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Higher Twists and Higher Gauss Sums

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Dedicated to Professor Hà Huy Khoái on the occasion of his 65th-birthday

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Abstract. We study two versions of higher twists of a Siegel modular form F viewed as a formal Fourier expansion

$$F(\mathcal{Z}) = \sum_{\mathcal{T} \in \mathbf{S}_{(t+1)n}} a(\mathcal{T}, F) q^{\mathcal{T}}$$

by a t-tuple $\boldsymbol{\varphi}$ of Dirichlet characters $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_t)$ modulo N, where $\mathbf{S}_{(t+1)n}$ denotes the set of half-integral semi-positive symmetric matrices of size $(t+1)n \times (t+1)n$. We show that both definitions are related via shift matrix operators and higher Gauss sums. We establish a spherical property of these higher Gauss sums (Proposition 2.1), and we prove automorphy properties of the higher twists (Proposition 3.1 and Proposition 3.2). These twists belong to tensor products of certain spaces of modular forms. This construction produces certain p-adic distributions with values in such tensor products applicable in various constructions of p-adic L-functions.

Introduction

Let F be a Siegel modular form of degree of degree $(t+1)n$ viewed as a formal Fourier expansion

$$F(\mathcal{Z}) = \sum_{\mathcal{T} \in \mathbf{S}_{(t+1)n}} a(\mathcal{T}, F) q^{\mathcal{T}}, \tag{1}$$

where $\mathbf{S}_{(t+1)n}$ denotes the set of half-integral semi-positive symmetric matrices of size $(t+1)n \times (t+1)n$. We study two following versions of higher twists of F by a t-tuple $\boldsymbol{\varphi}$ of Dirichlet characters $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_t)$ modulo N as the following functions of $(\tau, Z) \in \mathbb{H}_n \times \mathbb{H}_{nt}$:

$$F^{\boldsymbol{\varphi}}(\tau, Z) := \sum_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_t)\in\mathbb{Z}^{(n,tn)}/N\mathbb{Z}^{(n,tn)}} \varphi_1(\det(\alpha_1)) \times \cdots \times \varphi_t(\det(\alpha_t)) F\left(\begin{pmatrix} \tau & \frac{\alpha_1}{N} \cdots \frac{\alpha_t}{N} \\ \frac{\alpha_1^t}{N} & \\ \cdots & Z \\ \frac{\alpha_t^t}{N} & \end{pmatrix}\right), \quad (2)$$

where α_i runs over $\mathbb{Z}^{(n,n)}$ modulo N.

Moreover, let us define

$$F^{(\boldsymbol{\varphi})}(\tau, Z) := \sum_{h\in\mathbf{S}_n, \xi\in\mathbf{S}_{tn}} q_\tau^h q_Z^\xi \times \sum_{\mathcal{T}_{h,\xi}\in\mathbf{S}_{(t+1)n}} \varphi_1(\det(h_1))\dots\varphi_t(\det(h_t)) a(\mathcal{T}_{h,\xi}, F) \quad (3)$$

(where $\mathcal{T}_{h,\xi} = \begin{pmatrix} h & h_1 \cdots h_t \\ h_1^t & \\ \cdots & \xi \\ h_t^t & \end{pmatrix}$ and the inner sum is finite!).

Later on we refer to F^{φ} and $F^{(\varphi)}$ as the first and second verison of higher twists.
Such twists occur in our work [9], [10] on triple L-functions (in particular in the section about the "first integration") in [10] and in the work of C.-G.Schmidt and the first author on standard-L-functions [8].

In order to find some further applications, we would like to relate these definitions to modular distributions [50],[53], [52], [54]. In fact the partial series as above come up naturally in certain zeta distributions and Eisenstein distributions which can be extended to a very general setting of classical groups, see [21], [25].

The main motivation to study the modular distributions is that they provide a very general tool to construct p-adic L-functions (both bounded and unbounded, see [1], [38], [39], [22], [23], [24], [45], [44], [47], [48], [60].

Both definitions are related to shift matrix operators: the first twist via the values of the characters $\varphi_1, \dots, \varphi_t$, and the second twist is expressed through matrix operators using higher Gauss sums $G(\boldsymbol{\alpha}, \boldsymbol{\varphi})$ to be explained later on. The main aim of the paper is to explore properties of these twists.

We are grateful to the referee who pointed out to us that our constructions raise the hope (in the ordinary case) that an arithmetic p-adic measure in the sense of Hida can be defined, with values in a p-adic tensor product of Katz's spaces of p-adic Siegel forms.

1. Higher twists and matrix operators

In order to link the two definitions one needs an expression for the partial series

$$F_N^{(d_1,\cdots,d_t)}(\tau,Z):=\sum_{h\in\mathbf{S}_n,\xi\in\mathbf{S}_{tn}} q_\tau^h q_Z^\xi \sum_{\substack{\mathcal{T}_{h,\xi}\\ \det(h_1)\equiv d_1 \bmod N,\cdots,\det(h_t)\equiv d_t \bmod N}} a(\mathcal{T}_{h,\xi},F)$$
$$= \sum_{\substack{(h_1^0,\cdots,h_t^0) \bmod N\\ \det(h_1^0)\equiv d_1 \bmod N,\cdots,\det(h_t^0)\equiv d_t \bmod N}} F_{h_1^0,\cdots,h_t^0;N} \tag{4}$$

where

$$\mathcal{T}_{h,\xi}=\begin{pmatrix} h & h_1 \cdots h_t \\ h_1^t & \\ \cdots & \xi \\ h_t^t & \end{pmatrix}$$

and

$$F_{h_1^0,\cdots,h_t^0;N}(\tau,Z):=\sum_{h\in\mathbf{S}_n,\xi\in\mathbf{S}_{tn}} q_\tau^h q_Z^\xi \sum_{\substack{\mathcal{T}_{\xi,h}\in\mathbf{S}_{(t+1)n}\\ h_1\equiv h_1^0 \bmod N,\cdots,h_t\equiv h_t^0 \bmod N}} a(\mathcal{T}_{h,\xi},F). \tag{5}$$

In order to obtain the series (5) from the Fourier expansion (1) we use for any

$$\frac{\boldsymbol{\alpha}}{N}=(\frac{\alpha_1}{N},\ldots,\frac{\alpha_t}{N})\in\frac{1}{N}\mathbb{Z}^{(n,tn)}$$

the action of the matrix

$$\begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix}$$

(with the symmetric matrix $S(\frac{\boldsymbol{\alpha}}{N})=\begin{pmatrix} 0_n & \frac{\boldsymbol{\alpha}}{N} \\ \frac{\boldsymbol{\alpha}^t}{N} & 0_{tn} \end{pmatrix}\in\mathbb{Q}^{((t+1)n,(t+1)n)}$) on the formal Fourier expansion (1):

$$F(\mathcal{Z})=\sum_{\mathcal{T}\in\mathbf{S}_{(t+1)n}} a(\mathcal{T},F)q^{\mathcal{T}},$$

giving

$$F\mid_k \begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix}(\mathcal{Z}) = F((\mathcal{Z}+S(\frac{\boldsymbol{\alpha}}{N})) \tag{6}$$
$$= \sum_{\mathcal{T}\in\mathbf{S}_{(t+1)n}} a(\mathcal{T},F)e_{(t+1)n}(\mathcal{T}S(\frac{\boldsymbol{\alpha}}{N}))q^{\mathcal{T}}.$$

Here

$$\mathcal{T} \mapsto e_{(t+1)n}(\mathcal{T}S(\frac{\boldsymbol{\alpha}}{N})) = \exp(2\pi\sqrt{-1}\mathrm{tr}(\mathcal{T}S(\frac{\boldsymbol{\alpha}}{N}))) \tag{7}$$

is an additive character on the group $\mathbf{S}_{(t+1)n} \bmod N$ where

$$\mathbf{S}_{(t+1)n} = \Big\{S(\mathbf{h}; h, \xi)) = \begin{pmatrix} h & \mathbf{h} \\ \mathbf{h}^t & \xi \end{pmatrix} \in \mathbb{Q}^{((t+1)n,(t+1)n)} \,\Big|\, \mathbf{h} = (h_1, \cdots, h_t) \in \mathbb{Z}^{(n,tn)}, h \in \mathbf{S}_n, \xi \in \mathbf{S}_{tn}\Big\}$$

and for any $\mathbf{h} = (h_1, \cdots, h_t) \in \mathbb{Z}^{(n,tn)}$, $h \in \mathbf{S}_n, \xi \in \mathbf{S}_{tn}$ we use the notation $S(\mathbf{h}; h, \xi) = \begin{pmatrix} h & \mathbf{h} \\ \mathbf{h}^t & \xi \end{pmatrix} \in \mathbb{Z}^{((t+1)n,(t+1)n)}$. This character is non-trivial for $\boldsymbol{\alpha} \in \mathbb{Z}^{(n,tn)}$ if $\boldsymbol{\alpha} \not\equiv \mathbf{0} \bmod N$. Indeed, if we write

$$\mathcal{T}_{h,\xi} = \begin{pmatrix} h & h_1 \cdots h_t \\ h_1^t & \\ \cdots & \xi \\ h_t^t & \end{pmatrix} \in \mathbf{S}_{(t+1)n},$$

$$S(\frac{\boldsymbol{\alpha}}{N}) = \begin{pmatrix} 0_n & \frac{\alpha_1}{N} \cdots \frac{\alpha_t}{N} \\ \frac{\alpha_1^t}{N} & \\ \cdots & 0_{tn} \\ \frac{\alpha_t^t}{N} & \end{pmatrix} \in \mathbb{Q}^{((t+1)n,(t+1)n)},$$

then

$$\mathcal{T}S(\frac{\boldsymbol{\alpha}}{N}) = \begin{pmatrix} \frac{h_1\alpha_1^t}{N} + \cdots + \frac{h_t\alpha_t^t}{N} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{h_1\alpha_1^t}{N} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \frac{h_i\alpha_i^t}{N} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \frac{h_t\alpha_t^t}{N} \end{pmatrix} \in \mathbb{Q}^{((t+1)n,(t+1)n)},$$

hence

$$\mathrm{tr}(\mathcal{T}S(\frac{\boldsymbol{\alpha}}{N})) = 2\mathrm{tr}(\frac{h_1\alpha_1^t}{N} + \cdots + \frac{h_t\alpha_t^t}{N}) = \frac{2}{N}\sum_{i=1}^{t}\sum_{i',j'=1}^{n} h_{i;i',j'}\alpha_{i;i',j'}, \tag{8}$$

where

$$h_i = (h_{i;i',j'}), \quad \alpha_i = (\alpha_{i;i',j'}) \;\; (i = 1, \cdots, t; i', j' = 1, \cdots, n).$$

It follows from the formula (8) that the additive characters (7)

$$\mathcal{T} \mapsto e_{(t+1)n}(\mathcal{T}S(\frac{\boldsymbol{\alpha}}{N})) = \exp(2\pi\sqrt{-1}\mathrm{tr}(\mathcal{T}S(\frac{\boldsymbol{\alpha}}{N})))$$

on the group $\mathbf{S}_{(t+1)n} \bmod N$ factor through the finite quotient (the additive factor group)

$$\mathbf{S}_{(t+1)n} \bmod N/(\mathbf{S}_n \bmod N \times \mathbf{S}_{tn} \bmod N) \cong \mathbb{Z}^{(n,tn)} \bmod N$$

and that the characters (7) satisfy the following orthogonality relations: for any fixed

$$\mathcal{T} = \begin{pmatrix} h & h_1 \cdots h_t \\ h_1^t & \\ \cdots & \xi \\ h_t^t & \end{pmatrix}, \mathcal{T}_0 = \begin{pmatrix} h & h_1^0 \cdots h_t^0 \\ h_1^{0t} & \\ \cdots & \xi \\ h_t^{0t} & \end{pmatrix} \in \mathbf{S}_{(t+1)n}$$

we have that

$$\sum_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N} \exp(2\pi\sqrt{-1}\mathrm{tr}((-\mathcal{T}_0)S(\frac{\boldsymbol{\alpha}}{N})))e_{(t+1)n}(\mathcal{T}S(\frac{\boldsymbol{\alpha}}{N}))$$
$$= \sum_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N} \exp(2\pi\sqrt{-1}\mathrm{tr}((\mathcal{T}-\mathcal{T}_0)S(\frac{\boldsymbol{\alpha}}{N}))) \tag{9}$$
$$= \begin{cases} N^{n^2t}, \text{ if } h_1 \equiv h_1^0 \bmod N, \cdots, h_t \equiv h_t^0 \bmod N \\ 0, \quad \text{otherwise} \end{cases},$$

where

$$N^{n^2t} = \left|\frac{1}{N}\mathbb{Z}^{(n,tn)}/\mathbb{Z}^{(n,tn)}\right| = \left|\mathbf{S}_{(t+1)n} \bmod N/(\mathbf{S}_n \bmod N \times \mathbf{S}_{tn} \bmod N)\right|.$$

Our next task is to express the partial series $F_{h_1^0,\cdots,h_t^0;N}(\tau, Z)$ defined by the equality (5) using the action of the matrix $\begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix}$ on the formal Fourier expansion (1)

$$F(\mathcal{Z}) = \sum_{\mathcal{T}\in\mathbf{S}_{(t+1)n}} a(\mathcal{T}, F)q^{\mathcal{T}}$$

for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_t) \in \mathbb{Z}^{(n,tn)} \bmod N$, where the matrix $S(\frac{\boldsymbol{\alpha}}{N}) = \begin{pmatrix} 0_n & \frac{\boldsymbol{\alpha}}{N} \\ \frac{\boldsymbol{\alpha}^t}{N} & 0_{tn} \end{pmatrix} \in \mathbb{Q}^{((t+1)n,(t+1)n)}$.

Let us substitute the equality (8) for the trace and the orthogonality relations (9) to the formula for the action (6) giving directly the following equality: for any fixed

$$\mathcal{T}_{h,\xi}^0 = \begin{pmatrix} h & h_1^0 \cdots h_t^0 \\ h_1^{0t} & \\ \cdots & \xi \\ h_t^{0t} & \end{pmatrix} \in \mathbf{S}_{(t+1)n}$$

we have that for any choice of $\mathcal{T}_{h,\xi}^0$ with given $h_1^0, \cdots, h_t^0 \bmod N$ there is the equality

$$F_{h_1^0,\cdots,h_t^0;N}(\tau,Z)=N^{-n^2t}\times \tag{10}$$
$$\sum_{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N}\exp(2\pi\sqrt{-1}\mathrm{tr}((-\mathcal{T}^0_{h,\xi})S(\frac{\boldsymbol{\alpha}}{N})))F\mid_k\begin{pmatrix}1 & S(\frac{\boldsymbol{\alpha}}{N})\\ 0 & 1\end{pmatrix}(\mathcal{Z})$$
$$=N^{-n^2t}\sum_{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N}e_{(t+1)n}(-\mathcal{T}^0_{h,\xi}S(\frac{\boldsymbol{\alpha}}{N}))F\mid_k\begin{pmatrix}1 & S(\frac{\boldsymbol{\alpha}}{N})\\ 0 & 1\end{pmatrix}(\mathcal{Z}).$$

Using the equality (10) we obtain immediately an explicit expression for the series (4):

$$F_N^{(d_1,\cdots,d_t)}(\tau,Z):=\sum_{h\in\mathbf{S}_n,\xi\in\mathbf{S}_{tn}}q_\tau^h q_Z^\xi\sum_{\substack{\mathcal{T}_{h,\xi}\in\mathbf{S}_{(t+1)n}\\ \det(h_1)\equiv d_1 \bmod N,\cdots,\det(h_t)\equiv d_t \bmod N}}a(\mathcal{T}_{h,\xi},F) \tag{11}$$
$$=\sum_{\substack{(h_1^0,\cdots,h_t^0) \bmod N\\ \det(h_1^0)\equiv d_1 \bmod N,\cdots,\det(h_t^0)\equiv d_t \bmod N}}F_{h_1^0,\cdots,h_t^0;N}$$
$$=N^{-n^2t}\sum_{\substack{(h_1^0,\cdots,h_t^0) \bmod N\\ \det(h_1^0)\equiv d_1 \bmod N,\cdots,\det(h_t^0)\equiv d_t \bmod N}}$$
$$\sum_{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N}e_{(t+1)n}(-\mathcal{T}^0_{h,\xi}S(\frac{\boldsymbol{\alpha}}{N}))F\Big|_k\begin{pmatrix}1 & S(\frac{\boldsymbol{\alpha}}{N})\\ 0 & 1\end{pmatrix}(\mathcal{Z})\Big|_{\mathcal{Z}=\mathrm{diag}\{\tau,Z\}},$$

where $\mathcal{T}^0_{h,\xi}=\begin{pmatrix}h & h_1^0\cdots h_t^0\\ h_1^{0t} & \\ \cdots & \xi\\ h_t^{0t} & \end{pmatrix}$ is any element of $\mathbf{S}_{(t+1)n} \bmod N$ with given $h_1^0,\cdots,h_t^0 \bmod N$. Moreover, one can pass to the summation over all possible matrices $\mathcal{T}^0_{h,\xi}\in\mathbf{S}_{(t+1)n} \bmod N$ as above, with given h,ξ and $h_1^0,\cdots,h_t^0 \bmod N$. The total number of matrices $\mathcal{T}^0_{h,\xi} \bmod N$ with fixed h,ξ is equal to N^{tn^2}, and hence the expression (11) transforms to the following:

$$F_N^{(d_1,\cdots,d_t)}(\tau,Z)$$
$$:=\sum_{\substack{(h_1^0,\cdots,h_t^0) \bmod N\\ \det(h_1^0)\equiv d_1 \bmod N,\cdots,\det(h_t^0)\equiv d_t \bmod N}}F_{h_1^0,\cdots,h_t^0;N} \tag{12}$$
$$=N^{-n^2t}\sum_{\substack{\mathcal{T}^0_{h,\xi}\in\mathbf{S}_{(t+1)n} \bmod N\\ \det(h_1^0)\equiv d_1 \bmod N,\cdots,\det(h_t^0)\equiv d_t \bmod N}}\sum_{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N}$$
$$e_{(t+1)n}(-\mathcal{T}^0_{h,\xi}S(\frac{\boldsymbol{\alpha}}{N}))F\Big|_k\begin{pmatrix}1 & S(\frac{\boldsymbol{\alpha}}{N})\\ 0 & 1\end{pmatrix}(\mathcal{Z})\Big|_{\mathcal{Z}=\mathrm{diag}\{\tau,Z\}}$$

$$= N^{-tn^2} \sum_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N} \ \sum_{\substack{\mathcal{T}^0_{h,\xi}\in\mathbf{S}_{(t+1)n} \bmod N \\ \det(h^0_1)\equiv d_1 \bmod N,\cdots,\det(h^0_t)\equiv d_t \bmod N}}$$

$$e_{(t+1)n}(-\mathcal{T}^0_{h,\xi}S(\frac{\boldsymbol{\alpha}}{N}))F\Big|_k \begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix}(\mathcal{Z})\Big|_{\mathcal{Z}=\mathrm{diag}\{\tau,Z\}},$$

where $\mathcal{T}^0_{h,\xi} = \begin{pmatrix} h & h^0_1 \cdots h^0_t \\ h^{0t}_1 & \\ \cdots & \xi \\ h^{0t}_t & \end{pmatrix} \in \mathbf{S}_{(t+1)n}$.

In order to obtain an explicit expression for the series (3) it suffices to rewrite the definition of the higher twist of F by a t-tuple $\boldsymbol{\varphi}$ of *arbitrary* Dirichlet characters $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_t)$ modulo N as a function $F^{(\boldsymbol{\varphi})}$ on $\mathbb{H}_n \times \mathbb{H}_{tn}$ given by its Fourier expansion

$$F^{(\boldsymbol{\varphi})}(\tau, Z) := \sum_{h\in\mathbf{S}_n,\xi\in\mathbf{S}_{tn}} q^h_\tau q^\xi_Z \times \tag{13}$$

$$\sum_{\mathcal{T}_{h,\xi}\in\mathbf{S}_{(t+1)n}} \varphi_1(\det(h_1))\dots\varphi_t(\det(h_t))a(\mathcal{T}_{h,\xi}, F)$$

$$= \sum_{d_1 \bmod N,\cdots,d_t \bmod N} \varphi_1(d_1)\dots\varphi_t(d_t)F^{(d_1,\cdots,d_t)}_N(\tau, Z).$$

The series (13) now takes the form

$$F^{(\boldsymbol{\varphi})}(\tau, Z) = N^{-tn^2} \sum_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N} \ \sum_{\mathcal{T}^0_{h,\xi}\in\mathbf{S}_{(t+1)n} \bmod N} \tag{14}$$

$$\varphi_1(\det(h^0_1))\dots\varphi_t(\det(h^0_t))e_{(t+1)n}(-\mathcal{T}^0_{h,\xi}S(\frac{\boldsymbol{\alpha}}{N})) \times F\Big|_k \begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix}(\mathcal{Z})\Big|_{\mathcal{Z}=\mathrm{diag}\{\tau,Z\}}$$

$$= N^{-tn^2} \sum_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N} G(-\boldsymbol{\alpha}, \boldsymbol{\varphi})F\Big|_k \begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix}(\mathcal{Z})\Big|_{\mathcal{Z}=\mathrm{diag}\{\tau,Z\}}.$$

Here we use the notation (with fixed h and ξ)

$$G(\boldsymbol{\alpha}, \boldsymbol{\varphi}) = \sum_{\mathcal{T}^0_{h,\xi}\in\mathbf{S}_{(t+1)n} \bmod N} \varphi_1(\det(h^0_1))\dots\varphi_t(\det(h^0_t))e_{(t+1)n}(\mathcal{T}^0_{h,\xi}S(\frac{\boldsymbol{\alpha}}{N}))$$

$$= \sum_{\mathcal{T}^0_{h,\xi}\in\mathbf{S}_{(t+1)n} \bmod N} \boldsymbol{\varphi}(\mathbf{h}_0)e_{(t+1)n}(\mathcal{T}^0_{h,\xi}S(\frac{\boldsymbol{\alpha}}{N})), \tag{15}$$

for higher matrix Gauss sums, where $\mathcal{T}^0_{h,\xi} = \begin{pmatrix} h & \mathbf{h}_0 \\ \mathbf{h}^t_0 & \xi \end{pmatrix}$,

$$\mathbf{h}_0 = (h^0_1, \cdots, h^0_t), \quad \boldsymbol{\varphi}(\mathbf{h}_0) = \varphi_1(\det(h^0_1))\dots\varphi_t(\det(h^0_t)),$$

and

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_t) \in \mathbb{Z}^{(n,tn)} \bmod N.$$

Note that these Gauss sums do not depend at all on h or ξ.

2. Properties of higher matrix Gauss sums

2.1. Spherical property of higher Gauss sums

Let us consider two elements

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_t) \in \mathbb{Z}^{(n,tn)} \bmod N, \boldsymbol{\beta} = (\beta_1, \dots, \beta_t) \in \mathbb{Z}^{(n,tn)} \bmod N$$

related by the congruence

$$\boldsymbol{\alpha} \equiv a\boldsymbol{\beta}A^t \bmod N\mathbb{Z}^{(n,tn)} \tag{16}$$

for any

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \iota_{n,tn}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in \Gamma_0^{(t+1)n}(N^2),$$

where

$$\iota_{n,tn}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix}.$$

$$A \equiv \mathrm{diag}(A_1, \dots, A_t) \mod N, \tag{17}$$

and A_i are integral $n \times n$ matrices. We assume for our later application that

$$A \in \Gamma_1(M, N) := \{\gamma \in \Gamma_0(M) \mid A \equiv \text{ diag } (A_1, \dots A_t) \bmod N\}.$$

Then the map

$$\boldsymbol{\beta} \mapsto a\boldsymbol{\beta}A^t$$

is a permutation of $\mathbb{Z}^{(n,tn)}/N\mathbb{Z}^{(n,tn)}$.

Proposition 2.1. *Let*

$$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_t), \boldsymbol{\beta} = (\beta_1, \cdots, \beta_t)$$

be two elements in $\mathbb{Z}^{(n,tn)}/N\mathbb{Z}^{(n,tn)}$ *related by the congruence*

$$\boldsymbol{\alpha} \equiv a\boldsymbol{\beta}A^t \bmod N\mathbb{Z}^{(n,tn)}$$

for some

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \iota_{n,tn}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in \Gamma_0^{(t+1)n}(N^2)$$

with

$$A \equiv \mathrm{diag}(A_1, \dots, A_t) \mod N,$$

where the A_i are integral $n \times n$ matrices,

$$\iota_{n,tn}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix}.$$

Assume that

$$A \in \Gamma_1(M, N) := \{\gamma \in \Gamma_0(M) \mid A \equiv \mathrm{diag}(A_1, \dots A_t) \bmod N\},$$

then the following equality holds

$$G(\boldsymbol{\beta}, \boldsymbol{\varphi}) = \left(\prod_{j=1}^{t} \overline{\varphi_j\big(\det(a)\det(A)\big)} \right) G(\boldsymbol{\alpha}, \boldsymbol{\varphi}).$$

Proof of Proposition 2.1. . We begin with the following definitions: Let

$$G(\boldsymbol{\alpha}, \boldsymbol{\varphi}) = \sum_{\mathcal{T}^0_{h,\xi} \in \mathbf{S}_{(t+1)n} \bmod N} \varphi_1(\det(h^0_1)) \dots \varphi_t(\det(h^0_t)) e_{(t+1)n}(\mathcal{T}^0_{h,\xi} S(\frac{\boldsymbol{\alpha}}{N})),$$

$$G(\boldsymbol{\beta}, \boldsymbol{\varphi}) = \sum_{\tilde{\mathcal{T}}^0_{h,\xi} \in \mathbf{S}_{(t+1)n} \bmod N} \varphi_1(\det(\tilde{h}^0_1)) \dots \varphi_t(\det(\tilde{h}^0_t)) e_{(t+1)n}(\tilde{\mathcal{T}}^0_{h,\xi} S(\frac{\boldsymbol{\beta}}{N})), \tag{18}$$

and we choose for each $\mathcal{T}^0_{h,\xi} \in \mathbf{S}_{(t+1)n}$ mod N an element $\tilde{\mathcal{T}}^0_{h,\xi} \in \mathbf{S}_{(t+1)n}$ mod N in such a way that

$$\mathcal{T}^0_{h,\xi} \mapsto \tilde{\mathcal{T}}^0_{h,\xi}$$

is a permutation of $\mathbf{S}_{(t+1)n}$ mod N, and

$$e_{(t+1)n}(\mathcal{T}^0_{h,\xi} S(\frac{\boldsymbol{\alpha}}{N})) = e_{(t+1)n}(\tilde{\mathcal{T}}^0_{h,\xi} S(\frac{\boldsymbol{\beta}}{N})). \tag{19}$$

In order to satisfy the condition (19) it suffices to put

$$\tilde{\mathcal{T}}^0_{h,\xi} = \begin{pmatrix} a^t & 0 \\ 0 & A^t \end{pmatrix} \mathcal{T}_0 \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}$$

because of the congruence

$$\boldsymbol{\alpha} \equiv a \boldsymbol{\beta} A^t \bmod N\mathbb{Z}^{(n,tn)},$$

which implies that

$$S(\frac{\boldsymbol{\alpha}}{N}) = \begin{pmatrix} 0_n & \frac{\boldsymbol{\alpha}}{N} \\ \frac{\boldsymbol{\alpha}^t}{N} & 0_{tn} \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0_n & \frac{\boldsymbol{\beta}}{N} \\ \frac{\boldsymbol{\beta}}{N}^t & 0_{tn} \end{pmatrix} \begin{pmatrix} a^t & 0 \\ 0 & A^t \end{pmatrix} \bmod \mathbf{S}_{(t+1)n}.$$

It follows that

$$\operatorname{tr}\left(\mathcal{T}_0 S(\frac{\boldsymbol{\alpha}}{N}) - \tilde{\mathcal{T}}_0 S(\frac{\boldsymbol{\beta}}{N})\right) = \operatorname{tr}\left(\mathcal{T}_0 S(\frac{\boldsymbol{\alpha}}{N}) - \begin{pmatrix} a^t & 0 \\ 0 & A^t \end{pmatrix} \tilde{\mathcal{T}}_0 \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} S(\frac{\boldsymbol{\beta}}{N})\right) \bmod \mathbb{Z} \tag{20}$$

$$\equiv \operatorname{tr}\left(\mathcal{T}_0 \begin{pmatrix} a^t & 0 \\ 0 & A^t \end{pmatrix} S(\frac{\boldsymbol{\beta}}{N}) \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \mathcal{T}_0 \begin{pmatrix} a^t & 0 \\ 0 & A^t \end{pmatrix} S(\frac{\boldsymbol{\beta}}{N})\right) \bmod \mathbb{Z}$$

$$\equiv 0 \bmod \mathbb{Z}$$

implying the equality (19). In order to deduce Proposition 2.1 we notice that

$$\tilde{\mathcal{T}}_0 = \begin{pmatrix} \tilde{h} & \tilde{h}_1^0 \cdots \tilde{h}_t^0 \\ \tilde{h}_1^{0t} & \\ \cdots & \tilde{\xi} \\ \tilde{h}_t^{0t} & \end{pmatrix} \tag{21}$$

$$= \begin{pmatrix} a^t & 0 \\ 0 & A^t \end{pmatrix} \begin{pmatrix} h & h_1^0 \cdots h_t^0 \\ h_1^{0t} & \\ \cdots & \xi \\ h_t^{0t} & \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}.$$

From the congruence (17),

$$A \equiv \operatorname{diag}(A_1, \ldots, A_t) \mod N,$$

and we obtain that

$$\tilde{h} \equiv a^t h a \bmod N, \tilde{h}_1^0 \equiv a^t h_1^0 A_1 \bmod N, \cdots \tilde{h}_t^0 \equiv a^t h_t^0 A_t \bmod N. \tag{22}$$

The equality of Proposition 2.1 follows directly if we substitute the congruences (22) into the definitions (18). ■

2.2. Decomposition of the higher Gauss sums

Note that the higher Gauss sum $G(\boldsymbol{\alpha}, \boldsymbol{\varphi})$ as defined in (15) naturally decompose into Gauss sums for matrices of size n: Using the formula (8) we obtain

$$G(\boldsymbol{\alpha}, \boldsymbol{\varphi}) = \prod_{j=1}^{n} G_n(\alpha_j, \varphi_j) \tag{23}$$

with

$$G_n(\alpha, \varphi) = \sum_{h \bmod N} \varphi(\det(h)) e_n(\frac{h^t \cdot \alpha}{N}),$$

where h runs over all elements of $\mathbb{Z}^{(n,n)}$ mod N. Such matrix Gauss sums were studied in [8], section 6. In particular,

$$G_n(\alpha, \varphi) = N^{\frac{n(n-1)}{2}} \overline{\varphi}(\det(\alpha)) G(\varphi)^n \tag{24}$$

if φ is primitive mod N and $G(\varphi)$ denotes the standard Gauss sum attached to φ.
The formulas (23) and (24) together with standard properties of the classical Gauss sums imply the following properties

1. For a t-tuple $\boldsymbol{\varphi}$ of *primitive* Dirichlet characters $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_t)$ modulo N one has
$$|G(\boldsymbol{\alpha}, \boldsymbol{\varphi})| = N^{tn^2};$$
2. For a t-tuple $\boldsymbol{\varphi}$ of any Dirichlet characters $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_t)$ modulo N one has
$$\overline{G(\boldsymbol{\alpha}, \boldsymbol{\varphi})} = \varphi_1(-1)^n \ldots \varphi_t(-1)^n G(\boldsymbol{\alpha}, \bar{\boldsymbol{\varphi}}).$$

Question: Can one interpret $G(\boldsymbol{\alpha}, \boldsymbol{\varphi})$ as the constant of a transformation formula for a theta function (or of another nice modular form) ?

2.3. The second higher twist and distributions

Our second version of the higher twist is applicable for any N, so that one can replace N here by any Np^v, $(v \geq 0)$. Hence, this version of the twist produces a distribution $\Phi_{F,t}$ on the product

$$\underbrace{Y \times \cdots \times Y}_{t \text{ times}},$$

where $Y = Y_{N,p} = \lim\limits_{\overleftarrow{v}} Y_{Np^v}, \quad Y_{Np^v} = (\mathbb{Z}/Np^v\mathbb{Z})^\times$.
The value of this (modular) distribution is simply defined as

$$\Phi_{F,t}(\varphi_1 \times \cdots \times \varphi_t) := F^{(\boldsymbol{\varphi})}(\tau, Z).$$

3. On automorphy properties

Let us prove that these series belong to certain tensor product of spaces of modular forms:

Proposition 3.1 (automorphy propeerty of the first twist). *If* $F \in M_k^{(t+1)n}(\Gamma_0(M), \psi)$ *with* $N^2 \mid M$ *then we have that*

$$F^{\boldsymbol{\varphi}}(\tau, Z) \in M_k^n(\Gamma_0(M), \psi\varphi_1 \ldots \varphi_t) \otimes M_k^{tn}(\Gamma_1(M, N), \psi\varphi_1 \ldots \varphi_t),$$

where

$$\Gamma_1(M,N) := \{\gamma \in \Gamma_0(M) \mid A \equiv \mathrm{diag}(A_1,\dots A_t) \bmod N\}.$$

Proof of Proposition 3.1. For $\frac{\boldsymbol{\alpha}}{N} = (\frac{\alpha_1}{N},\dots,\frac{\alpha_t}{N}) \in \frac{1}{N}\mathbb{Z}^{(n,tn)}$ we put

$$S(\frac{\boldsymbol{\alpha}}{N}) = \begin{pmatrix} 0_n & \frac{\boldsymbol{\alpha}}{N} \\ \frac{\boldsymbol{\alpha}^t}{N} & 0_{tn} \end{pmatrix} \in \mathbb{Q}^{((t+1)n,(t+1)n)}.$$

We try to find $\frac{\boldsymbol{\alpha}^*}{N} \in \frac{1}{N}\mathbb{Z}^{(n,tn)}$ such that

$$\begin{pmatrix} 1_{(t+1)n} & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0_{(t+1)n} & 1_{(t+1)n} \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} 1_{(t+1)n} & S(\frac{\boldsymbol{\alpha}^*}{N}) \\ 0_{(t+1)n} & 1_{(t+1)n} \end{pmatrix} =$$

$$\begin{pmatrix} \mathcal{A} + S(\frac{\boldsymbol{\alpha}}{N})\mathcal{C} & -\mathcal{A}S(\frac{\boldsymbol{\alpha}^*}{N}) + \mathcal{B} - S(\frac{\boldsymbol{\alpha}}{N})\mathcal{C}S(\frac{\boldsymbol{\alpha}^*}{N}) + S(\frac{\boldsymbol{\alpha}}{N})\mathcal{D} \\ \mathcal{C} & -\mathcal{C}S(\frac{\boldsymbol{\alpha}^*}{N}) + \mathcal{D} \end{pmatrix}$$

is in $\Gamma_0(M)$. The first (evident) condition is that $\mathcal{C} \equiv 0 \bmod M$. It is easy to see that the two block matrices on the diagonal

$$-\mathcal{C}S(\frac{\boldsymbol{\alpha}}{N}^*) + \mathcal{D} = \begin{pmatrix} d & -c\frac{\boldsymbol{\alpha}^*}{N} \\ -C\frac{\boldsymbol{\alpha}^{*t}}{N} & D \end{pmatrix}$$

and

$$\mathcal{A} + S(\frac{\boldsymbol{\alpha}}{N})\mathcal{C} = \begin{pmatrix} a & \frac{\boldsymbol{\alpha}}{N}C \\ \frac{\boldsymbol{\alpha}^t}{N}c & A \end{pmatrix}$$

are integral, if both c and C are congruent to 0 modulo M. The remaining condition is that

$$-\mathcal{A}S(\frac{\boldsymbol{\alpha}^*}{N}) + \mathcal{B} - S(\frac{\boldsymbol{\alpha}}{N})\mathcal{C}S(\frac{\boldsymbol{\alpha}^*}{N}) + S(\frac{\boldsymbol{\alpha}}{N})\mathcal{D} =$$

$$\begin{pmatrix} b - \frac{\boldsymbol{\alpha}}{N}C\frac{\boldsymbol{\alpha}^{*t}}{N} & -a\frac{\boldsymbol{\alpha}^*}{N} + S(\alpha)D \\ -A\frac{\boldsymbol{\alpha}^{*t}}{N} + \frac{\boldsymbol{\alpha}^t}{N}d & B - \frac{\boldsymbol{\alpha}}{N}^t c\frac{\boldsymbol{\alpha}^*}{N} \end{pmatrix}$$

is integral, which is satisfied if $C \equiv 0 \bmod N^2$, $c \equiv 0 \bmod N^2$ and

$$-a\frac{\boldsymbol{\alpha}^*}{N} + \frac{\boldsymbol{\alpha}}{N}D \quad \text{and} - A\frac{\boldsymbol{\alpha}^{*t}}{N} + \frac{\boldsymbol{\alpha}^t}{N}d \quad \text{are both integral.}$$

Therefore we should choose

$$\boldsymbol{\beta} = \bar{a}\boldsymbol{\alpha}D \bmod N$$

(with $\bar{a}$ any inverse of a mod N) . With this choice the second integrality condition above is automatically satisfied:

$$-A\boldsymbol{\beta}^t + \boldsymbol{\alpha}^t d = -AD^t\boldsymbol{\alpha}^t\bar{a}^t + \boldsymbol{\alpha}^t d \equiv \boldsymbol{\alpha}^t(-\bar{a}^t + d) \quad \bmod \quad N\mathbb{Z}^{(tn,n)}$$

(this expression is then integral itself!). By the above,

$$\sum_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N} \varphi_1(\det(\alpha_1))\dots\varphi_t(\det(\alpha_t))$$
$$F\mid_k \begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}(\mathcal{Z})\Big|_{\mathcal{Z}=\mathrm{diag}\{\tau,Z\}} =$$
$$\sum_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N} \varphi_1(\det(\alpha_1))\dots\varphi_t(\det(\alpha_t))$$
$$F\mid_k \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix}\begin{pmatrix} 1 & S(\frac{\boldsymbol{\beta}}{N}) \\ 0 & 1 \end{pmatrix}(\mathcal{Z})\Big|_{\mathcal{Z}=\mathrm{diag}\{\tau,Z\}}.$$

We assume that $F\mid_k \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix}$ is again periodic. Using

$$\boldsymbol{\alpha} \equiv a\boldsymbol{\beta}A^t \bmod N$$

we can sum over the β_j mod N instead of the α_j and we obtain the following expression

$$\left(\prod_{j=1}^{t}\varphi_j\big(\det(a)\det(A)\big)\right)\sum_{\boldsymbol{\beta}=(\beta_1,\cdots,\beta_t)\in\mathbb{Z}^{(n,tn)}/N\mathbb{Z}^{(n,tn)}}\varphi_1(\beta_1)\dots\varphi_t(\beta_t)F\mid_k \begin{pmatrix} 1 & S(\boldsymbol{\beta}) \\ 0 & 1 \end{pmatrix}$$

proving the automorphy property of Proposition 3.1. ∎

Proposition 3.2 (automorphy property of the second twist). *If* $F \in M_k^{(t+1)n}(\Gamma_0(M),\psi)$ *with* $N^2 \mid M$ *then we have that*

$$F^{(\boldsymbol{\varphi})}(\tau,Z) \in M_k^n(\Gamma_0(M),\psi\varphi_1\dots\varphi_t)\otimes M_k^{tn}(\Gamma_1(M,N),\psi\varphi_1\dots\varphi_t),$$

where
$$\Gamma_1(M,N) := \{\gamma\in\Gamma_0(M) \mid A\equiv \mathrm{diag}(A_1,\dots A_t) \bmod N\}.$$

Proof of Proposition 3.2. Proof of Proposition 3.2 will be deduced from the explicit expression of the new higher twist (3) through the higher Gauss sums, given by the equality (15), and from the propery of the higher Gauss sums given in Proposition 2.1. Then the proof goes in a similar way as for previously defined first version of the twist (given by the equality (1)). By the above,

$$F^{(\boldsymbol{\varphi})}(\tau,Z) = N^{-tn^2}\sum_{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_t)\in\mathbb{Z}^{(n,tn)} \bmod N} G(-\boldsymbol{\alpha},\boldsymbol{\varphi})F\Big|_k \begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix}(\mathcal{Z})\Big|_{\mathcal{Z}=\mathrm{diag}\{\tau,Z\}}. \tag{25}$$

Then for any

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \iota_{n,tn}(\begin{pmatrix} a & b \\ c & d \end{pmatrix},\begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in \Gamma_0^{(t+1)n}(N^2)$$

with

$$A \equiv \operatorname{diag}(A_1, \dots, A_t) \mod N$$

(where the A_i are $n \times n$ matrices),

$$\begin{aligned} &F^{\boldsymbol{\varphi}}(\tau, Z) \mid_k^{\tau} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid_k^{Z} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \left(\prod_{j=1}^{t} \left(\varphi_j(\det(a)) \varphi_j(\det(A_j)) \right) \right) \left(F \mid_k \left(\begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix}, \right)^{\boldsymbol{\varphi}} \operatorname{diag}\{\tau, Z\}. \end{aligned}$$

We deduce that for an appropriate β there is the following matrix equality

$$\begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix} \begin{pmatrix} 1 & S(\frac{\boldsymbol{\beta}}{N}) \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix} \in \Gamma_0^{(t+1)n}(N^2)$$

with

$$\tilde{\mathcal{C}} = \mathcal{C}$$

and

$$\tilde{\mathcal{A}} \equiv \mathcal{A} \equiv \operatorname{diag}\,(a, A_1, \dots, A_t) \mod N$$

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \Gamma_0^n(N^2) \quad \text{and} \quad \tilde{a} \equiv a \bmod N.$$

Moreover one has

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \in \Gamma_1(M, N) := \{\gamma \in \Gamma_0(M) \mid A \equiv \operatorname{diag}(A_1, \dots A_t) \bmod N\}$$

and $\tilde{A} \equiv A \mod N$.

One obtains then the following equality for the higher twists (3):

$$\begin{aligned} &N^{-tn^2} \sum_{\frac{\boldsymbol{\alpha}}{N} = (\frac{\alpha_1}{N}, \dots, \frac{\alpha_t}{N}) \in \frac{1}{N}\mathbb{Z}^{(n,tn)}/\mathbb{Z}^{(n,tn)}} G(-\boldsymbol{\alpha}, \boldsymbol{\varphi}) \times \\ &F \mid_k \begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} (\mathcal{Z}) \Big|_{\mathcal{Z} = \operatorname{diag}\{\tau, Z\}} \\ &= N^{-tn^2} \sum_{\frac{\boldsymbol{\alpha}}{N} = (\frac{\alpha_1}{N}, \dots, \frac{\alpha_t}{N}) \in \frac{1}{N}\mathbb{Z}^{(n,tn)}/\mathbb{Z}^{(n,tn)}} G(-\boldsymbol{\alpha}, \boldsymbol{\varphi}) \times \\ &F \mid_k \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix} \begin{pmatrix} 1 & S(\frac{\boldsymbol{\beta}}{N}) \\ 0 & 1 \end{pmatrix} (\mathcal{Z}) \Big|_{\mathcal{Z} = \operatorname{diag}\{\tau, Z\}}. \end{aligned}$$

Under the assumption of Proposition 3.1 we have that the function $F \mid_k \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix}$

is again periodic because of its automorphy property. Using

$$\boldsymbol{\alpha} \equiv a\boldsymbol{\beta}A^t \bmod N$$

we can sum over the β_j mod N instead of the α_j and we obtain

$$\left(\prod_{j=1}^{t} \overline{\varphi_j\big(\det(a)\det(A)\big)}\right) N^{-tn^2} \times \sum_{\boldsymbol{\beta}=(\beta_1,\ldots,\beta_t)\in\mathbb{Z}^{(n,tn)} \bmod N} G(-\boldsymbol{\beta},\boldsymbol{\varphi})F \mid_k \begin{pmatrix} 1 & S(\frac{\boldsymbol{\beta}}{N}) \\ 0 & 1 \end{pmatrix}$$

proving the automorphy property of Proposition 3.2. ■

Corollary 3.3. *Iteration of the above gives a similar function as in the version given by* (2)*. Let* $\varphi_1, \varphi_2, \varphi_3$ *be three Dirichlet characters mod* N *and* $F \in M_k^3(\Gamma_0(M), \psi)$ *with* $N^2 \mid M$*. Then*

$$h^{(\varphi_1,\varphi_2,\varphi_3)}(\tau_1,\tau_2,\tau_3) := \sum_{h_1,h_2,h_3\in\mathbf{S}_1\times\mathbf{S}_1\times\mathbf{S}_1} q_{\tau_1}^{h_1} q_{\tau_2}^{h_2} q_{\tau_3}^{h_3} \times \times \sum_{\mathcal{T}\in\mathbf{S}_{3n}} \varphi_1(\alpha)\varphi_2(\beta)\varphi_3(\gamma))a(\mathcal{T},F),$$

with the notation $\mathcal{T} = \begin{pmatrix} h_1 & \alpha & \beta \\ \alpha & h_2 & \gamma \\ \beta & \gamma & h_3 \end{pmatrix}$*, is an element of the following tensor product of spaces of modular forms:*

$$M_k(\Gamma_0(M), \psi\varphi_1\varphi_2) \otimes M_k(\Gamma_0(M), \psi\varphi_1\varphi_3) \otimes M_k(\Gamma_0(M), \psi\varphi_2\varphi_3).$$

Some questions and remarks about the higher twists

1. What are differential operators (or elements of the universal envelopping algebra) which correspond to the higher twists? (like the Ramanujan operator or the Gauss-Manin connection coresponding to the usual twist; maybe these analogues are given by the differential operators in Bcherer-Satoh and Ibukiyama, see [7], [27], [28]), and also [14], [15].
2. Group theoretic interpretation of the differential operators in Boecherer-Satoh and Ibukiyama (see [7], [27], [28]), and also [14], [15].
3. Note that the Hecke algebra for a Siegel parabolic subgroup $P_n \subset Sp_n$ acts on periodic functions. Thus its a non-commutative Hecke algebra containing the usual Hecke algebra for Sp_n.

   Interpret commutation rules of the type

$$F\mid_k \begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = F\mid_k \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix} \begin{pmatrix} 1 & S(\frac{\boldsymbol{\beta}}{N}) \\ 0 & 1 \end{pmatrix}$$

for the action of the matrices $\begin{pmatrix} 1 & S(\frac{\boldsymbol{\alpha}}{N}) \\ 0 & 1 \end{pmatrix}$ (where $S(\frac{\boldsymbol{\alpha}}{N})$ denotes the symmetric matrix

$$S(\frac{\boldsymbol{\alpha}}{N}) = \begin{pmatrix} 0_n & \frac{\boldsymbol{\alpha}}{N} \\ \frac{\boldsymbol{\alpha}^t}{N} & 0_{tn} \end{pmatrix} \in \mathbb{Q}^{((t+1)n,(t+1)n)})$$

in terms of the non-commutative (parabolic) extension of the Hecke algebra (see [2]), or possibly in terms of an even more complicated Hecke algebra.

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