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The Subextension Problem for the Class $\mathcal{E}^{\psi}$

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Dedicated to Professor Hà Huy Khoái on the occasion of his 65th-birthday

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Abstract. The aim of this note is to give some results on the class $\mathcal{E}^{\psi}$ and to investigate the subextension problem for this class.

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1. Introduction

Let $\Omega \subset \widetilde{\Omega}$ be domains in $\mathbb{C}^n$ and u a plurisubharmonic function on Ω (briefly, $u \in \text{PSH}(\Omega)$). A function $\widetilde{u} \in \text{PSH}(\widetilde{\Omega})$ is said to be a subextension of u if for all $z \in \Omega, \widetilde{u}(z) \leq u(z)$. In 1980, El Mir gave an example of plurisubharmonic function on the unit bidisc for which the restriction to any smaller bidisc admits no subextension to the whole space (see [13]). The subextension problem in the class $\mathcal{F}(\Omega)$ introduced and investigated by U.Cegrell (see [7]) has recently been studied by U.Cegrell and A.Zeriahi. In [11] the authors proved that if $\Omega, \widetilde{\Omega}$ are bounded hyperconvex domains in $\mathbb{C}^n$ with $\Omega \Subset \widetilde{\Omega}$ and $u \in \mathcal{F}(\Omega)$, then there exists $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega})$ such that $\widetilde{u} \leq u$ on Ω and

$$\int_{\widetilde{\Omega}} (dd^c \widetilde{u})^n \leq \int_{\Omega} (dd^c u)^n.$$

Next, in 2005, U.Cegrell, S.Kolodziej and A. Zeriahi have shown that plurisubharmonic functions in the class $\mathcal{F}(\Omega)$ admit global subextension to $\mathbb{C}^n$ with logarithmic growth at infinity (see Theorem 5.1 in [10]). For the class $\mathcal{E}_p(\Omega), p > 0$ the subextension problem was investigated by P.H.Hiep. He proved in [17] that if $\Omega \subset \widetilde{\Omega} \subset \mathbb{C}^n$ are hyperconvex domains and $u \in \mathcal{E}_p(\Omega), p > 0$, then there exists a function $\widetilde{u} \in \mathcal{E}_p(\widetilde{\Omega})$ such that $\widetilde{u} \leq u$ on Ω and $e_p(\widetilde{u}) = \int\limits_{\widetilde{\Omega}} (-\widetilde{u})^p (dd^c\widetilde{u})^n \leq \int\limits_{\Omega} (-u)^p (dd^c u)^n$. Recently, in [4] when studying the classes of weighted pluricomplex energy $\mathcal{E}_\chi(\Omega)$, S. Belnelkourchi has proved a result on subextension in these classes. Note that for some particular forms of χ, the class $\mathcal{E}_\chi(\Omega)$ is $\mathcal{F}(\Omega)$ or $\mathcal{E}_p(\Omega)$ (for detail, see Section 2). The subextension problem concerning to boundary values was considered in recent years. Namely, in 2008, in [12] the authors showed that if Ω_1 and Ω_2 are two bounded hyperconvex domains such that $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n, n \geq 1$ and $u \in \mathcal{F}(\Omega_1)$ with boundary values $F \in \mathcal{E}(\Omega_1)$ has subextension $v \in \mathcal{F}(\Omega_2)$ with boundary values $G \in \mathcal{E}(\Omega_2) \cap \mathcal{MPSH}(\Omega_2)$, where $\mathcal{F}(\Omega)$ (resp. $\mathcal{E}(\Omega)$) are the classes of plurisubharmonic functions are given in Section 2 and $\mathcal{MPSH}(\Omega)$ denotes the set of maximal plurisubharmonic functions on Ω. However, the subextension problem is not always possible. In 2006, in [19], J.Wiklund proved that the subextension problem is not possible in the class $\mathcal{E}(\Omega)$. Namely, he constructed for any hyperconvex domain Ω a function u in $\mathcal{E}(\Omega)$ that can not be subextended to a larger domain(see [19]). Recently, based on an idea of J.Wiklund in [19], L.Hed gave an example showing that the subextension problem is not possible in the narrower subclass $\mathcal{N}(\Omega)$ of the class $\mathcal{E}(\Omega)$ (see Example 5.2 in [14]). Thus, the subextension problem holds in $\mathcal{F}(\Omega)$, $\mathcal{E}_p(\Omega)$ and $\mathcal{E}_\chi(\Omega)$, but not always in $\mathcal{E}(\Omega)$. In this note we deal with the class $\mathcal{E}^\psi(\Omega)$ for a negative plurisubharmonic function ψ in a bounded hyperconvex domain Ω. In Section 2 we will give the definition of this class and by relying on our recent result in [15], we explain the relation between this class and the class $\mathcal{E}_\chi$. The main result in this note is to prove the subextension problem still holds in the class $\mathcal{E}^\psi(\Omega)$.

Namely we prove the following.

Theorem 4.1. *Let $\Omega \subset \widetilde{\Omega} \Subset \mathbb{C}^n$ be hyperconvex domains and $\psi \in PSH^-(\widetilde{\Omega}), \psi \not\equiv 0$ and $u \in \mathcal{E}^\psi(\Omega)$. Then there exists $\widetilde{u} \in \mathcal{E}^\psi(\widetilde{\Omega})$ such that $\widetilde{u} \leq u$ on Ω and*

$$e_\psi(\widetilde{u}) = \int\limits_{\widetilde{\Omega}} (-\psi)(dd^c\widetilde{u})^n \leq \int\limits_{\Omega} (-\psi)(dd^c u)^n = e_\psi(u).$$

The note is organized as follows. In Section 2 we recall some classes of plurisubharmonic functions introduced and recently investigated by Cegrell, as well as, the class $\mathcal{E}^\psi(\Omega)$. In the next section we prove some facts about the class $\mathcal{E}^\psi(\Omega)$ and give some examples. The proof of Theorem 4.1 is presented in Section 4.

2. Preliminaries

Now we recall some classes of plurisubharmonic functions on which the complex Monge-Ampère operator $(dd^c.)^n$ is well defined. These classes were introduced and investigated by U.Cegrell (see [6]-[8]).

2.1. Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain. By $\mathrm{PSH}^-(\Omega)$ we denote the class of negative plurisubharmonic functions on Ω. Put

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \{u \in \mathrm{PSH}^-(\Omega) \cap \mathrm{L}^\infty(\Omega) : \lim_{z\to\partial\Omega} u(z) = 0, \quad \int\limits_\Omega (dd^c u)^n < \infty\},$$

and

$$\mathcal{F} = \mathcal{F}(\Omega) = \big\{u \in \mathrm{PSH}^-(\Omega) : \exists \ \ \mathcal{E}_0 \ni u_j \searrow u, \ \ \sup_j \int\limits_\Omega (dd^c u_j)^n < \infty\big\}.$$

For each $p > 0$ we define

$$\mathcal{E}_p = \mathcal{E}_p(\Omega) = \big\{u \in \mathrm{PSH}^-(\Omega) : \exists \ \ \mathcal{E}_0 \ni u_j \searrow u, \ \ \sup_j \int\limits_\Omega (-u_j)^p (dd^c u_j)^n < \infty\big\}$$

and

$$\mathcal{E} = \mathcal{E}(\Omega) = \big\{u \in \mathrm{PSH}^-(\Omega) : \forall z_0 \in \Omega, \exists \ \text{ a neighborhood } \ \omega \ni z_0, \ \text{ and } \ \mathcal{E}_0 \ni u_j \searrow u \ \text{ on } \ \omega, \sup_j \int\limits_\Omega (dd^c u_j)^n < \infty\big\}.$$

As in [7], Cegrell proved that

$$\mathcal{E} = \mathcal{E}(\Omega) = \big\{u \in \mathrm{PSH}^-(\Omega) : \ \forall K \Subset \Omega \ \exists u_K \in \mathcal{F}(\Omega) \ \text{such that} \ u_K = u \ \text{on} \ K\big\}.$$

From the above definitions, it is easy to see that

$$\mathcal{E}_0(\Omega) \subset \mathcal{F}(\Omega) \subset \mathcal{E}(\Omega)$$

and

$$\mathcal{E}_0(\Omega) \subset \mathcal{E}_p(\Omega) \subset \mathcal{E}(\Omega).$$

Moreover, Lemma 2.9 in [1] implies that if $u \in \mathcal{F}(\Omega)$, then

$$\int\limits_\Omega (dd^c u)^n < \infty,$$

and Corollary 3.5 there shows that $u^*(\xi) = \limsup\limits_{\Omega\ni z\to\xi} u(z) = 0, \quad \forall \xi \in \partial\Omega$.

2.2. We recall the class $\mathcal{N}(\Omega)$ introduced in [9]. Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and $\{\Omega_j\}_{j\geq 1}$ an increasing sequence of strictly pseudoconvex subsets Ω_j

of Ω such that $\Omega_j \Subset \Omega_{j+1}$ and $\bigcup\limits_{j=1}^{\infty} \Omega_j = \Omega$. Let $\varphi \in \mathrm{PSH}^-(\Omega)$. For each $j \geq 1$, put

$$\varphi^j = \sup\{u : u \in \mathrm{PSH}^-(\Omega), \quad u \leq \varphi \ \text{ on } \ \Omega \setminus \bar{\Omega}_j\}.$$

As in [9], the function $\widetilde{\varphi} = \left(\lim\limits_{j\to\infty} \varphi^j\right)^* \in \mathrm{PSH}(\Omega)$ and $\widetilde{\varphi} \in \mathcal{MPSH}(\Omega)$. Set

$$\mathcal{N} = \mathcal{N}(\Omega) = \{\varphi \in \mathcal{E} : \widetilde{\varphi} = 0\},$$

or equivalently,

$$\mathcal{N} = \mathcal{N}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \varphi^j \uparrow 0\}.$$

It is easy to see $\mathcal{E}_0 \subset \mathcal{F} \subset \mathcal{N}$.

2.3. We deal with the class $\mathcal{E}_\chi$ introduced and recently investigated in [5]. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing function. By $\mathcal{E}_\chi$ we denote the set of all functions $u \in \mathrm{PSH}^-(\Omega)$ for which there exists a sequence $\{u_j\} \subset \mathcal{E}_0$ decreasing to u in Ω and satisfying

$$\sup_{j\geq 1} \int\limits_{\Omega} -\chi(u_j)(dd^c u_j)^n < \infty.$$

This definition clearly contains the classes of U.Cegrell:

- $\mathcal{E}_\chi(\Omega) = \mathcal{F}(\Omega)$ if χ is bounded and $\chi(0) \neq 0$.
- $\mathcal{E}_\chi(\Omega) = \mathcal{E}_p(\Omega), p > 0$, if $\chi(t) = -(-t)^p$.

2.4. Now we introduce a new class of plurisubharmonic functions defined as follows. Let $\psi \in \mathrm{PSH}^-(\Omega), \quad \psi \not\equiv 0$. According to ideas of Cegrell in [8], we denote by $\mathcal{E}^\psi(\Omega)$ the set

$$\mathcal{E}^\psi = \mathcal{E}^\psi(\Omega) = \big\{u \in \mathrm{PSH}^-(\Omega) : \ \exists \ \mathcal{E}_0 \ni u_j \searrow u, \ \sup_{j\geq 1} \int\limits_{\Omega} (-\psi)(dd^c u_j)^n < +\infty\big\}.$$

Proposition 3.1 in [8] implies that $\mathcal{E}^\psi \subset \mathcal{E}$ and, hence, the complex Monge - Ampère operator is well defined on this class.

2.5. Remark. In the above definition 2.4 we may replace the sequence $\mathcal{E}_0 \supset \{u_j\} \searrow u$ satisfying the condition $\sup\limits_{j\geq 1} \int\limits_{\Omega} (-\psi)(dd^c u_j)^n < +\infty$, by a sequence $\mathcal{F} \supset \{u_j\} \searrow u$ with the same condition. Indeed, let $\mathcal{F} \supset \{u_j\} \searrow u$ with $\sup\limits_{j\geq 1} \int\limits_{\Omega} (-\psi)(dd^c u_j)^n < +\infty$. Take $\varphi \in \mathcal{E}_0$ and put $v_j = \max(u_j, j\varphi)$. Then $v_j \in \mathcal{E}_0$ and $v_j \searrow u$. On the other hand, since $v_j \geq u_j$ and using the comparison principle, we have

$$\int\limits_{\Omega} -\psi(dd^c v_j)^n \leq \int\limits_{\Omega} -\psi(dd^c u_j)^n$$

and the desired conclusion follows.

2.6. About the relation between the class $\mathcal{E}_\chi(\Omega)$ and $\mathcal{E}^\psi(\Omega)$ we may say that they are quite different. First we note that the definitions of the two $\mathcal{E}^\psi(\Omega)$ and $\mathcal{E}_\chi(\Omega)$ are different. Moreover, in Remark 3.6 below, we show that there exist plurisubharmonic functions $\psi \in \mathrm{PSH}^-(\Omega)$ such that $\mathcal{E}^\psi(\Omega) \not\subset \mathcal{F} \cup \Big[\bigcup\limits_{p>0} \mathcal{E}_p(\Omega)\Big]$. Also in Corollary 3.3 in [15] it is proved that if $\chi(t) < 0$ for all $t < 0$ then the class $\mathcal{E}_\chi(\Omega) \subset \mathcal{N}(\Omega)$ and, hence, for all $u \in \mathcal{E}_\chi$ with such χ we have $u^*(\xi) = \lim\limits_{\Omega \ni z \to \xi} \sup u(z) = 0,\ \forall\ \xi \in \partial\Omega$. However, in [16] P.H.Hiep constructed a function $u \in \mathcal{E}^\psi(\Omega_1 \times \Omega_2)$ for some $\psi \in \mathrm{PSH}^-(\Omega_1 \times \Omega_2), \psi \not\equiv 0$ with $u^*|_{\partial\Omega_1 \times \Omega_2 \cup \Omega_1 \times \partial\Omega_2} < 0$ (see Proposition 3.4 in [16]).

2.7. We recall the definition of the pluricomplex Green function $g_{\Omega,a}$ on a domain $\Omega \subset \mathbb{C}^n$. Let $\Omega \subset \mathbb{C}^n$ and $a \in \Omega$. The pluricomplex Green function with a pole at a is defined by

$$g_{\Omega,a}(z) = \sup\{u(z) : u \in \mathrm{PSH}^-(\Omega), u(z) - \log\|z - a\| \le 0(1), \quad \text{as} \quad z \to a\}.$$

From the definition of the function $g_{\Omega,a}$ and the definition of the class $\mathcal{F}(\Omega)$ in 2.1, it is easy to see that $g_{\Omega,a} \in \mathcal{F}(\Omega)$ for every hyperconvex domain $\Omega \subset \mathbb{C}^n$.

We need the following results which will be used in Section 3.

Proposition 2.8. *Let Ω be a hyperconvex domain and $\psi \in PSH^-(\Omega)$. Assume that $u_1, \ldots, u_m \in \mathcal{F}(\Omega)$. Then the following inequality holds*

$$\sqrt[n]{\int\limits_\Omega -\psi(dd^c(u_1 + \cdots + u_m))^n} \le \sqrt[n]{\int\limits_\Omega -\psi(dd^c u_1)^n} + \cdots + \sqrt[n]{\int\limits_\Omega -\psi(dd^c u_m)^n}. \tag{2.1}$$

Proof. From Theorem 5.5 in [7] it follows that if $u_1, \ldots, u_n \in \mathcal{F}(\Omega)$ and $\psi \in \mathrm{PSH}^-(\Omega)$ then

$$\begin{aligned} &\int\limits_\Omega -\psi dd^c u_1 \wedge \ldots \wedge dd^c u_n \\ &\le \Big(\int\limits_\Omega -\psi(dd^c u_1)^n\Big)^{\frac{1}{n}} \cdots \Big(\int\limits_\Omega -\psi(dd^c u_n)^n\Big)^{\frac{1}{n}}. \end{aligned} \tag{2.2}$$

We prove (2.1) by induction on m. First we check (2.1) holds when $m = 2$. We have

$$\begin{aligned} &\int\limits_\Omega -\psi\Big(dd^c(u_1 + u_2)\Big)^n \\ = &\int\limits_\Omega -\psi \sum_{k=1}^n \binom{n}{k} (dd^c u_1)^k \wedge (dd^c u_2)^{n-k} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \binom{n}{k} \int_{\Omega} -\psi (dd^c u_1)^k \wedge (dd^c u_2)^{n-k} \\
&\le \sum_{k=1}^{n} \binom{n}{k} \Big[\int_{\Omega} -\psi (dd^c u_1)^n\Big]^{\frac{k}{n}} \Big[\int_{\Omega} -\psi (dd^c u_2)^n\Big]^{\frac{n-k}{n}} \\
&= \left(\sqrt[n]{\int_{\Omega} -\psi (dd^c u_1)^n} + \sqrt[n]{\int_{\Omega} -\psi (dd^c u_2)^n} \right)^n,
\end{aligned}
$$

where the third inequality follows from (2.2).

Hence, (2.1) holds for $m = 2$. Assume that (2.1) holds for some m. We have to prove (2.1) for $m+1$. Assume that $u_1, \dots, u_{m+1} \in \mathcal{F}$. Then $u_1 + \cdots + u_m \in \mathcal{F}$. Applying the case $m = 2$ we have

$$
\begin{aligned}
&\sqrt[n]{\int_{\Omega} -\psi (dd^c (u_1 + \cdots + u_{m+1}))^n} \\
&= \sqrt[n]{\int_{\Omega} -\psi (dd^c (u_1 + \cdots + u_m + u_{m+1}))^n} \\
&\le \sqrt[n]{\int_{\Omega} -\psi (dd^c (u_1 + \cdots + u_m))^n} + \sqrt[n]{\int_{\Omega} -\psi (dd^c u_{m+1})^n} \\
&\le \sqrt[n]{\int_{\Omega} -\psi (dd^c u_1)^n} + \cdots + \sqrt[n]{\int_{\Omega} -\psi (dd^c u_m)^n} + \sqrt[n]{\int_{\Omega} -\psi (dd^c u_{m+1})^n}.
\end{aligned}
$$

and we are done. ■

Proposition 2.9. *Let $u, v \in \mathcal{N}$ with $u \le v$. Then for $\psi \in PSH^-(\Omega)$ satisfying $\int_{\Omega} -\psi (dd^c u)^n < \infty$ the inequality holds*

$$
\int_{\Omega} -\psi (dd^c u)^n \ge \int_{\Omega} -\psi (dd^c v)^n. \tag{2.3}
$$

Proof. The proof relies on Lemma 3.3 in [2]. Indeed, Lemma 3.3 in [2] implies that

$$
\int_{\Omega} -\psi (dd^c u)^n \ge \int_{\Omega} -\psi (dd^c v) \wedge (dd^c u)^{n-1}. \tag{2.4}
$$

Again once more applying Lemma 3.3 on the right hand side of (2.4) we have

$$\begin{aligned}\int_{\Omega} -\psi(dd^c v)\wedge(dd^c u)^{n-1} &= \int_{\Omega} -\psi(dd^c u)\wedge(dd^c v)\wedge(dd^c u)^{n-2}\\ &\geq \int_{\Omega} -\psi(dd^c v)\wedge(dd^c v)\wedge(dd^c u)^{n-2}\\ &= \int_{\Omega} -\psi(dd^c v)^2\wedge(dd^c u)^{n-2}.\end{aligned}$$

Continuing this process we get the desired conclusion. ■

3. Some results on the class $\mathcal{E}^{\psi}$

In this section we give some results on the class $\mathcal{E}^{\psi}$. Proofs of these results follow from [7].

Proposition 3.1. *Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain and $\psi \in PSH^{-}(\Omega), \psi \neq 0$. Then*

a) *$\mathcal{E}^{\psi}$ is a convex cone;*

b) *If $u, v \in \mathcal{E}^{\psi}$ then $\max(u, v) \in \mathcal{E}^{\psi}$.*

Proposition 3.2. *Let $\psi_1, \psi_2 \in PSH^{-}(\Omega), \psi_1 \neq 0, \psi_2 \neq 0$ and $\psi_1 \leq \psi_2$ on Ω. Then $\mathcal{E}^{\psi_1} \subset \mathcal{E}^{\psi_2}$. Consequently, $\mathcal{F} \subset \mathcal{E}^{h^*_{K,\Omega}}$ for all compact subsets $K \subset \Omega$, where $h^*_{K,\Omega}$ denotes the relatively extremal function of the pair (K, Ω):*

$$h_{K,\Omega}(z) = \sup\{v(z) : v \in PSH^{-}(\Omega), v\,|_K \leq -1\}$$

and $h^*_{K,\Omega}$ denotes the upper semicontinuous regularization of $h_{K,\Omega}$.

Next we have the following.

Proposition 3.3. *Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain and $\psi \in PSH^{-}(\Omega),\ \psi \neq 0$. Assume that $u \in PSH^{-}(\Omega)$. If $u \in \mathcal{E}^{\psi}$ then $\int_{\Omega}(-\psi)(dd^c u)^n < +\infty$. Conversely, assume that $u \in \mathcal{N}$ and $\int_{\Omega}(-\psi)(dd^c u)^n < +\infty$ then $u \in \mathcal{E}^{\psi}$.*

Proof. Necessity. Let $u \in \mathcal{E}^{\psi}$. There exists a sequence $\{u_j\} \subset \mathcal{E}_0,\ \ u_j \searrow u$ such that

$$\sup_j \int_{\Omega}(-\psi)(dd^c u_j)^n < +\infty.$$

Since $u \in \mathcal{E}$ then $(dd^c u_j)^n$ weakly converges to $(dd^c u)^n$. On the other hand, because $-\psi$ is lower semi-continuous, then

$$\int_{\Omega}(-\psi)(dd^c u)^n \leq \liminf_j \int_{\Omega}(-\psi)(dd^c u_j)^n < +\infty,$$

and the desired conclusion follows.

Sufficiency. Assume that $u \in \mathcal{N}$ and $\int\limits_{\Omega}(-\psi)(dd^c u)^n < +\infty$. From Theorem 2.1 in [7] it follows that we can choose a decreasing sequence $\{u_j\} \subset \mathcal{E}_0$ such that $\lim\limits_j u_j = u$ on Ω. Since $u_j \geq u$ and $u_j, u \in \mathcal{N}$ then the inequality (2.3) implies that

$$\int\limits_{\Omega}(-\psi)(dd^c u_j)^n \leq \int\limits_{\Omega}(-\psi)(dd^c u)^n,$$

and we are done. ■

Proposition 3.4. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and $\psi \in PSH^-(\Omega), \psi \not\equiv 0$. Then*

a) *The following statements are equivalent:*

(i) $\psi \in \mathcal{L}^\infty(\Omega)$;

(ii) $\int\limits_{\Omega} -\psi(dd^c u)^n < \infty$ *for all* $u \in \mathcal{F}(\Omega)$;

(iii) $\exists\ C > 0$ *such that* $\int\limits_{\Omega} -\psi(dd^c u)^n \leq C\int\limits_{\Omega}(dd^c u)^n$ *for all* $u \in \mathcal{F}(\Omega)$;

(iv) $\sup\{\int\limits_{\Omega} -\psi(dd^c u)^n : u \in \mathcal{F}(\Omega),\ \ \int\limits_{\Omega}(dd^c u)^n \leq 1\} < \infty$.

b) *The following statements are equivalent:*

(i) $\exists\ A > B > 0$ *such that* $-B \leq \psi(z) \leq -A$ *for* $z \in \Omega$;

(ii) $\mathcal{E}_\psi = \mathcal{F}$.

Proof. a) (i)⇒ (ii). Since $u \in \mathcal{F}(\Omega)$ it follows that $\int\limits_{\Omega}(dd^c u)^n < \infty$. Hence, (i)⇒ (ii) is clear.

(ii)⇒ (iii). To get a contradiction we assume that (iii) does not hold. Then there exists a sequence $u_k \in \mathcal{F}(\Omega)$ such that

$$\int\limits_{\Omega} -\psi(dd^c u_k)^n > 2^{kn}\int\limits_{\Omega}(dd^c u_k)^n \quad \forall\ k \geq 1. \tag{3.1}$$

Note that $u_k \not\equiv 0$. Put $v_k = \dfrac{1}{2^k}\dfrac{u_k}{\sqrt[n]{\int\limits_{\Omega}(dd^c u_k)^n}}$. Then $v_k \in \mathcal{F}(\Omega)$. We have

$$\int\limits_{\Omega}(dd^c v_k)^n = \frac{1}{2^{kn}} \quad \forall\ k \geq 1 \tag{3.2}$$

and by (3.1)

$$\int\limits_{\Omega} -\psi(dd^c v_k)^n = \frac{\int\limits_{\Omega} -\psi(dd^c u_k)^n}{2^{kn}\int\limits_{\Omega}(dd^c u_k)^n} > 1.$$

Let $v = \sum\limits_{k=1}^{\infty} v_k$. Since $\mathcal{F} \ni \sum\limits_{j=1}^{k} v_j \searrow v$ and by Proposition 2.8 we have

$$\sqrt[n]{\int_{\Omega} (dd^c(\sum_{j=1}^{k} v_j)^n} \leq \sum_{j=1}^{k} \sqrt[n]{\int_{\Omega} (dd^c v_j)^n} = \sum_{j=1}^{k} \frac{1}{2^j} \leq 1,$$

it follows that $v \in \mathcal{F}(\Omega)$. We have

$$(dd^c v)^n \geq \sum_{k=1}^{\infty} (dd^c v_k)^n$$

and, hence, by the hypothesis (ii)

$$+\infty > \int_{\Omega} -\psi(dd^c v)^n \geq \sum_{k=1}^{\infty} \int_{\Omega} -\psi(dd^c v_k)^n > \sum_{k=1}^{\infty} 1 = +\infty.$$

A contradiction!.

(iii)$\Rightarrow$ (iv) is clear.

(iv)$\Rightarrow$ (i). To get a contradiction we assume that ψ is not bounded on Ω. Then there exists a sequence $\{z_k\} \subset \Omega$ such that $\psi(z_k) \to -\infty$, as $k \to +\infty$. The Green's function $g_{\Omega, z_k}(z) \in \mathcal{F}(\Omega)$ and

$$\int_{\Omega} \Big(dd^c \frac{g_{\Omega, z_k}}{2\pi} \Big)^n = 1.$$

Hence, by the hypothesis $\sup\limits_{k} \int\limits_{\Omega} -\psi \Big(dd^c \frac{g_{\Omega, z_k}}{2\pi} \Big)^n < +\infty$. This is absurd, because

$$\sup_{k} \int_{\Omega} (-\psi) \Big(dd^c \frac{g_{\Omega, z_k}}{2\pi} \Big)^n = \sup_{k} \frac{1}{(2\pi)^n} (-\psi(z_k)) = +\infty.$$

b) (i)$\Rightarrow$ (ii). Assume that $u \in \mathcal{F}(\Omega)$. Then there exits $\{u_j\} \in \mathcal{E}_0(\Omega), \quad u_j \searrow u$ and

$$\sup_{j} \int_{\Omega} (dd^c u_j)^n < \infty.$$

Then

$$\int_{\Omega} -\psi(dd^c u_j)^n \leq B \int_{\Omega} (dd^c u_j)^n < \infty \ \text{ for all } \ j \geq 1.$$

Hence, $u \in \mathcal{E}^{\psi}(\Omega)$. Similarly, it is easy to prove if $u \in \mathcal{E}^{\psi}$, then $u \in \mathcal{F}$. Thus $\mathcal{F} = \mathcal{E}^{\psi}$.

(ii)$\Rightarrow$ (i). First we show that ψ is bounded below. Using a) it suffices to check $\int_\Omega -\psi(dd^c u)^n < \infty$ for all $u \in \mathcal{F}$. Indeed, let $u \in \mathcal{F}$. By the hypothesis, $u \in \mathcal{E}^\psi$. Hence, Proposition 3.3 implies that $\int_\Omega -\psi(dd^c u)^n < \infty$ and the conclusion follows. Next we have to prove $\sup_\Omega \psi < 0$. Seeking a contradiction we assume that $\sup_\Omega \psi = 0$. There exists a sequence $\{z_k\} \subset \Omega$ with $\psi(z_k) \to 0$ as $k \to \infty$. We may assume that $|\psi(z_k)| \leq \frac{1}{(2\pi)^n 2^{kn}}$ for all $k \geq 1$. Let $g_k = g_{\Omega,z_k}$ be the Green's function with pole at z_k. Then we have

$$\int_\Omega -\psi(dd^c g_k)^n = (2\pi)^n(-\psi(z_k)) = (2\pi)^n|\psi(z_k)| \leq \frac{1}{2^{kn}}.$$

Put $g = \sum_{k=1}^{\infty} g_k$. We prove $g \in \mathcal{E}^\psi$ but $g \notin \mathcal{F}$ and we get a contradiction. Indeed, since $g_k \in \mathcal{F}$ then it follows that $\mathcal{F} \ni \sum_{j=1}^{k} g_j \searrow g$. Proposition 2.8 implies that

$$\begin{aligned}\sqrt[n]{\int_\Omega -\psi\Big(dd^c(\sum_{j=1}^k g_j)\Big)^n} &\leq \sum_{j=1}^k \sqrt[n]{\int_\Omega -\psi(dd^c g_j)^n}\\ &\leq \sum_{j=1}^k \frac{1}{2^j} \leq 1.\end{aligned}$$

Hence, Remark 2.5 implies that $g \in \mathcal{E}^\psi$. However $g \notin \mathcal{F}$ because

$$\begin{aligned}\int_\Omega (dd^c g)^n &\geq \sum_{k=1}^{\infty} \int_\Omega (dd^c g_k)^n\\ &\geq (2\pi)^n \sum 1 = +\infty.\end{aligned}$$

Proposition 3.4 is completely proved. ■

As is well known, if $u \in \mathcal{F}(\Omega)$ then $u^*|_{\partial\Omega} = 0$ and

$$\int_\Omega (dd^c u)^n < +\infty.$$

However, the following result, contrary to the above result, shows that there exists $u \in \mathcal{E}^\psi$ with $u^*|_{\partial\Omega} = 0$ such that

$$\int_\Omega (dd^c u)^n = +\infty.$$

Example 3.5. There exists $u \in \mathcal{E}^\psi$ for some $\psi \in \mathrm{PSH}^-(\Omega)$ such that $u|_{\partial\Omega} = 0$ but

$$\int_\Omega (dd^c u)^n = +\infty.$$

Indeed, let $\Omega = \mathbb{B} = \mathbb{B}(0,1) = \{z \in \mathbb{C}^n : \ \|z\| < 1\}$ be the unit ball in $\mathbb{C}^n$. Take $\psi(z) = \log\|z\| \in \mathcal{F}(\mathbb{B})$. For each $k \geq 1$, put

$$\psi_k(z) = \max\{\log\|z\|, -k\}.$$

Then

$$\int_\Omega (dd^c \psi_k)^n = (2\pi)^n.$$

Moreover, $\psi_k \in \mathcal{E}_0(\Omega)$, $\ \psi_k \searrow \psi$ as $k \to \infty$. For each $j = 1, 2, \cdots$, let $u_j(z) = \max\{\log\|z\|, -\frac{1}{2^j}\}$. Then $u_j \in \mathcal{E}_0(\Omega)$ and

$$\int_\Omega (-\psi)(dd^c u_j)^n \leq \frac{1}{2^j}(2\pi)^n \longrightarrow 0$$

as $j \to \infty$.

We construct a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$\int_\Omega (-\psi)\big(dd^c(\sum_{j=1}^k u_{n_j})\big)^n < (2\pi)^n \quad \text{for all} \quad k \geq 1.$$

We choose $n_1 = 1$. Then

$$\int_\Omega (-\psi)(dd^c u_{n_1})^n = \int_\Omega (-\psi)(dd^c u_1)^n \leq \frac{1}{2}(2\pi)^n < (2\pi)^n.$$

Assume that $n_1 < n_1 < \cdots < n_k$ are chosen. Then

$$\int_\Omega (-\psi)\big(dd^c(\sum_{j=1}^k u_{n_j})\big)^n < (2\pi)^n.$$

Let $v = \sum_{j=1}^k u_{n_j}$. For $m > n_k$, $\ q \geq 1$ and $j = 0, 1, \cdots, n-1$ we have

$$\begin{aligned}\int_\Omega (-\psi_q)(dd^c v)^j \wedge (dd^c u_m)^{n-j} &\leq \Big(\int_\Omega (-\psi_q)(dd^c v)^n\Big)^{\frac{j}{n}} \Big(\int_\Omega (-\psi_q)(dd^c u_m)^n\Big)^{\frac{n-j}{n}} \\ &\leq \Big(\int_\Omega (-\psi)(dd^c v)^n\Big)^{\frac{j}{n}} \Big(\int_\Omega (-\psi)(dd^c u_m)^n\Big)^{1-\frac{j}{n}}\end{aligned}$$

Letting $q \to \infty$ we get

$$\int\limits_{\Omega} (-\psi)(dd^c v)^j \wedge (dd^c u_m)^{n-j} \leq (2\pi)^j \Big[\frac{(2\pi)^n}{2^m}\Big]^{1-\frac{j}{n}} \longrightarrow 0 \quad \text{as} \quad m \to +\infty.$$

However,

$$\int\limits_{\Omega} (-\psi)\big(dd^c(v+u_m)\big)^n = \sum_{j=0}^{n} \binom{n}{j} \int\limits_{\Omega} (-\psi)(dd^c v)^j \wedge (dd^c u_m)^{n-j}$$
$$= \int\limits_{\Omega} (-\psi)(dd^c v)^n + \sum_{j=0}^{n-1} \mathrm{C}_j^n \int\limits_{\Omega} (-\psi)(dd^c v)^j \wedge (dd^c u_m)^{n-j}.$$

From the inductive hypothesis and the above estimation, we have

$$\int\limits_{\Omega} (-\psi)(dd^c v)^n < (2\pi)^n$$

and

$$\sum_{j=0}^{n-1} \binom{n}{j} \int\limits_{\Omega} (-\psi)(dd^c v)^j \wedge (dd^c u_m)^{n-j} \longrightarrow 0 \quad \text{as} \quad m \to +\infty.$$

Hence, we can choose $m = n_{k+1}$ large enough such that

$$\int\limits_{\Omega} (-\psi)\big(dd^c(\sum_{j=1}^{k+1} u_{n_j})\big)^n = \int\limits_{\Omega} (-\psi)\big(dd^c(v + u_{n_{k+1}})\big)^n < (2\pi)^n.$$

Set

$$u = \sum_{j=1}^{\infty} u_{n_j} = \sum_{j=1}^{\infty} \max\big(\log\|z\|, -\frac{1}{2^{n_j}}\big).$$

Then $v_k = \sum\limits_{j=1}^{k} u_{n_j} \searrow u$ on Ω and from the above construction we infer that $u \in \mathcal{E}^{\psi}, \;\; u|_{\partial\Omega} = 0$. However, we have

$$\int\limits_{\Omega} (dd^c u)^n \geq \sum_{j=1}^{k} \int\limits_{\Omega} (dd^c u_{n_j})^n = k \cdot (2\pi)^n \longrightarrow +\infty \quad \text{as} \quad k \to +\infty.$$

Remark 3.6. For u and ψ as in Example 3.5, let $g_{\Omega,a}$ be the Green's function with pole at $a \neq 0$. Then $g_{\Omega,a} \in \mathcal{F}(\mathbb{B})$ and by Proposition 3.3, $g_{\Omega,a} \in \mathcal{E}^{\psi}$. Then the function $v = u + g_{\Omega,a} \in \mathcal{E}^{\psi}$ but $v \notin \mathcal{F} \bigcup \bigcup\limits_{p>0} \mathcal{E}_p$. Indeed, if $v \in \mathcal{F}$, then $\max\{u, v\} = u \in \mathcal{F}$, which is impossible because $\int\limits_{\Omega} (dd^c u)^n = +\infty$. Moreover, since $(dd^c v)^n \geq (dd^c g_{\Omega,a})^n = (2\pi)^n \delta_a$, it implies that $v \notin \mathcal{E}_p$ for all $p > 0$

because if $v \in \mathcal{E}_p$ then $(dd^c v)^n$ vanishes on every pluripolar set. This shows that the class $\mathcal{E}^{\psi}$ is quite different from the class $\mathcal{F}$ and $\mathcal{E}_p, p > 0$.

4. Subextension in the class $\mathcal{E}^{\psi}$

In this section we deal with subextension in the class $\mathcal{E}^{\psi}$. As in Example 3.5 we note that the class $\mathcal{E}^{\psi}$ is quite different from the classes $\mathcal{F}$ and $\mathcal{E}_p$. Hence, studying the subextension problem in the class $\mathcal{E}^{\psi}$ is possible. Namely we prove the following.

Theorem 4.1. *Let $\Omega \subset \widetilde{\Omega} \Subset \mathbb{C}^n$ be hyperconvex domains, $\psi \in PSH^-(\widetilde{\Omega}), \psi \not\equiv 0$ and $u \in \mathcal{E}^{\psi}(\Omega)$. Then there exists $\widetilde{u} \in \mathcal{E}^{\psi}(\widetilde{\Omega})$ such that $\widetilde{u} \le u$ on Ω and*

$$e_{\psi}(\widetilde{u}) = \int\limits_{\widetilde{\Omega}} (-\psi)(dd^c\widetilde{u})^n \le \int\limits_{\Omega} (-\psi)(dd^c u)^n = e_{\psi}(u).$$

Proof. From the definition of $\mathcal{E}^{\psi}$ it follows that there exists $\{u_j\} \subset \mathcal{E}_0(\Omega),\ \ u_j \searrow u$ on Ω and

$$\alpha = \sup_j \int\limits_{\Omega} (-\psi)(dd^c u_j)^n < +\infty.$$

Moreover, as in the proof of Proposition 3.3 we may assume that

$$e_{\psi}(u) = \lim_j \int\limits_{\Omega} (-\psi)(dd^c u_j)^n.$$

Fix $j \ge 1$. Since $u_j \in \mathcal{E}_0(\Omega)$ then by the definition of the class $\mathcal{E}_0(\Omega)$ in **2.1** it follows that $(dd^c u_j)^n(\Omega) < +\infty$. Hence the measure $\mu_j = 1_{\Omega}(dd^c u_j)^n$ is a Borel measure in $\widetilde{\Omega}$ which is finite and puts no mass on pluripolar sets of $\widetilde{\Omega}$ (see [3]), where 1_{Ω} denotes the characteristic function of the set Ω. Lemma 5.14 in [7] implies that there exists an unique function $g_j \in \mathcal{F}(\widetilde{\Omega})$ with $(dd^c g_j)^n = 1_{\Omega}(dd^c u_j)^n$ as a measure on $\widetilde{\Omega}$. We claim that $g_j \le u_j$ on Ω. Theorem 5.11 in [7] implies that there exist $\psi_j \in \mathcal{E}_0(\widetilde{\Omega}),\ \ 0 \le f_j \in \mathrm{L}^1(\widetilde{\Omega}, (dd^c\psi_j)^n)$ such that

$$\mu_j = f_j(dd^c\psi_j)^n$$

as a measure on $\widetilde{\Omega}$. Let $\{\Omega_k\}_{k\ge1}$ be an increasing exhaustion sequence of relatively compact open sets of Ω, $\Omega_k \Subset \Omega_{k+1} \Subset \Omega$, $\bigcup\limits_k \Omega_k = \Omega$. For each $k \ge 1$, let

$$\mu_{j,k} = 1_{\Omega_k} \inf\{k, f_j\}(dd^c\psi_j)^n$$

be a Borel measure on $\widetilde{\Omega}$. Theorem (C) in [18] implies that there exists a function $u_{j,k} \in \mathcal{E}_0(\Omega)$ such that

$$(dd^c u_{j,k})^n = 1_{\Omega_k} \inf\{k, f_j\}(dd^c\psi_j)^n$$

on Ω and $g_{j,k} \in \mathcal{E}_0(\widetilde{\Omega})$ such that

$$(dd^c g_{j,k})^n = 1_{\Omega_k} \inf\{k, f_j\}(dd^c \psi_j)^n$$

on $\widetilde{\Omega}$. Indeed, the existence of $g_{j,k}$ is clear. In order to prove the existence of $u_{j,k}$ we use the following argument. For each k consider the function

$$\widetilde{\psi}_{j,k} = \sup\{\varphi \in \mathrm{PSH}^-(\Omega) : \varphi \leq \psi_j \quad \text{on} \quad \Omega_k\}.$$

Then $\psi_j \leq \widetilde{\psi}_{j,k}$ on Ω, and $\widetilde{\psi}_{j,k} = \psi_j$ on Ω_k. Note that $\widetilde{\psi}_{j,k} \in \mathcal{E}_0(\Omega)$ because $\widetilde{\psi}_{j,k} \geq \|\psi_j\|_{\mathcal{L}^\infty(\Omega_k)} h_{\Omega_j,\Omega}$ where $\|\psi_j\|_{\mathcal{L}^\infty(\Omega_k)} = \sup\{|\psi_j(z)| : z \in \Omega_k\}$ and $h_{\Omega_j,\Omega}$ is the relatively extremal function of the pair (Ω_k, Ω). Now we have

$$1_{\Omega_k} \min\{k, f_j\}(dd^c \psi_j)^n = 1_{\Omega_k} \min\{k, f_j\}(dd^c \widetilde{\psi}_{j,k})^n \leq k(dd^c \widetilde{\psi}_{j,k})^n$$

and the desired conclusion follows. From the comparison theorem it follows that the sequences $\{u_{j,k}\}_k$ and $\{g_{j,k}\}_k$ are decreasing sequences of plurisubharmonic functions in the classes $\mathcal{E}_0(\Omega)$ and $\mathcal{E}_0(\widetilde{\Omega})$ respectively. On the other hand, if $\xi \in \partial\Omega$, then we have

$$\liminf_{\Omega \ni z \to \xi} (u_{j,k}(z) - g_{j,k}(z)) = -g_{j,k}(\xi) \geq 0$$

and

$$(dd^c u_{j,k})^n \leq (dd^c g_{j,k})^n$$

on Ω. By the comparison principle it follows that

$$g_{j,k} \leq u_{j,k}$$

on Ω. Assume that $g_{j,k} \searrow h_j$ as $k \to \infty$. Then $h_j \in \mathcal{F}(\widetilde{\Omega})$. Indeed, we have

$$\sup_k \int_{\widetilde{\Omega}} (dd^c g_{j,k})^n \leq \int_{\widetilde{\Omega}} f_j 1_\Omega (dd^c \psi_j)^n = \int_{\Omega} f_j (dd^c \psi_j)^n \leq \int_{\widetilde{\Omega}} f_j (dd^c \psi_j)^n < +\infty,$$

and the desired conclusion follows. Moreover,

$$(dd^c h_j)^n = f_j 1_\Omega (dd^c \psi_j)^n = 1_\Omega \mu_j = 1_\Omega (dd^c u_j)^n = (dd^c g_j)^n,$$

and Lemma 5.14 in [7] implies that $h_j = g_j$ on $\widetilde{\Omega}$. Similarly, $u_{j,k} \searrow u_j$ as $k \to \infty$ on Ω. For each $j \geq 1$, put

$$\widetilde{g}_j = [\sup_{k \geq j} g_k]^*.$$

Then $\widetilde{g}_j \in \mathrm{PSH}^-(\widetilde{\Omega})$, and from $\widetilde{g}_j \geq g_j$ it follows that $\widetilde{g}_j \in \mathcal{F}(\widetilde{\Omega})$. It is easy to see that $\{\widetilde{g}_j\}$ is decreasing. On the other hand, from $\widetilde{g}_j \geq g_j$ on $\widetilde{\Omega}$ then by integration by parts, it follows that

$$\int_{\widetilde{\Omega}}(-\psi)(dd^c\widetilde{g}_j)^n \leq \int_{\widetilde{\Omega}}(-\psi)(dd^c g_j)^n = \int_{\widetilde{\Omega}}(-\psi)1_{\Omega}(dd^c u_j)^n = \int_{\Omega}(-\psi)(dd^c u_j)^n.$$

Hence,

$$\sup_j \int_{\widetilde{\Omega}}(-\psi)(dd^c\widetilde{g}_j)^n \leq \alpha < +\infty.$$

For $k \geq j$ we notice that $g_k \leq u_k \leq u_j$ on Ω. Thus

$$\widetilde{g}_j = [\sup_{k\geq j} g_k]^* \leq u_j$$

on Ω. Put $\widetilde{u} = \lim_j \widetilde{g}_j$. We prove that $\widetilde{u} \in \mathcal{E}^{\psi}(\widetilde{\Omega})$. Obviously, $\widetilde{u} \in \mathrm{PSH}^-(\Omega)$. By [7] there exists a sequence $\{\varphi_j\} \subset \mathcal{E}_0(\widetilde{\Omega})$ such that $\varphi_j \searrow \widetilde{u}$ on $\widetilde{\Omega}$. By replacing φ_j by $\widetilde{u}_j = \max\{\varphi_j, \widetilde{g}_j\} \in \mathcal{E}_0(\widetilde{\Omega})$ and $\widetilde{u}_j \searrow \widetilde{u}$ we may assume that $\varphi_j \geq \widetilde{g}_j$ on $\widetilde{\Omega}$. Then

$$\int_{\widetilde{\Omega}}(-\psi)(dd^c\varphi_j)^n \leq \int_{\widetilde{\Omega}}(-\psi)(dd^c\widetilde{g}_j)^n \leq \int_{\Omega}(-\psi)(dd^c u_j)^n \leq \alpha < +\infty.$$

Thus

$$\sup_j \int_{\widetilde{\Omega}}(-\psi)(dd^c\varphi_j)^n < +\infty,$$

and, hence, $\widetilde{u} \in \mathcal{E}^{\psi}(\widetilde{\Omega})$. Since $\widetilde{g}_j \leq u_j$ on Ω, it follows that $\widetilde{u} \leq u$ on Ω. On the other hand,

$$e_{\psi}(\widetilde{u}) = \int_{\widetilde{\Omega}}(-\psi)(dd^c\widetilde{u})^n \leq \liminf_j \int_{\widetilde{\Omega}}(-\psi)(dd^c\varphi_j)^n \leq \liminf_j \int_{\Omega}(-\psi)(dd^c u_j)^n = e_{\psi}(u).$$

Theorem 4.1 is completely proved. ■

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