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On the Reinhardt Conjecture ⋆

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Dedicated to Professor Hà Huy Khoái on the occasion of his 65th-birthday

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Abstract. In 1934, Reinhardt asked for the centrally symmetric convex domain in the plane whose best lattice packing has the lowest density. He conjectured that the unique solution up to an affine transformation is the smoothed octagon (an octagon rounded at corners by arcs of hyperbolas). This article offers a detailed strategy of proof. In particular, we show that the problem is an instance of the classical problem of Bolza in the calculus of variations. A minimizing solution is known to exist. The boundary of every minimizer is a differentiable curve with Lipschitz continuous derivative. If a minimizer is piecewise analytic, then it is a smoothed polygon (a polygon rounded at corners by arcs of hyperbolas). To complete the proof of the Reinhardt conjecture, the assumption of piecewise analyticity must be removed, and the conclusion of smoothed polygon must be strengthened to smoothed octagon.

1. Introduction

A contract requires a miser to make payment with a tray of identical gold coins filling the tray as densely as possible. The contract stipulates the coins to be convex and centrally symmetric. What shape coin should the miser choose in order to part with as little gold as possible?

Let K be a centrally symmetric convex domain in the Euclidean plane. If Λ is a lattice such that the translates of K under Λ have disjoint interiors, then

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the packing density of $\Lambda + K$ is the ratio of the area of K to the co-area of the lattice Λ. Let $\delta(K)$ be the maximum density of any lattice packing of K. A lattice realizing this density exists for each K.

Let $\delta_{\min}$ be the infimum of $\delta(K)$, as K ranges over all convex domains in the Euclidean plane. Reinhardt proves that there exists K for which $\delta(K) = \delta_{\min}$. Reinhardt's problem is to determine the constant $\delta_{\min}$ and to describe K explicitly for which $\delta(K) = \delta_{\min}$.

Reinhardt conjectured that $\delta(K) = \delta_{\min}$ when K is a smoothed octagon. The smoothed octagon is constructed by taking a regular octagon and clipping the corners with hyperbolic arcs. The hyperbolic arcs are chosen so that the smoothed octagon has no corners; that is, so that there is a unique tangent at each point of the boundary. The asymptotes of each hyperbola are lines extending two sides of the regular octagon. The density of the smoothed octagon is

$$\delta(K) = \frac{8 - \sqrt{32} - \ln 2}{\sqrt{8} - 1} \approx 0.902414.$$

Reinhardt's original article contains many useful facts about his conjecture. The main facts from his article have been summarized in Section 2.. Beyond Reinhardt's article, various lower bounds for $\delta_{\min}$ have been published. See [6], [8], [9], [10], [3], [15], [1]. The special case of centrally symmetric decagons is considered in [7]. Nazarov has proved the local optimality of the smoothed octagon [11]. The problem is discussed further in [12], where it is referred to as a "famous conjecture." It is also known that non-lattice packings of centrally symmetric convex domains in the plane cannot have greater density than $\delta(K)$ [2]. The Ulam conjecture, which is the corresponding conjecture in three dimensions, posits the sphere as solution [4].

In this article we take the following approach. The boundary of any optimal K is a C^1 curve. We express the boundary of K in the calculus of variations. We will see that the problem is a special instance of the classical problem of Bolza. The circle is the unique solution to the Euler-Lagrange equations (up to an affine transformation), but an examination of second-order conditions shows that the circle does not minimize. This means that optimal K is not an interior point of the configuration space.

This leads to a study of the constraint that K must be convex. Piecewise analytic solutions have a well-defined (local) invariant, called the rank. Calculus of variations further reduces the problem to the study of rank one. We show that rank-one solutions K are structurally similar to the smoothed octagon. In particular, K is a smoothed polygon, whose boundary consists of finitely many linear segments connected by hyperbolic arcs. We propose a nonlinear optimization problem in a small number of variables over certain rank-one configurations. The successful solution of this nonlinear optimization problem (and eliminating the assumption of piecewise analyticity) would complete the proof of the Reinhardt conjecture.

2. Reinhardt's article

We give a brief review a series of lemmas in Reinhardt's original article. The proofs are generally elementary.

2.1. Balanced hexagons

Definition 2.1 (balanced hexagon). We call a centrally symmetric hexagon G a *balanced hexagon* of a centrally symmetric convex domain K if G contains K and if the midpoint of each side of G is a point on the boundary of K. These six points on the boundary of K are called the midpoints of G.

Lemma 2.2. *Let G be a balanced hexagon of a centrally symmetric convex domain K without corners. Then G does not degenerate to a parallelogram. That is, the six vertices are distinct* [14].

Lemma 2.3. *Let K be a centrally symmetric convex domain. Each point of K is a midpoint of at most one balanced hexagon* [14, p.228].

Lemma 2.4. *Let G be a balanced hexagon of a centrally symmetric convex domain K. Assume that the center of symmetry of K is the origin. Let u_j, $j \in \mathbb{Z}/6\mathbb{Z}$, be the midpoints of the sides of G listed in cyclic order around the hexagon. Then*

- $u_j + u_{j+2} + u_{j+4} = 0$;
- $u_{j+3} = -u_j$;
- *The area of G is $4/3$ the area of the hexagon H formed by the convex hull of $\{u_j\}$;*
- *The six segments from the origin to the six midpoints u_j breaks H into six congruent triangles. In particular, the area of G is 8 times the area of the triangle $\{0, u_j, u_{j+2}\}$.*

Proof. If the vertices of the centrally symmetric hexagon G are w_j, with $w_{j+3} = -w_j$, then

$$u_j = (w_j + w_{j+1})/2.$$

The first two statements are then immediate. The other statements appear in [14, p.219,p.222]. ■

2.2. Miserly domains

Lemma 2.5. *There exists a centrally symmetric convex domain K for which $\delta(K) = \delta_{\min}$.*

Proof. This follows by Blaschke's selection lemma [14, p.220]. ■

Definition 2.6 (miserly domain). Any centrally symmetric convex domain K that realizes the lower bound $\delta(K) = \delta_{\min}$ is called a *miserly domain.*

Lemma 2.7. *If K is a miserly domain, then it has no corners. That is, there is a unique tangent through each point on the boundary.* [14, p.221]

Lemma 2.8. *Let K be a centrally symmetric domain without corners. Assume that for each point p on the boundary of K there exists a balanced hexagon G_p on which p is a midpoint. Assume further that the area of G_p is independent of p. Then K has no other balanced hexagons and*

$$\delta(K) = \text{area}(K)/\text{area}(G) \tag{1}$$

for every balanced hexagon G of K. Moreover, if K is any miserly domain, then it satisfies the given assumptions of this lemma.

Proof. The facts asserted without proof in this proof appear in [14, pp.219–222]. Let K be a centrally symmetric convex domain in the plane. Let G be a smallest centrally symmetric hexagon that contains K. Such a hexagon exists, and $\delta(K)$ equals the ratio of the area of K to that of G. Call any such hexagon a *fitting* hexagon. Every fitting hexagon of K is a balanced hexagon. By Lemma 2.3, there are no balanced hexagons other than the G_p. Hence $G = G_p$ for some p. The first part of the lemma now follows.

Now let K be a miserly domain. Reinhardt proves that each boundary point p of K lies on a balanced hexagon that is also a fitting hexagon of K, although p is not necessarily a midpoint of the balanced hexagon. Next, he shows that each boundary point of K is in fact the midpoint of a balanced hexagon that is also a fitting hexagon of K. Since there are no other balanced hexagons, there are no other fitting hexagons. The set of balanced hexagons coincides with the set of fitting hexagons. All fitting hexagons have the same area. Thus, the assumptions of the first part of the lemma are all satisfied for a miserly domain. ■

3. The boundary curve

3.1. Hexameral domains. If we combine the properties of miserly domains that were established by Reinhardt, we are led to the following definition.

Definition 3.1 (hexameral domain). We say that K is a hexameral domain if the following conditions hold.

- K is a centrally symmetric domain whose center of symmetry is the origin;
- K has no corners;
- Each point on the boundary of K is a midpoint of a balanced hexagon G. Moreover, these balanced hexagons all have the same area.

By the preceding lemmas, if K is a miserly domain, then (after recentering at the origin) it is a hexameral domain. The packing density $\delta(K)$ of a hexameral domain is computed by Formula (1) and Lemma 2.4. The smoothed octagon and the circle are examples of hexameral domains. The class of hexameral domains is much larger than the class of miserly domains. We consider the optimization problem of determining the miserly domains within the class of hexameral domains. If K is a hexameral domain, then each point of the boundary is a midpoint of a *unique* balanced hexagon.

Let K be a hexameral domain. Give a continuous parametrization $t \mapsto \sigma_0(t)$ of the boundary curve. We follow the convention of parametrizing the boundary in a counterclockwise direction. Since K has no corners, we may assume that σ_0 is C^1. At each time t, there is a uniquely determined balanced hexagon with midpoint $\sigma_0(t)$. Let the other midpoints be listed in (counterclockwise) order as $\sigma_j(t)$, $j \in \mathbb{Z}/6\mathbb{Z}$.

If u and v are ordered pairs of real numbers, write $u \wedge v$ for the 2×2 determinant with columns u and v.

Lemma 3.2. *Let K be a hexameral domain with C^1 boundary parametrization σ_0. Then the curves σ_2 and σ_4 are also C^1 parametrizations of the boundary, oriented in the same way as σ_0.*

Proof. Reinhardt shows that the boundary parametrizations σ_2, σ_4 are continuous if σ_0 is continuous, and that they are oriented in the same way as σ_0 [14, p.222]. Let us check that σ_2 is C^1, whenever σ_0 is. Since K has no corners, the unit tangent $n_2(t)$ to $\sigma_2(t)$, with the orientation given by σ_0, is a continuous function of t. It is enough to check that the speed of σ_2 is continuous in t. By Lemma 2.4, $\sigma_0(t) \wedge \sigma_2(t)$ is a fixed fraction of the area of the balanced hexagon, and does not depend on t.

We claim that $\sigma_0(t) \wedge n_2(t) \neq 0$. Let $H(t)$ be the hexagon given by the convex hull of $\{\sigma_j(t)\}$. If $\sigma_0(t) \wedge n_2(t) = 0$, then the tangent line to σ_2 at t contains the edge of $H(t)$ through $\sigma_2(t)$ and $\sigma_1(t)$. Then also, $\sigma_0'(t) \wedge \sigma_2(t) = 0$ and the tangent line to σ_0 lies along another edge of $H(t)$. This forces a corner at $\sigma_1(t)$, which is contrary to Lemma 2.7.

This nonvanishing result and the fact that $\sigma_0(t) \wedge \sigma_2(t)$ is independent of t imply that there exists a function $v_2 : \mathbb{R} \to \mathbb{R}$ such that

$$\sigma_0'(t) \wedge \sigma_2(t) + \sigma_0(t) \wedge n_2(t) v_2(t) = 0. \tag{2}$$

The function $v_2(t)$ is the speed, and from the form of this equation, it is necessarily continuous in t. ∎

3.2. Multi-curve. There is no harm in rescaling a hexameral domain so that its balanced hexagon has area $\sqrt{12}$, which is the area of a regular hexagon of inradius 1. For this normalization, Lemma 2.4 gives

$$\sigma_j(t) \wedge \sigma_{j+2}(t) = \sqrt{3}/2. \tag{3}$$

This suggests the following definition.

Definition 3.3 (multi-point, multi-curve). A function $u : \mathbb{Z}/6\mathbb{Z} \to \mathbb{R}^2$ such that

$$u_j + u_{j+2} + u_{j+4} = 0, \quad u_{j+3} = -u_j, \quad u_j \wedge u_{j+2} = \sqrt{3}/2 \tag{4}$$

is called a *multi-point*. An indexed set of C^1 curves

$$\sigma : \mathbb{Z}/6\mathbb{Z} \times [t_0, t_1] \to \mathbb{R}^2$$

is a *multi-curve* if for all $t \in [t_0, t_1]$, $\sigma(t)$ is a multi-point. That is,

- $\sigma_j(t) + \sigma_{j+2}(t) + \sigma_{j+4}(t) = 0$,
- $\sigma_{j+3}(t) = -\sigma_j(t)$,
- $\sigma_j(t) \wedge \sigma_{j+2}(t) = \sqrt{3}/2$.

By differentiation, a multi-curve also satisfies for all j:

$$\sigma_j(t) \wedge \sigma'_{j+2}(t) + \sigma'_j(t) \wedge \sigma_{j+2}(t) = 0. \tag{5}$$

The boundary of a hexameral domain admits a parametrization as a triple curve. The converse does not hold because a multi-curve has no convexity constraint and no constraint for the curves σ_j to fit seamlessly into a simple closed curve containing the origin in the interior.

3.3. Lipschitz continuity.

Lemma 3.4. *Let K be a miserly domain and let σ_j be a multi-curve parametrization on the boundary of K. Assume that σ_0 is parametrized by arclength s. Then σ'_0 is Lipschitz continuous.*

Proof. For each s, let H_s be the hyperbola through $\sigma_0(s)$ whose asymptotes are the lines in direction $\sigma'_j(s)$ through $\sigma_j(s)$, for $j = \pm 1$. By Reinhardt [14, p.220], near $\sigma_0(s)$, the an arc of H_s lies inside K. As s varies, by continuity over the compact boundary, the curvatures of the hyperbolas H_s at $\sigma_0(s)$ are bounded above by some $\kappa \in \mathbb{R}$. This means that a disk of fixed curvature κ can be placed locally in K at each point $\sigma_0(s)$ so that $\sigma'_0(s)$ is tangent to the disk. The curve σ_0 is constrained on the other side by convexity, so that σ_0 is wedged between the tangent line to σ_0 at s and the disk.

If we parametrize the curve by arclength, then $\sigma'_0(s)$ has unit length. Lipschitz continuity now follows from this bound κ on the curvature. ∎

Lemma 3.5. *Let K be a miserly domain and let σ_j be a multi-curve parametrization on the boundary of K. Assume that σ_0 is parametrized by arclength s. Then σ'_j is Lipschitz continuous for all j.*

Proof. By evident symmetries, it is enough to consider $j = 2$. Let t be the arclength parameter for the curve σ_0 and let s be the arclength parameter for the curve σ_2. We consider s as a function of t. By Lemma 3.2, the function s is C^1.

We show that ds/dt is Lipschitz continuous function of t. In fact, $ds/dt = v_2(t)$, given by (2). The coefficient $\sigma_0(t) \wedge n_2(t)$ of $v_2(t)$ in (2) is nonzero and by continuity is bounded away from 0. Thus, the Lipschitz continuity of v_2 follows from the Lipschitz continuity of the other functions σ_0', σ_2, σ_0, and n_2 in that equation.

Now we show that σ_2' is a Lipschitz continuous function of t. Write

$$\sigma_2'(t) = \frac{d\sigma_2}{ds}\frac{ds}{dt}.$$

The first term on the right is Lipschitz continuous by Lemma 3.4. The second term on the right has just been shown to be Lipschitz continuous. Hence the result. ■

Since σ_j' is Lipschitz continuous, Rademacher's theorem implies that σ_j' is differentiable almost everywhere (or directly we have that the angular argument of σ_j' is monotonic, hence differentiable almost everywhere). Thus, we may express the convexity constraint locally at $\sigma_j(t)$ by a second derivative:

$$\sigma_j'(t) \wedge \sigma_j''(t) \geq 0. \tag{6}$$

3.4. Special linear group. The special linear group $SL_2(\mathbb{R})$ acts on $\mathbb{R}^2$ by linear transformations and preserves the wedge product:

$$gu \wedge gv = \det(g)(u \wedge v).$$

Conversely any affine transformation fixing the origin and fixing some $u \wedge v \neq 0$ must be given by some $g \in SL_2(\mathbb{R})$.

The group $SL_2(\mathbb{R})$ acts on the data of the Reinhardt problem, on the set of miserly domains, on the set of multi-curves, and so forth.

Given a multi-curve σ and multi-point u, there exists a unique C^1 curve $\phi : [t_0, t_1] \to SL_2(\mathbb{R})$, such that

$$\sigma_j(t) = \phi(t)u_j \tag{7}$$

for $j \in \mathbb{Z}/6\mathbb{Z}$.

The transformed multi-curve $\phi(t_0)^{-1}\sigma_j$ starts at $\sigma_j(t_0) = u_j$. It is often convenient to use the multi-point formed by roots of unity:

$$u_j^* = \exp(\pi i j/3), \quad i = \sqrt{-1}. \tag{8}$$

In particular, any hexameral domain is equivalent under $SL_2(\mathbb{R})$ to a hexameral domain that starts at the multi-point u^* on the unit circle. We call this a *circle representation* of the hexameral domain or multi-curve.

Let σ be a multi-curve. Define $X : [t_0, t_1] \to \mathfrak{gl}_2(\mathbb{R})$ by

$$\sigma_j'(t) = X(t)\sigma_j(t) \quad \text{for } j = 0, 2.$$

Then also,

$$\sigma_j'(t) = X(t)\sigma_j(t) \quad \text{for } j \in \mathbb{Z}$$

and

$$\phi'(t) = X(t)\phi(t), \tag{9}$$

where ϕ is given by Equation (7). Equation (5) implies that $X(t) \in \mathfrak{sl}_2(\mathbb{R})$, the Lie algebra of $SL_2(\mathbb{R})$. The tangent lines to the curves σ_j are determined by the image of $X(t)$ in the projective space $\mathbb{P}(\mathfrak{sl}_2(\mathbb{R}))$ over the vector space $\mathfrak{sl}_2(\mathbb{R})$.

If we transform σ_j to $g\sigma_j$, for some $g \in SL_2(\mathbb{R})$, then $X(t)$ transforms to $\operatorname{Ad} g\, X = gX(t)g^{-1} \in \mathfrak{sl}_2(\mathbb{R})$. By Lemma 3.5, if σ_0 is parametrized by arclength, then X is Lipschitz continuous.

We have seen that the parametrized boundary of a hexameral domain determines a curve in $SL_2(\mathbb{R})$. Conversely, a curve in SL_2 determines a hexameral domain in the following sense.

Lemma 3.6. *Let K be a centrally symmetric convex domain (with center of symmetry 0) with a multi-point u on the boundary. Let $\phi : [t_0, t_1] \mapsto SL_2(\mathbb{R})$ be a C^1 curve. Define curves σ_j by Equation (7). Assume that σ_j parametrizes the boundary of K, for $j \in \mathbb{Z}/6\mathbb{Z}$. Then K is a hexameral domain.*

Proof. We check the balanced hexagon condition. At time t, let $w_j(t)$ be the point of intersection of the tangent line to $\sigma_j(t)$ with the tangent line to $\sigma_{j+1}(t)$. The condition that $w_j(t)$ are the vertices of a balanced hexagon generates a system of six linear equations and three unknowns. Consistency of this system of equations imposes three constraints. The first two constraints are expressed as a single vector equation with two components:

$$\begin{aligned} 0 &= \sigma_0(t) + \sigma_2(t) + \sigma_4(t), \\ 0 &= \sigma_0(t) \wedge \sigma_2'(t) + \sigma_0'(t) \wedge \sigma_2(t). \end{aligned}$$

Integrating the final constraint, gives that $\sigma_0(t) \wedge \sigma_2(t)$ is constant. These conditions hold for a curve coming from ϕ. Thus, solving for $w_j(t)$, we find that each point of the simple closed curve is a midpoint of a balanced hexagon with vertices $w_j(t)$. Since $\sigma_0(t) \wedge \sigma_2(t)$ is constant, these balanced hexagons have the same area. Thus, K is a hexameral domain. ■

3.5. Star conditions. The convexity of a hexameral domain places certain constraints on the tangent $X \in \mathfrak{sl}_2(\mathbb{R})$. We normalize the curve by applying an affine transformation so that $\phi(0) = I$ in the circle representation. This means that the roots of unity u_j^* lie on the boundary of the hexameral domain. We form a hexagram through these six points. Specifically, we construct the six equilateral triangles, each with three vertices:

$$u_j^*, \quad u_{j+1}^*, \quad (u_j^* + u_{j+1}^*).$$

For the boundary curve to be convex, the tangent direction Xu_j^* at time $t = 0$ must lie between the the secant lines joining u_j^* with $u_{j\pm1}^*$, hence must point into this triangle for each j. If we write

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

we have the following constraints on X:

$$\sqrt{3}|a| < c, \quad 3b + c < 0. \tag{10}$$

Lemma 3.7. *In this context,*

$$\det(X) > 0.$$

Proof.

$$\det(X) = -bc - a^2 \geq -bc - \frac{c^2}{3} = \frac{-c(3b+c)}{3} > 0.$$

■

4. Rank of multi-curves

Definition 4.1 (rank). Let σ be a multi-curve. We say that it has *well-defined rank* if the multi-curve is C^2, parametrized by $[t_0, t_1]$, with the property that for each curve σ_j one of the two conditions hold:

- It is a line segment;
- The curvature of σ_j is nonzero on the open interval (t_0, t_1).

The *rank* of such a multi-curve is the number $k \in \{0, 1, 2, 3\}$ of curves σ_j ($j \in 2\mathbb{Z}/6\mathbb{Z}$) that are *not* line segments.

For example, a multi-curve parametrizing a circle has rank 3. The smoothed octagon is parametrized by finitely many multi-curves of rank 1.

Lemma 4.2. *Let $\phi : [t_0, t_1] \to SL_2(\mathbb{R})$ be twice differntiable at t. Then there exists a j such that the curvature constraint* (6) *is a strict inequality at t.*

Proof. Without of loss of generality, we may take $t = 0$ and may apply an affine transformation, so that the boundary is given by ϕu_j^*, with $\phi(0) = I$. Let $\phi' u_j^* = X\phi u_j^*$. The constraint (6) becomes

$$\phi' u_j^* \wedge \phi'' u_j^* = X u_j^* \wedge (X' + X^2) u_j^*.$$

A short calculation assuming the vanishing of this curvature inequality for $j = 0, 2$ gives for $j = 4$:

$$X u_4^* \wedge (X' + X^2) u_4^* = \frac{3\sqrt{3}(a^2+bc)^2}{3a+\sqrt{3}c}.$$

The star conditions in Section 3. imply that the numerator and denominator are both positive. ∎

Lemma 4.3. *No multi-curve has rank* 0*.*

Proof. This is a corollary of Lemma 4.2. A direct proof can be given as follows. The tangent lines to a multi-curve, by the argument in the proof of Lemma 3.6, determines a balanced hexagon. If the rank is zero, the tangent lines, the balanced hexagon, and its midpoints are fixed. Thus, the curve degenerates to a stationary curve at the fixed midpoints. ∎

It is natural to consider hexameral domains whose boundary is parametrized by a finite number of analytic multi-curves.

Lemma 4.4. *Suppose that a hexameral domain K has a parametrization by a finite number of analytic multi-curves σ. Then K also admits a parametrization by a finite number of triple curves satisfying the hypotheses of Definition 4.1, each admitting a well-defined rank.*

Proof. The curvature of an analytic curve vanishes identically, or has at most finitely many zeroes on a compact interval $[t_0, t_1]$. Subdividing the intervals at the finitely many zeroes, we may assume that the only zeroes appear at the endpoints of the intervals. ∎

5. Rank three and the ellipse

The following sections analyze the multi-curves according to rank, starting with rank three in this section. The primary method will be the calculus of variations to search for a curve $\phi(t)$ in $SL_2(\mathbb{R})$ that minimizes the area of a hexameral domain K.

5.1. First variation. We consider a curve $\phi : [t_0, t_1] \to SL_2(\mathbb{R})$. Form corresponding curves $\sigma_j(t) = \phi(t)u_j$ and $\sigma_{j+3}(t) = -\sigma_j(t)$, for $j \in \mathbb{Z}/6\mathbb{Z}$ and u_j satisfying Conditions (4). Consider the closed curve that follows the line segment from $(0,0)$ to $\sigma_j(t_0)$, the curve $\sigma_j(t)$ from $t_0 \le t \le t_1$, and then the line segment from $\sigma_j(t_1)$ to $(0,0)$. Assume that this closed curve is simple, and let I_j be the area enclosed by the curve. Set

$$I(\phi) = \sum_{j=0}^{5} I_j.$$

Let

$$\phi(t) = \phi(t_0) \cdot \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}.$$

If we express the integrals I_j in terms of ϕ, a short calculation gives

$$I(\phi) = 3 \int_{t_0}^{t_1} (\alpha d\gamma - \gamma d\alpha) + (\beta d\delta - \delta d\beta). \tag{11}$$

Lemma 5.1. *Let K be a miserly domain. Pick a C^1 parametrization $\phi(t) \in SL_2(\mathbb{R})$ of the boundary of K. Suppose that for all $t \in [t_0, t_1]$, the multi-curve associated with the curve ϕ has a well-defined rank 3. Then the first variation of $I(\phi)$ (with fixed boundary conditions) vanishes.*

Proof. Assume for a contradiction that the first variation is nonzero. Under the conditions necessary for the rank to be defined, on any compact interval $[s_0, s_1]$ with $t_0 < s_0 < s_1 < t_1$, the curvatures of the curves σ_j are bounded away from zero. Thus, a sufficiently small C^∞-variation of the functional preserves the convexity condition. We can assume the small variation gives a simple curve. By Lemma 3.6, the small variation is again the boundary of a hexameral domain. If the first variation is nonzero, the area of can be decreased, holding the area of the balanced hexagon constant. By Lemma 2.8, K is not a miserly domain. ■

Basic results about variations now imply that the Euler-Lagrange equations must hold on (t_0, t_1). As we are working in $SL_2(\mathbb{R})$, we may take the variation of the form

$$\exp(\epsilon X(t)) \cdot \phi(t) \tag{12}$$

for some curve

$$X(t) = \begin{pmatrix} u(t) & w(t) + v(t) \\ w(t) - v(t) & -u(t) \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}), \tag{13}$$

the Lie algebra of $SL_2(\mathbb{R})$. The Euler-Lagrange equations are

$$\begin{aligned} 0 &= (\delta^2 + \gamma^2)' \\ 0 &= (\alpha^2 + \beta^2)' \\ 0 &= (\gamma\alpha + \delta\beta)' \end{aligned}$$

with boundary conditions

$$\begin{aligned} 1 &= \alpha(t_0) = \delta(t_0), \\ 0 &= \beta(t_0) = \gamma(t_0) \end{aligned}$$

Integrating, we have

$$\begin{aligned} 1 &= \delta^2 + \gamma^2 \\ 1 &= \alpha^2 + \beta^2 \\ 0 &= \gamma\alpha + \delta\beta. \end{aligned}$$

We also have the determinant condition $\alpha\delta - \beta\gamma = 1$. These are the defining conditions of a special orthogonal matrix. Hence, there is a function $\theta : [t_0, t_1] \to \mathbb{R}$ such that

$$\phi(t) = \phi(t_0) \cdot \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}.$$

Thus, the curves σ_j trace out arcs of an ellipse. We summarize in the following lemma.

Lemma 5.2. *Let K be a miserly domain. Suppose that some portion of the boundary has well-defined rank* 3. *Then up to a special linear transformation, that portion of the boundary consists of three arcs of a unit circle.*

5.2. Second variation. Next we study the second variation of the area functional $I(\phi)$. For this, we may confine our attention to the unit circle:

$$\phi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Again, we consider variations of the form of Equations (12) and (13). A calculation of the second variation around ϕ gives

$$\int_{t_0}^{t_1} 4u(t)w'(t)dt.$$

This does not have fixed sign. Therefore, the solution to the Euler-Lagrange equations is a saddle point. It follows that an arc of a circle cannot form part of a miserly domain. This gives the following result.

Theorem 5.3. *Let K be a miserly domain. Then there is no segment of the boundary parametrized by a multi-curve of rank three.*

6. Rank two

In this section, we consider the variation of a Reinhardt curve of well-defined rank two. We prove that the first variation is never zero in the rank two situation. This leads to the following theorem.

Theorem 6.1. *Let K be a miserly domain. Then no segment of the boundary is parametrized by a rank two multi-curve. In fact, the first variation in area, along a rank-preserving variation, is always non-zero* (*on the space of multi-curves*).

Proof. By applying a special linear transformation, we may assume that σ_0 moves along the line $y = -1$. At any given time t, there is a skew transformation

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

that makes the first coordinates of $\sigma_1(t)$ and $\sigma_2(t)$ equal. The vertices of the midpoint hexagon (after this transformation) are

$$\pm(r-z,-1), \quad \pm(r,s), \quad \pm(r-z,1),$$

with $r > z > 0$, and $-1 < s < 1$. This hexagon has area $\sqrt{12}$ provided

$$z = 2r - 2\rho,$$

where $\rho = \sqrt{3}/2$. We regard r as a function of z through this relation. We may solve for the midpoints of this hexagon, then reapply the skew transformation to obtain the coordinates of σ_j. We then have

$$\sigma_1(t) = \tfrac{1}{2}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}2\rho\\ -1+s\end{pmatrix},$$
$$\sigma_2(t) = \tfrac{1}{2}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}2\rho\\ +1+s\end{pmatrix}.$$

The condition that σ is a multi-curve forces the tangent to σ_2 to be an edge of the midpoint hexagon:

$$x'(s^2-1) - s'z = 0. \tag{14}$$

If $s'(t) = 0$, then $x'(t) = 0$ and the curve σ_1 is not regular at t, which is contrary to the definition of a multi-curve. Thus, s' is everywhere nonzero, and we can define z in terms of x and s by the relation (14). We can pick the sign of s' so that it is positive. We have $z > 0$. Then x' is everywhere negative.

We let I_j be the area bounded by the three curves: the segment from 0 to $\sigma_j(t_0)$, the curve σ_j on $[t_0, t_1]$, and the segment from $\sigma_j(t_1)$ to 0. A short calculation gives

$$I_1 + I_2 = \int_{t_0}^{t_1} \frac{1}{4}(\sqrt{3}s' - (1+s^2)x')\,dt.$$

The integral I_0 is not relevant for a compactly supported variation since σ_0 remains linear under any small rank-preserving variation, and I_0 remains constant. The first variation of this integral in x clearly does not vanish, under the established condition $s' > 0$. ∎

7. Hyperbolic links (rank one)

We have shown that the boundary of a miserly domain cannot contain multi-curves of rank $\neq 1$. This section analyses triple curves of rank one.

7.1. Square representation. Let K be a hexameral domain with boundary parametrized by a multi-curve σ. Suppose that for some fixed index j, the curves σ_{j+2} and σ_{j+4} travel along straight lines. By applying a special linear transformation, we may assume that σ_{j+2} moves along $x = a$ and σ_{j+4} moves along $y = a$, for some $a > 0$. By reparametrizing the curves, we may assume that $\sigma_{j+2}(t) = a(1,t)$ and $\sigma_{j+4}(t) = a(s(t),1)$, for some function s. The fixed area condition on balanced hexagons,

$$\sigma_{j+2}(t) \wedge \sigma_{j+4}(t) = \frac{\sqrt{3}}{2}, \tag{15}$$

gives $1 - ts = k$, where $k = \sqrt{3}/(2a^2) > 0$. This determines the function s. The condition $\sigma_j = -\sigma_{j+2} - \sigma_{j+4}$ gives $\sigma_j(t) = a(-1-s, -1-t)$. The curve σ_j traces the hyperbola $(x+a)(y+a) = a^2(1-k)$, whose asymptotes are lines $x = -a$ and $y = -a$ containing the curves σ_{j+5} and σ_{j+1}. (This calculation shows why hyperbolic arcs play a special role.) The curve traced by σ_j does not form the arc of a convex region containing the origin unless

$$0 < k < 1, \quad s < 0, \quad t < 0. \tag{16}$$

We assume this condition. Also, $k < 1$ implies $a^2 > \sqrt{3}/2$. The balanced hexagon G degenerates to a quadrilateral when $s \le -1$ or $t \le -1$. We therefore assume that

$$-1 < s < 0, \qquad -1 < t < 0. \tag{17}$$

This implies that

$$-1 < s < -ts = k - 1, \qquad -1 < t < k - 1. \tag{18}$$

The parameter t thus ranges over an interval $[t_0, t_1]$ with $-1 < t_0 < t_1 < k - 1$. The parameter s runs over $[s_0, s_1]$ with $s_i = (1-k)/t_i$. We may write

$$t_1 = t_0 + \tau(k - 1 - t_0), \tag{19}$$

for some $\tau \in (0, 1)$.

In summary, up to a special linear transformation, and reparametrization of the curve, a rank one multi-curve is uniquely determined by the index j of the hyperbolic arc, the initial parameters (a, t_0), and the terminal parameter τ, where

$$a > \sqrt{3}/2, \quad k = \sqrt{3}/(2a^2), \quad -1 < t_0 < k - 1, \quad 0 < \tau < 1. \tag{20}$$

We call this the *square representation* of the multi-curve. The starting and terminal point s_i for the curve $\sigma_{j+4}(t_i) = a(s_i, 1)$ and the constant k are determined by Equation (3). The curve σ_j, is determined by σ_{j+2} and σ_{j+4}.

Conversely, if we are given three parameters a, t_0, τ satisfying the conditions (20), and given the hyperbolic index j, there exists a multi-curve whose square representation has these parameters. This gives us a convenient way to construct multi-curves.

7.2. Area. Let $I_j(a, t_0, t_1)$ be the area bounded by the line segment from $(0,0)$ to $\sigma_j(t_0)$, the curve σ_j, $t_0 \le t \le t_1$, and the line segment from $\sigma_j(t_1)$ to $(0,0)$. An easy calculation gives

$$\begin{aligned} I(a, t_0, t_1) &= a^2((s_0 - s_1) + (t_1 - t_0) - (1-k)\ln(s_1/s_0)) \\ &= a^2((1-k)(1/t_0 - 1/t_1) + (t_1 - t_0) - (1-k)\ln(t_0/t_1)), \end{aligned} \tag{21}$$

where $I = I_0 + I_2 + I_4$, and the parameters s_1, s_2, k are given in terms of a, t_0, t_1 as above.

For example, the boundary of the smoothed octagon is parametrized by eight multi-curves (one for each hyperbolically rounded corner of the octagon). The parameters for each of the eight multi-curves of the smoothed octagon are

$$\begin{aligned} a &= \frac{12^{1/4}}{\sqrt{4-\sqrt{2}}}, \\ t_0 &= -1/\sqrt{2}, \\ t_1 &= -1/2, \\ I &= \frac{\sqrt{3}\left(8 - 8\sqrt{2} + \sqrt{2}\log(2)\right)}{4\left(-4+\sqrt{2}\right)}. \end{aligned}$$

This gives density

$$8I/\sqrt{12} \approx 0.902414,$$

mentioned in the introduction to this article.

7.3. The set of initial states. The initial state for a multi-curve is specified by a matrix $\phi(t_0) \in SL_2(\mathbb{R})$ and a velocity $X(t_0) \in \mathfrak{sl}_2(\mathbb{R})$, given by Equation (9). We wish to allow reparametrizations of the curve, so that the velocity is given only up to a scalar, giving a point in projective space: $[X(t_0)] \in \mathbb{P}(\mathfrak{sl}_2(\mathbb{R}))$. The space of initial states then has dimension five:

$$\dim S = 3 + 2, \qquad \text{where } S = SL_2(\mathbb{R}) \times \mathbb{P}(\mathfrak{sl}_2(\mathbb{R})).$$

These five dimensions correspond to the three dimensional group of transformations that can be used to transform a rank-one multi-curve to its square representation, together with the two parameters (a, t_0) giving the initial state in the square representation.

Likewise, the terminal state for a multi-curve is given by a point in the same five dimensional space.

7.4. Hyperbolic chains and smoothed polygons

Definition 7.1. An analytic multi-curve of rank one is called a *hyperbolic link* (because the of hyperbolic arc σ_j). A piecewise analytic multi-curve of rank one is called a *hyperbolic chain.* A hexameral domain K whose boundary is a hyperbolic chain is is called a smoothed polygon.

We are now in a position to state the main result of this article.

Theorem 7.2. *Let K be a miserly domain. If the boundary of K is piecewise analytic, then K is a smoothed polygon.*

Proof. In earlier sections, we have ruled out the existence of segments on the boundary of ranks zero, two, or three. Thus, it must consist of segments of rank one. ■

We consider a multi-curve that consists of a finite number of rank one triples, joined one to another to form C^1 curves. There is no variational problem here, because there are no functional degrees of freedom for segments of rank one. When an initial state for a hyperbolic link is fixed, there is exactly one degree of freedom, the parameter $\tau \in (0,1)$ in the square representation.

Suppose that we have a multi-curve σ consisting of a finite number of hyperbolic links. Along each hyperbolic link, exactly one of the three curves σ_j follows a hyperbolic arc; the other two are linear. Set $\Delta = 2\mathbb{Z}/6\mathbb{Z}$. We break the domain of the multi-curve into finitely many subintervals, each labeled with an index $j \in \Delta$, according to which arc σ_j is the hyperbola. The first link of σ is entirely specified by (s_1, τ_1, j_1), where $s_1 \in S$ is an initial state, $\tau_1 \in (0,1)$ determines the length of the arc, and $j_1 \in \Delta$ specifies the index of the hyperbolic arc. The triple (s_1, τ_1, j_1) determines the terminal state $s_2 \in S$ of the first hyperbolic link. Similarly, the second link of σ is determined by (s_2, τ_2, j_2), for some $\tau_2 \in (0,1)$ and $j_2 \in \Delta$. Working through the hyperbolic chain, link by link, we obtain a sequence

$$s_0 \in S, \quad ((\tau_0, j_0), (\tau_1, j_1), \ldots, (\tau_n, j_n)), \quad \tau_i \in (0,1), \quad j_i \in \Delta \tag{22}$$

that uniquely determines the hyperbolic chain.

Conversely, given a sequence of parameters (22), an induction on n shows that there is a unique hyperbolic chain with those parameters. We can extend the parameters $\tau \in (0,1)$ to include $\tau = 0$, with the understanding that this corresponds to a degenerate link consisting of a single point. If two consecutive parameters (τ_i, j) and (τ_{i+1}, j) have the same index $j \in \Delta$, then they can be combined into a single hyperbolic link. Thus, there is no loss in generality in assuming that consecutive links carry distinct hyperbolic indices j. In fact, we may insert degenerate parameters (τ, j), with $\tau = 0$ so that the parameters take the special form

$$(\tau_i, j_i), \text{ where } j_i = j_0 + 2i. \tag{23}$$

If we are given a hyperbolic chain σ with parameters (22), we may extract the square representation $(a(i), t_0(i), t_1(i))$ of each link i from these parameters. We may then sum Equation (21) over the set of links to obtain the area

$$I(\sigma) = I(s_0, ((u_0, j_0), \ldots)) = \sum_i I(a(i), t_0(i), t_1(i)) \tag{24}$$

represented by the entire chain.

7.5. Closed curves. Consider a multi-curve σ that gives the boundary of a hexameral domain. In the circle representation, write $\sigma_j(t) = \phi(t) u_j^*$. We have

$$u_{j+1}^* = \rho u_j^*, \quad \rho = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta = \pi/3.$$

Let the initial point of the curve be given at time t_0 and let $t_1 > t_0$ be the first time at which $\sigma_0(t_1) = \sigma_1(t_0)$:

$$t_1 = \min\{t \,:\, \sigma_0(t) = \sigma_1(t_0), \quad t > t_0\}. \tag{25}$$

Then for the boundary of the hexameral domain to be closed and C^1, we must have

$$\phi(t_1) = \phi(t_0)\rho \quad \text{and} \quad X(t_1) = X(t_0). \tag{26}$$

The angular argument must also satisfy

$$0 \leq \arg(\phi(t_0)^{-1}\phi(t)u_0^*) \leq \pi/3, \quad t \in [t_0, t_1]. \tag{27}$$

Given a choice u of multi-point on the boundary of K, the parameters $s \in S$ and (τ_i, j_i) are uniquely determined. Conversely, the parameters uniquely determine the boundary of K.

When we choose to do so, we may pick the starting multi-point on the boundary of K in such a way that the first parameter is

$$(\tau_0, j_0) \text{ with } j_0 = 0. \tag{28}$$

We may also arrange that the multipoint is the endpoint of a hyperbolic link.

Definition 7.3 (link length). Assume that the boundary of a hexameral domain is a hyperbolic chain that satisfies conditions (28), (23). Let t_0, t_1 be given as in (25). We may extend σ_j to a periodic function $\mathbb{R} \to \mathbb{R}^2$. We may extend the parameters (τ_i, j_i) to all of $i \in \mathbb{Z}$ with $j_i = 2i$. The curve σ_j, restricted to $[t_0, t_1]$ has parameters

$$((\tau_0, 0), (\tau_1, 2), \ldots, (\tau_n, 2n)),$$

for some n. When n is chosen to give the shortest representation of this form, we call $n+1$ the *link length* of the hexameral domain.

For example, the boundary of the smoothed octagon contains eight hyperbolic arcs, one at each corner of the octagon. They appear in centrally symmetric pairs. There exists a choice of initial multi-point such that the smoothed octagon is given by the following data:

$$(I, [X]) \in S, \quad ((\tau, 0), (\tau, 2), (\tau, 4), (\tau, 0)),$$

for some $\tau > 0$ that is independent of the link and some $X \in \mathfrak{sl}_2(\mathbb{R})$. The link length is 4.

Lemma 7.4. *Let K be a hexameral domain whose boundary is a hyperbolic chain. Let $n+1$ be the link length of K. Then*

$$n \equiv 0 \mod 3.$$

(*In particular, the smoothed octagon minimizes the link length.*)

Proof. If we consider the multi-curve σ, the link $(\tau_{n+1}, 2n+2)$ lies along the same part of the multi-curve as $(\tau_0, 0)$, so that $\tau_{n+1} = \tau_0$. However, the hyperbolic index shifts by 1 as we pass from σ_0 to σ_1. Therefore,

$$(\tau_{n+1}, 2n+2) = (\tau_0, 2) \in \mathbb{R} \times \Delta.$$

The congruence

$$2n + 2 \equiv 2 \in \Delta = 2\mathbb{Z}/6\mathbb{Z}$$

gives the result. ∎

7.6. Link reduction. With this background in place, we now return to a discussion of the Reinhardt conjecture. Lemma 7.4 shows that the conjectural solution to the Reinhardt problem minimizes link length. This leads to the intuition that decreases in $\delta(K)$ should have the effect of shortening the link length. This is the motivation for the Conjecture 7.5, which asserts that we may simultaneously decrease areas and eliminate links from some hyperbolic chains.

We consider all possible hyperbolic chains σ with the same initial and terminal states s_0, s_1. These chains are given by parameters

$$s_0, \quad ((\tau_1, j_1), \ldots).$$

Since s_1 lies in a five-dimensional space, the terminal state places five constraints on the parameters set $(\tau_1, \tau_2, \ldots)$. By counting dimensions, we might guess that for generic parameters s_0, s_1, the terminal state cuts out a set of hyperbolic chains of codimension five. Generically, it should take at least five links to match the terminal state s_1.

If, instead of fixing both endpoints, we may impose the closed curve condition (26). We may use the action of the special linear group to force $\phi(t_0) = I$. The free parameters are X and $(\tau_0, \ldots, \tau_n)$, or $n + 3$ free variables.

Conjecture 7.5 (Link Reduction). Let K be a hexameral domain with multi-curve σ around the boundary. Let t_0, t_1 be the parameters (25). Suppose that a portion $[t_0', t_1']$ of the boundary (with $t_0 \le t_0' \le t_1' \le t_1$) is a hyperbolic chain with six links, given in the form (22). (We do not assume (28), (23).) Let $s_0, s_1 \in S$ be the initial and terminal states for the curve at t_0' and t_1'. Then there is another hyperbolic chain with five links that

- fits the same initial and terminal states $s_0, s_1 \in S$,
- satisfies the angle condition (27),
- has no greater area $I(\sigma)$ as defined by (24),
- and in fact has strictly smaller area, unless the chain is already (degenerately) a five-link chain.

In other words, the conjecture claims that one of the links can be removed, decreasing the number of links to five, while simultaneously decreasing area.

The conditions on the parameters t_0, t_1, t'_0, t'_1 are there to ensure that the hyperbolic chain is short enough that the corresponding curves $\sigma_0, \ldots, \sigma_5$ parametrize distinct portions of the boundary.

The condition that the hyperbolic chain should bound part of a hexameral domain places constraints on s_0, s_1. They cannot be arbitrary states of S.

The number of parameters in the implied optimization is eight: six links and the five-dimensional initial state, reduced by the three dimensional group $SL_2(\mathbb{R})$. This conjecture may be explored by computer, but I have only done so in a very limited way.

The following is a very special case of the Reinhardt conjecture.

Conjecture 7.6 (Five Link). Let K be a hexameral domain with multi-curve σ around the boundary. Let t_0, t_1 be the parameters (25). Suppose that the entire non-repeating boundary $[t_0, t_1]$ is a hyperbolic chain with five links. Then $\delta(K) = \delta_{\min}$ exactly when K is the smoothed octagon, up to a transformation by $SL_2(\mathbb{R})$.

The smoothed octagon belongs to this family of hexameral domains, with parameters $\tau_3 = 0$ and $\tau_i = \tau_j$, if $i, j \neq 3$.

Nazarov's proof of the local optimality of the smoothed octagon gives the conjecture for hexameral domains sufficiently close to the smoothed octagon [11].

This is an optimization problem on a seven dimensional space. (There is the five dimensional initial state $s \in S$ and five links (τ_i, j_i), reduced by the action of the three-dimensional group SL_2.) For a generic choice of parameters $s, \{(\tau_i, j_i)\}$ the hyperbolic chain will not form a simple closed curve, and can be discarded.

Again, we might hope to test this conjecture by computer, and perhaps even to prove it with interval arithmetic.

Lemma 7.7. *Assume Conjectures 7.5 and 7.6. Then up to affine transformation, the smoothed octagon uniquely minimizes $\delta(K)$ over the class of all smoothed polygons K.*

Proof. The first conjecture successively reduces the number of links to five. The second conjecture treats the case of five links (which includes as degenerate cases, fewer than five links). ■

8. Bolza

Viewed as a problem in the calculus of variations or control theory, the Reinhardt problem is an instance of the classical problem of Bolza with nonholonomic inequality constraints, autonomous, fixed endpoints, and no isoperimetric constraints. Lacking isoperimetric constraints, it is an instance of the classical problem of Mayer with inequality constraints [5, Ch.7].

We have established the existence of a minimizer with Lipschitz continuous derivative (when considered as a second-order system; the minimizer itself is Lipschitz continuous when converted to a first-order system). This is sufficiently regular to match the hypotheses in standard treatments of the subject, such as [13].

We search for minimizers of the integral (11):

$$I(\phi) = 3\int_{t_0}^{t_1} (\alpha d\gamma - \gamma d\alpha) + (\beta d\delta - \delta d\beta). \tag{29}$$

over the class of curves $\phi : [t_0, t_1] \to SL_2(\mathbb{R})$ such that

- the special linear condition holds: $\alpha\delta - \beta\gamma = 1$, where

$$\phi(t) = \phi(t_0) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

- ϕ is C^1
- The derivative ϕ' is Lipschitz continuous.
- Convexity constraints hold:

$$\sigma_j'(t) \wedge \sigma_j''(t) \geq 0, \quad j = 0, 2, 4, \quad t \in [t_0, t_1] \ a.e.$$

where u_j^* is as in (8) and $\phi(t_0)\sigma_j(t) = \phi(t)u_j^*$.

The variational problem has several symmetries, which lead to conserved quantities by Noether's theorem. The convexity constraints and area are invariant under reparametrization of $\phi : [t_0, t_1] \to SL_2(\mathbb{R})$. The problem is autonomous. Finally, the group $SL_2(\mathbb{R})$ acts on the set of minimizers.

We may convert to a first order problem by setting $\phi(t_0)Y = \phi'$. The convexity constraints become linear in Y':

$$y_j \wedge y_j' \geq 0, \tag{30}$$

where $y_j = Yu_j^*$.

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