alpha_s, \alpha_s)\phi X + \phi \nabla^{\perp}_X \alpha_s. \tag{66}$$

Therefore, by hypothesis and $\nabla^{\perp}_X \alpha_s \in \Gamma(L_1^{\perp})$, then from (66), it follows that $\phi X = 0$ for all $X \in \Gamma(D_0)$. Hence $D_0 = \{0\}$, which completes the proof. ■

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