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On the Asymptotic Behavior of Solutions and Positive Almost Periodic Solution for Predator-Prey System with the Holling Type II Functional Response

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Abstract. In this paper, we investigate a predator-prey population which is modeled by a system of differential equations modified Leslie-Gower and Holling type II schemes with time-dependent parameters. We establish a sufficient condition posed on the behavior at infinity of coefficients for globally asymptotic stability of solutions. The existence and uniqueness of positive bounded solution of system are also proved. In the case where the coefficients of equations are almost periodic functions, it is showed that there exists a unique positive almost periodic solution which attracts every solution starting in $\operatorname{int}\mathbb{R}^2_+$.

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1. Introduction

In mathematical ecology, one of the popular models is a model consisting of two different species where one of them provides food to the other. The interaction between species in this type is very universal in nature and is called "Predator-Prey" relation. The predator-prey system plays an important role both in theory and practice and has been studied by many authors. Recently, there are many works revealing the dynamics of predator-prey systems for a so-called semi-ratio-dependent class with functional responses. This class consists of models which are described by the following system

$$\begin{aligned} \dot{x} &= x[a - bx] - c(x)y, \\ \dot{y} &= y\big[d - e\frac{y}{x}\big], \end{aligned} \tag{1}$$

where x and y stand for the quantity (or density) of the prey and the predator, respectively. The function $c(x)$ is called the predator functional response. The biological signification of a, d, e and a/b has been explained in [4]. The predator consumes the prey according to the functional response $c(x)$ and carrying capacity $x(t)/e$ proportional to the population size of prey (or prey abundance).

The predator-prey equation (1) with functional response was first proposed by Leslie and Gower [11]. Since then, there has been much interest both in theory and application of this model. Based on experiments, Holling [8] suggested some kinds of functional responses to model the phenomena of predation, which made the standard Lotka-Volterra system more realistic. Depending on the form of the functional responses, these models are classified into five types (see [4]). If $c(x) = mx$, we have type I. These models have been studied by Leslie and Gower [11], Holling [8], Hsu and Huang [9]. The functional response $c(x)$ is of type II if $c(x) = mx/(A+x)$. There are some papers studying stability, furcation behavior,... of these models. For example, we can refer to Colling [7], Berryman [5], Cheng et al. [6], Hsu et al. [13]...When $c(x) = mx^n/(A + x^n) \quad n \geq 2$, it is called of type III, which was suggested by Holling [8]. The general form of functional response of this type was introduced by Kazarinov and Driessche [10]. If $c(x) = mx^2/(A+x)(B+x)$, it is concerned with type IV. This model can be found in Colling [7], Tanner [12],...The type V is also called Ivlev's functional response. In this case, $c(x) = m(1 - e^{-Ax})$.

The model that Alaoui and Okiye have dealt with in [1], is the model of type II of the form:

$$\begin{aligned} \dot{x} &= x\Big[r_1 - bx - \frac{a_1}{x + k_1}y\Big], \\ \dot{y} &= y\Big[r_2 - \frac{a_2}{x + k_2}y\Big], \end{aligned} \tag{2}$$

where r_1, a_1, b, k_1, r_2, a_2 and k_2 are the model's parameters, assuming to be positive. It was proven in [1] that the system (2) is ultimately bounded with respect to $\mathbb{R}^2_+$. Now we consider a time-varying predator-prey system:

$$\begin{aligned} \dot{x} &= x\Big[r_1(t) - b(t)x - \frac{a_1(t)}{x + k_1(t)}y\Big], \\ \dot{y} &= y\Big[r_2(t) - \frac{a_2(t)}{x + k_2(t)}y\Big], \end{aligned} \tag{3}$$

where b, k_1, r_i, $a_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous and bounded above and below by some positive constants; $k_2 : \mathbb{R} \to \mathbb{R}$ is supposed to be nonnegative, continuous and bounded, $x(t)$, $y(t)$ denote quantity prey, predator at time t, respectively.

The paper is organized as follows: In Section 2, we discuss about globally asymptotic stability of system (3). In the last section, we prove the existence of a unique positive bounded solution for system (3) and study the existence of a positive almost periodic solution.

2. The Stability

Let $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}^2_+ = [0, +\infty) \times [0, +\infty)$, $\mathrm{int}\mathbb{R}^2_+ = (0, +\infty) \times (0, +\infty)$.

Definition 2.1. System (3) is said to be globally asymptotically stable, if for any two solutions $(x_i(t), y_i(t))$, $i = 1, 2$ of (3) with initial condition $(x_i(t_0), y_i(t_0)) \in \mathrm{int}\mathbb{R}^2_+$ one has $\lim\limits_{t\to+\infty} (|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|) = 0$.

Lemma 2.2. *Assume that g, $h : \mathbb{R} \to \mathbb{R}$ are continuous, bounded above and below by positive constants. Then*

(i) *The logistic equation*

$$\dot{z} = z[g(t) - h(t)z] \tag{4}$$

has a unique solution $Z^0(t)$ which is defined on $\mathbb{R}$ and is bounded above and below by positive constants. Moreover, $\lim\limits_{t\to+\infty} |z(t) - Z^0(t)| = 0$ for any solution of (4) with $z(t_0) > 0$ for some $t_0 \in \mathbb{R}$.

(ii) *If in addition, g, h are almost periodic, then the above solution $Z^0(t)$ is also almost periodic.*

(iii) *For any solution $z(t)$ of (4) with $z(t_0) > 0$ for some $t_0 \in \mathbb{R}$, we have*

$$\limsup_{t\to+\infty} z(t) \le \limsup_{t\to+\infty} \frac{g(t)}{h(t)}, \quad \liminf_{t\to+\infty} z(t) \ge \liminf_{t\to+\infty} \frac{g(t)}{h(t)}.$$

Proof. The proof of the assertions (i) and (ii) can be found in [2].

In order to prove the assertion (iii), let $z(t)$ be a solution of (4) with $z(t_0) > 0$. Suppose that

$$\limsup_{t\to+\infty} z(t) > \limsup_{t\to+\infty} \frac{g(t)}{h(t)}.$$

Then there exist $\varepsilon > 0$ and $T \ge t_0$ such that $g(t) - h(t)z(t) \le -\varepsilon h(t)$, for all $t \ge T$. Thus,

$$z(t) \le z(T) \exp\{-\int_T^t \varepsilon h(s)ds\}, \ t \ge T.$$

This implies that $\lim\limits_{t\to+\infty} z(t) = 0$, which contradicts the fact that

$$\limsup_{t\to+\infty} z(t) = \limsup_{t\to+\infty} Z^0(t) > 0 \quad \text{(by the assertion (i))}.$$

The contradiction implies that

$$\limsup_{t\to+\infty} z(t) \le \limsup_{t\to+\infty} \frac{g(t)}{h(t)}.$$

Similarly, we can prove that

$$\liminf_{t\to+\infty} z(t) \ge \limsup_{t\to+\infty} \frac{g(t)}{h(t)}.$$

The proof is complete. ■

By Lemma 2.2, the following logistic equation

$$\dot{x} = x[r_1(t) - b(t)x] \tag{5}$$

has a unique solution, throughout this paper, we denote it by $X^0(t)$, which is defined on $\mathbb{R}$ and is bounded above and below by positive constants. Furthermore, $\lim\limits_{t\to+\infty} |x(t) - X^0(t)| = 0$, for any solution $x(t)$ of (5) with $x(t_0) > 0$ for some $t_0 \in \mathbb{R}$.

Lemma 2.3. *Let $(x(t), y(t))$ be a solution of* (3) *with initial condition $x(t_0) > 0$, $y(t_0) > 0$ and $\bar{x}(t)$ be a solution of* (5) *with $\bar{x}(t_0) > 0$. If $x(t_0) \le \bar{x}(t_0)$ (or $x(t_0) \ge \bar{x}(t_0)$) then $x(t) < \bar{x}(t)$ for all $t > t_0$ (or $x(t) > \bar{x}(t)$ for all $t < t_0$, respectively) belonging to a common interval of existence of $(x(t), y(t))$ and $\bar{x}(t)$.*

Proof. It is easy to see that for each $\bar{t} \in \mathbb{R}$ such that $x(\bar{t}) = \bar{x}(\bar{t})$ then $\dot{\bar{x}}(\bar{t}) - \dot{x}(\bar{t}) > 0$. Thus, there exists $\varepsilon > 0$ such that $\bar{x}(t) - x(t) > 0$, for all $t \in (\bar{t}, \bar{t} + \varepsilon)$ and $\bar{x}(t) - x(t) < 0$, for all $t \in (\bar{t} - \varepsilon, \bar{t})$. Thus, the lemma follows. ■

We denote

$$M_1^+ := \limsup_{t\to+\infty} X^0(t), \quad M_1^- := \limsup_{t\to-\infty} X^0(t)$$

$$M_2^+ := \limsup_{t\to+\infty} \frac{r_2(t)(M_1^+ + k_2(t))}{a_2(t)}, \; M_2^- := \limsup_{t\to-\infty} \frac{r_2(t)(M_1^- + k_2(t))}{a_2(t)},$$

$$m_1^+ := \liminf_{t\to+\infty} \frac{1}{b(t)}[r_1(t) - \frac{a_1(t)M_2^+}{k_1(t)}], \; m_1^- := \liminf_{t\to-\infty} \frac{1}{b(t)}[r_1(t) - \frac{a_1(t)M_2^-}{k_1(t))}],$$

$$m_2^+ := \liminf_{t\to+\infty} \frac{r_2(t)(m_1^+ + k_2(t))}{a_2(t)}, \; m_2^- := \liminf_{t\to-\infty} \frac{r_2(t)(m_1^- + k_2(t)}{a_2(t)}.$$

Throughout this paper, we assume that

$$\liminf_{t\to+\infty}[r_1(t) - \frac{a_1(t)M_2^+}{k_1(t)}] > 0. \tag{6}$$

With assumption (6), we see that $m_1^+ > 0$ and $m_2^+ > 0$. There exist $M_2 > M_2^+$, $M_1 > M_1^+$, $m_1 > 0$, $m_2 > 0$ such that

$$\begin{aligned} &\liminf_{t\to+\infty}[r_1(t) - \frac{a_1(t)M_2}{k_1(t)}] > 0,\ \limsup_{t\to+\infty}\frac{r_2(t)(M_1+k_2(t))}{a_2(t)} < M_2, \\ m_1 < &\liminf_{t\to+\infty}\frac{1}{b(t)}[r_1(t) - \frac{a_1(t)M_2}{k_1(t)}],\ m_2 < \liminf_{t\to+\infty}\frac{r_2(t)(m_1+k_2(t))}{a_2(t)}. \end{aligned} \tag{7}$$

Clearly that $m_1^+ > m_1$ and $m_2 > m_2^+$. Thus, there exists $t_1 \in \mathbb{R}$ such that for all $t \ge t_1$ we have

$$\begin{aligned} X^0(t) &< \frac{M_1 + M_1^+}{2}, \quad \frac{r_2(t)(M_1+k_2(t))}{a_2(t)} < M_2, \\ m_1 &< \frac{1}{b(t)}[r_1(t) - \frac{a_1(t)M_2}{k_1(t)}],\ \ m_2 < \frac{r_2(t)(m_1+k_2(t))}{a_2(t)}. \end{aligned} \tag{8}$$

For $0 < \eta < \frac{M_1 - M_1^+}{2}$ and $t \ge t_1$, we denote

$$\mathcal{A}_0(t) = \{(x,y):\ m_1 \le x \le X^0(t),\ m_2 \le y \le M_2\},$$

$$\mathcal{A}_\eta(t) = \{(x,y):\ m_1 \le x \le X^0(t)+\eta,\ m_2 \le y \le M_2\}.$$

Theorem 2.4. *Suppose that the condition* (6) *holds.*

(i) *If* $(x(t),y(t))$ *is a solution of* (3) *with* $(x(t_0),y(t_0)) \in \mathcal{A}_0(t_0)$, *for some* $t_0 \ge t_1$ *then* $(x(t),y(t)) \in \mathcal{A}_0(t)$, *for all* $t \ge t_0$.

(ii) *If* $(x(t),y(t))$ *is a solution of* (3) *with initial condition* $x(t_0) > 0, y(t_0) > 0$, *then there exists* $T \ge t_1$ *such that* $(x(t),y(t)) \in \mathcal{A}_\eta(t)$, *for all* $t \ge T$.

Proof. (a) In order to see the assertion (i), we consider the vector field $(\dot{x},\dot{y})$ of system (3) on the boundary of the rectangle $\mathcal{A}_0(t) = [m_1, X^0(t)] \times [m_2, M_2]$ in $\mathbb{R}_+^2$ for each $t \ge t_0$.
On the interval $\{(x, M_2):\ m_1 \le x \le X^0(t)\}$, the vector field $(\dot{x},\dot{y})$ of (3) has

$$\dot{y} = y[r_2(t) - \frac{a_2(t)y}{x+k_2(t)}] \le y[r_2(t) - \frac{a_2(t)M_2}{x+k_2(t)}] < 0,\ \text{(by (8))}.$$

On the interval $\{(x, m_2):\ m_1 \le x \le X^0(t)\}$, the vector field $(\dot{x},\dot{y})$ of (3) has

$$\dot{y} = y[r_2(t) - \frac{a_2(t)y}{x+k_2(t)}] \ge y[r_2(t) - \frac{a_2(t)m_2}{m_1+k_2(t)}] > 0,\ \text{(by (8))}.$$

On the interval $\{(m_1, y):\ m_2 \le y \le M_2)\}$, by (8), the vector field $(\dot{x},\dot{y})$ of (3) has

$$\dot{x} = x[r_1(t) - b(t)x - \frac{a_1(t)y}{x + k_2(t)}] \geq x[r_1(t) - b(t)m_1 - \frac{a_1(t)M_2}{m_1 + k_2(t)}] > 0.$$

On the interval $\{(X^0(t), y) : \ m_2 \leq y \leq M_2)\}$, the vector field $(\dot{x}, \dot{y})$ of (3) has

$$\begin{aligned}\dot{x} = x[r_1(t) - b(t)x - \frac{a_1(t)y}{x + k_1(t)}] &= X^0(t)[r_1(t) - b(t)X^0(t) - \frac{a_1(t)y}{X^0(t) + k_1(t)}] \\ &< X^0(t)[r_1(t) - b(t)X^0(t)] = \dot{X}^0.\end{aligned}$$

Thus, since

$$\begin{aligned}\partial\mathcal{A}_0(t) =& \{(x, M_2) : \ m_1 \leq x \leq X^0(t)\} \cup \{(x, m_2) : \ m_1 \leq x \leq X^0(t)\} \\ &\cup \{(m_1, y) : \ m_2 \leq y \leq M_2)\} \cup \{(X^0(t), y) : \ m_2 \leq y \leq M_2)\},\end{aligned}$$

the assertion (i) is deduced.
(b) Let $(x(t), y(t))$ be the solution of (3) with initial condition $x(t_0) > 0, y(t_0) > 0$. Let $\bar{x}(t)$ be the solution of (5) with $\bar{x}(t_0) = x(t_0)$. According to Lemma 2.3, we obtain $\bar{x}(t) > x(t)$ for all $t \geq t_0$. On the other hand, applying Lemma 2.2 we receive $\lim\limits_{t\to+\infty} |\bar{x}(t) - X^0(t)| = 0$. Thus, there exists $t_2 > t_1$ such that

$$x(t) < X^0(t) + \eta, \text{ for all } t \geq t_2. \tag{9}$$

Thus, (9) implies that for $t \geq t_2$

$$\begin{aligned}\dot{y} = y[r_2(t) - \frac{a_2(t)y}{x + k_2(t)}] &\leq y[r_2(t) - \frac{a_2(t)y}{X^0(t) + \eta + k_2(t)}] \\ &\leq y[r_2(t) - \frac{a_2(t)y}{M_1 + k_2(t)}].\end{aligned}$$

Let $\bar{y}(t)$ be the solution of $\dot{\bar{y}} = \bar{y}[r_2(t) - \frac{a_2(t)\bar{y}}{M_1 + k_2(t)}]$ with $\bar{y}(t_2) = y(t_2)$, then $y(t) \leq \bar{y}(t)$ for all $t \geq t_2$. Thus, by Lemma 2.2 (iii) and (8), we have

$$\limsup_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} \bar{y}(t) \leq \limsup_{t\to+\infty} \frac{r_2(t)(M_1 + k_2(t))}{a_2(t)} < M_2.$$

This implies that there exists $t_3 \geq t_2$ such that

$$y(t) < M_2, \text{ for all } t \geq t_3. \tag{10}$$

By (10), for $t \geq t_3$ we have

$$\dot{x} = x[r_1(t) - b(t)x - \frac{a_1(t)y}{x + k_1(t)}] \geq x[r_1(t) - b(t)x - \frac{a_1(t)M_2}{k_1(t)}].$$

Let $\tilde{x}(t)$ be the solution of $\dot{\tilde{x}} = \tilde{x}\left[r_1(t) - b(t)x - \dfrac{a_1(t)M_2}{k_1(t)}\right]$, with $\tilde{x}(t_3) = x(t_3)$, then $x(t) \geq \tilde{x}(t)$ for all $t \geq t_3$. Thus, by Lemma 2.2 (iii) and (8), we have

$$\liminf_{t\to+\infty} x(t) \geq \liminf_{t\to+\infty} \tilde{x}(t) \geq \liminf_{t\to+\infty} \frac{1}{b(t)}\left[r_1(t) - \frac{a_1(t)}{k_1(t)}M_2\right] > m_1.$$

Thus, there exists $t_4 \geq t_3$ such that

$$x(t) > m_1, \text{ for all } t \geq t_4. \tag{11}$$

By (11), for $t \geq t_4$ we have

$$\dot{y} = y[r_2(t) - \frac{a_2(t)y}{x + k_2(t)}] \leq y[r_2(t) - \frac{a_2(t)y}{m_1 + k_2(t)}].$$

Let $\tilde{y}(t)$ be the solution of

$$\dot{\tilde{y}} = \tilde{y}\left[r_2(t) - \frac{a_2(t)\tilde{y}}{m_1 + k_2(t)}\right],$$

with $\tilde{y}(t_4) = y(t_4)$, then $\tilde{y}(t) \geq y(t)$ for all $t \geq t_4$. Once again using Lemma 2.2 (iii) and (8), we get

$$\liminf_{t\to+\infty} y(t) \geq \liminf_{t\to+\infty} \tilde{y}(t) \geq \liminf_{t\to+\infty} \frac{r_2(t)(m_1 + k_2(t))}{a_2(t)} > m_2.$$

Hence, there exists $T \geq t_4$ such that

$$y(t) > m_2, \quad \text{for all} \quad t \geq T. \tag{12}$$

From (9), (10), (11) and (12), it follows that $(x(t), y(t)) \in \mathcal{A}_\eta(t)$ for all $t \geq T$. The proof is complete. ∎

Theorem 2.5. *Suppose that the condition* (6) *holds. If*

$$\liminf_{t\to+\infty}\left[\left(b(t) - \frac{a_1(t)M_2^+}{(m_1^+ + k_1(t))^2}\right)\frac{(m_1^+ + k_2(t))^2}{M_2^+} - \frac{a_1(t)(X^0(t) + k_2(t))}{m_1^+ + k_1(t)}\right] > 0, \tag{13}$$

then system (3) *is globally asymptotically stable.*

Proof. It is easy to see that there exist positive numbers $M_1,\ M_2,\ m_1,\ m_2$ and η with $\eta < \frac{M_1 - M_1^+}{2}$, $m_1 < m_1^+,\ M_1^+ < M_1,\ M_2^+ < M_2,\ m_2^+ > m_2$ such that (7) and

$$\liminf_{t\to+\infty}\left[\left(b(t) - \frac{a_1(t)M_2}{(m_1 + k_1(t))^2}\right)\frac{(m_1 + k_2(t))^2}{M_2} - \frac{a_1(t)(X^0(t) + \eta + k_2(t))}{m_1 + k_1(t)}\right] > 0 \tag{14}$$

hold. Thus, there exist numbers $t_1 \in \mathbb{R}$, $\alpha > 0$, $\varepsilon > 0$, $(\alpha > \varepsilon)$ such that (8) holds and the following inequalities are satisfied

$$\begin{aligned} &\Big(b(t) - \frac{a_1(t)M_2}{(m_1+k_1(t))^2}\Big) \cdot \frac{(m_1+k_2(t))^2}{a_2(t)M_2} > \alpha + \varepsilon, \\ &\alpha - \varepsilon > \frac{a_1(t)(X^0(t)+\eta+k_2(t))}{a_2(t)(m_1+k_1(t))}, \text{ for all } t \geq t_1. \end{aligned} \tag{15}$$

Hence, for $t \geq t_1$

$$\begin{aligned} -b(t) + \frac{a_1(t)M_2}{(m_1+k_1(t))^2} + \alpha\frac{a_2(t)M_2}{(m_1+k_2(t))^2} &< -\varepsilon\frac{a_2(t)M_2}{(m_1+k_2(t))^2}, \\ \frac{a_1(t)}{m_1+k_1(t)} - \alpha\frac{a_2(t)}{X^0(t)+\eta+k_2(t)} &< -\varepsilon\frac{a_2(t)}{X^0(t)+\eta+k_2(t)}. \end{aligned} \tag{16}$$

Since, $X^0(t)$, $a_2(t)$ are bounded above and below by positive constants, $k_2(t)$ is nonnegative and bounded, there exists $\gamma > 0$ such that

$$\inf_{t\in\mathbb{R}} \frac{a_2(t)M_2}{(m_1+k_2(t))^2} > \gamma, \quad \inf_{t\in\mathbb{R}} \frac{a_2(t)}{X^0(t)+\eta+k_2(t)} > \gamma. \tag{17}$$

We now prove that system (3) is globally asymptotically stable. Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ be two solutions of (3) with the initial conditions $x_i(t_0) > 0$, $y_i(t_0) > 0$, $(i = 1, 2)$. According to Theorem 2.4, there exists $T \geq t_1$ such that $(x_i(t), y_i(t)) \in \mathcal{A}_\eta(t)$, for all $t \geq T$, $(i = 1, 2)$. Let

$$V = |\ln\frac{x_1(t)}{x_2(t)}| + \alpha|\ln\frac{y_1(t)}{y_2(t)}|, \quad t \geq T.$$

Calculating the upper right derivative of $|\ln\frac{x_1(t)}{x_2(t)}|$, we have

$$\begin{aligned} &D^+\left|\ln\frac{x_1(t)}{x_2(t)}\right| \\ &= \Big[-b(t)\,(x_1(t)-x_2(t)) - \frac{a_1(t)y_1(t)}{x_1(t)+k_1(t)} + \frac{a_1(t)y_2(t)}{x_2(t)+k_1(t)}\Big]\,\mathrm{sgn}\,(x_1(t)-x_2(t)). \end{aligned}$$

By using the mean value theorem of differential calculus to the function $f(x,y) = \frac{y}{x+k_1}$, we obtain

$$\frac{y_1(t)}{x_1(t)+k_1(t)} - \frac{y_2(t)}{x_2(t)+k_2(t)} = -\eta_1(t)\frac{x_1(t)-x_2(t)}{(\xi_1(t)+k_1(t))^2} + \frac{y_1(t)-y_2(t)}{\xi_1(t)+k_1(t)},$$

where $\xi_1(t)$ lies between $x_1(t)$ and $x_2(t)$, $\eta_1(t)$ lies between $y_1(t)$ and $y_2(t)$. Therefore, for $t \geq T$ we have

$$D^+\left|\ln\frac{x_1(t)}{x_2(t)}\right| \le \left[-b(t)+\frac{a_1(t)M_2}{(m_1+k_1(t))^2}\right]|x_1(t)-x_2(t)|+\frac{a_1(t)|y_1(t)-y_2(t)|}{m_1+k_1(t)}.$$

Similarly, there exist $\xi_2(t)$ lying between $x_1(t)$ and $x_2(t)$ and $\eta_2(t)$ lying between $y_1(t)$ and $y_2(t)$ such that

$$\begin{aligned}
&D^+\left|\ln\frac{y_1(t)}{y_2(t)}\right|\\
&=[-\frac{a_2(t)y_1(t)}{x_1(t)+k_2(t)}+\frac{a_2(t)y_2(t)}{x_2(t)+k_2(t)}]\operatorname{sgn}(y_1(t)-y_2(t))\\
&=[-\frac{a_2(t)\eta_2(t)(x_2(t)-x_1(t))}{(\xi_2(t)+k_2(t))^2}+\frac{a_2(t)(y_2(t)-y_1(t))}{\xi_2(t)+k_2(t)}]\operatorname{sgn}(y_1(t)-y_2(t))\\
&\le\frac{a_2(t)M_2}{(m_1+k_2(t))^2}|x_1(t)-x_2(t)|-\frac{a_2(t)}{X^0(t)+\eta+k_2(t)}|y_1(t)-y_2(t)|.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
D^+V\le&\left[-b(t)+\frac{a_1(t)M_2}{(m_1+k_1(t))^2}+\alpha\frac{a_2(t)M_2}{(m_1+k_2(t))^2}\right]|x_1(t)-x_2(t)|\\
&+\left[\frac{a_1(t)}{m_1+k_1(t)}-\alpha\frac{a_2(t)}{X^0(t)+\eta+k_2(t)}\right]|y_1(t)-y_2(t)|.
\end{aligned}$$

By (16) and (17), for $t\ge T$ we obtain

$$\begin{aligned}
D^+V&\le-\varepsilon\gamma[|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|]\\
&\le-\varepsilon\gamma\left[m_1|\ln\frac{x_1(t)}{x_2(t)}|+\frac{m_2}{\alpha}\alpha|\frac{y_1(t)}{y_2(t)}|\right]\\
&\le-\varepsilon\gamma\delta V,\ \text{where } \delta=\min\{m_1,\frac{m_2}{\alpha}\}.
\end{aligned}$$

Thus, $V(t)\le V(T)\exp\{-\varepsilon\gamma\delta(t-T)\}\to 0,$ as $t\to+\infty$. Therefore,

$$\lim_{t\to+\infty}(|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|)=0.$$

The theorem is proved. ■

3. The Existence of Positive Almost Periodic Solution

Definition 3.1. A solution $(x(t),y(t))$ of (3) is said to be positively bounded if it is defined on $\mathbb{R}$ and

$$0<\inf_{t\in\mathbb{R}}x(t)\le\sup_{t\in\mathbb{R}}x(t)<+\infty,\quad 0<\inf_{t\in\mathbb{R}}y(t)\le\sup_{t\in\mathbb{R}}y(t)<+\infty.$$

Theorem 3.2. *Let the condition* (6) *hold. If*

$$\liminf_{t\to-\infty}[r_1(t) - \frac{a_1(t)M_2^-}{k_1(t)}] > 0, \tag{18}$$

then the system (3) *has at least one positively bounded solution.*

From (18), it is easy to see that there exist positive constants $\bar{M}_2,\ \bar{M}_1,\ \bar{m}_1,\ \bar{m}_2$ such that

$$\begin{gathered}\bar{M}_1 > M_1^-,\ \bar{M}_2 > M_2^-,\ \bar{m}_1 < m_1^-,\ \bar{m}_2 < m_2^-,\\ \liminf_{t\to-\infty}(r_1(t) - \frac{a_1(t)\bar{M}_2}{k_1(t)}) > 0,\ \ \limsup_{t\to-\infty}\frac{r_2(t)(\bar{M}_1 + k_2(t))}{a_2(t)}) < \bar{M}_2,\\ \liminf_{t\to-\infty}\frac{1}{b(t)}(r_1(t) - \frac{a_1(t)\bar{M}_2}{k_1(t)}) > \bar{m}_1,\ \liminf_{t\to-\infty}\frac{r_2(t)(\bar{m}_1 + k_2(t))}{a_2(t)}) > \bar{m}_2.\end{gathered} \tag{19}$$

Thus, there exists $t_2 \in \mathbb{R}$ such that for $t \leq t_2$ we have

$$\begin{aligned}&X^0(t) < \frac{\bar{M}_1 + M_1^-}{2}, \quad \frac{r_2(t)(\bar{M}_1 + k_2(t))}{a_2(t)} < \bar{M}_2,\\ &\bar{m}_1 < \frac{1}{b(t)}[r_1(t) - \frac{a_1(t)\bar{M}_2}{k_1(t)}], \quad \bar{m}_2 < \frac{r_2(t)(\bar{m}_1 + k_2(t))}{a_2(t)}.\end{aligned} \tag{20}$$

For $0 < \beta < \frac{\bar{M}_1 - M_1^-}{2}$ and $t \leq t_2$, let us denote

$$\begin{aligned}\mathcal{A}_0^-(t) &= \{(x,y):\ \bar{m}_1 \leq x \leq X^0(t),\ \bar{m}_2 \leq y \leq \bar{M}_2\},\\ \mathcal{A}_\beta^-(t) &= \{(x,y):\ \bar{m}_1 \leq x \leq X^0(t) + \beta,\ \bar{m}_2 \leq y \leq \bar{M}_2\}.\end{aligned} \tag{21}$$

Proof of Theorem 3.1. By the same argument as in the proof of Theorem 2.4 (i), we have that if $(x(t), y(t))$ is a solution of (3) with $(x(t_0), y(t_0)) \in \mathcal{A}_0^-(t_0)$ for some $t_0 < t_2$, then $(x(t), y(t)) \in \mathcal{A}_0^-(t)$ for all $t \in [t_0, t_2]$. Let n_0 be a positive integer such that $-n_0 \leq t_2$. For each positive integer $n \geq n_0$, let $(x_n(t), y_n(t))$ be a solution of (3) with $(x_n(-n), y_n(-n)) = (\bar{m}_1, \bar{m}_2)$. We have that the solution $(x_n(t), y_n(t))$ is defined on $[-n, t_2]$ and $(x_n(t), y_n(t)) \in \mathcal{A}_0^-(t),\ \forall t \in [-n, t_2],\ \forall n \geq n_0$. In particular $(x_n(t_2), y_n(t_2)) \in \mathcal{A}_0^-(t_2), \forall n \geq n_0$. Therefore, there exists a sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that $(x_{n_j}(t_2), y_{n_j}(t_2)) \to (\xi, \eta) \in \mathcal{A}_0^-(t_2)$, as $j \to \infty$. Let $(x^*(t), y^*(t))$ be the solution of (3) with initial condition $(x^*(t_2), y^*(t_2)) = (\xi, \eta)$. Since $(x_{n_j}(t_2), y_{n_j}(t_2)) \to (\xi, \eta)$, as $j \to \infty$, it follows that $\{(x_{n_j}(t), y_{n_j}(t))\}_{j=1}^{\infty}$ uniformly converges to $(x^*(t), y^*(t))$ on any compact subset of the domain of $(x^*(t), y^*(t))$.

We now prove that $(-\infty, t_2]$ is contained in the domain of $(x^*(t), y^*(t))$. Suppose that there exists $t_0 \in (-\infty, t_2)$ such that $(x^*(t_0), y^*(t_0)) \notin \mathcal{A}_0^-(t_0)$. Let j_0 be a positive integer such that $-n_{j_0} < t_0$. We have that $\{(x_{n_j}(t_0), y_{n_j}(t_0))\}_{j=j_0}^{\infty}$ converges to $(x^*(t_0), y^*(t_0))$ as $j \to \infty$. Hence, $(x^*(t_0), y^*(t_0)) \in \mathcal{A}_0^-(t_0)$, which is a contradiction. Thus $(-\infty, t_2]$ is a subset of the domain of $(x^*(t), y^*(t))$.

For each $\bar{t} \in (-\infty, t_2]$, we have $\{(x_{n_j}(\bar{t}), y_{n_j}(\bar{t}))\}_{j=1}^{\infty}$ converges to $(x^*(\bar{t}), y^*(\bar{t}))$, as $j \to \infty$. Thus, $(x^*(\bar{t}), y^*(\bar{t})) \in \mathcal{A}_0^-(\bar{t})$, for all $\bar{t} \in (-\infty, t_2]$. According to Theorem 2.4, $(x^*(t), y^*(t))$ is defined on $\mathbb{R}$ and its components are bounded above and below by positive constants. The theorem is proved. ■

Theorem 3.3. *Let the conditions* (6) *and* (18) *hold. If*

$$\liminf_{t\to-\infty}\left[\left(b(t)-\frac{a_1(t)M_2^-}{(m_1^-+k_1(t))^2}\right)\frac{(m_1^-+k_2(t))^2}{M_2^-}-\frac{a_1(t)(X^0(t)+k_2(t))}{m_1^-+k_1(t)}\right]>0, \tag{22}$$

then the system (3) *has a unique positively bounded solution.*

Proof. It is easy to see that there exist positive numbers $\bar{M}_1,\ \bar{M}_2,\ \bar{m}_1,\ \bar{m}_2$ and β with $\beta<\dfrac{\bar{M}_1-\bar{M}_1^-}{2},\ \bar{m}_1<m_1^-,\ M_1^-<\bar{M}_1,\ M_2^-<\bar{M}_2,\ \bar{m}_2<m_2^-$ such that (19) and

$$\liminf_{t\to-\infty}\left[\left(b(t)-\frac{a_1(t)\bar{M}_2}{(\bar{m}_1+k_1(t))^2}\right)\frac{(\bar{m}_1+k_2(t))^2}{\bar{M}_2}-\frac{a_1(t)(X^0(t)+\beta+k_2(t))}{\bar{m}_1+k_1(t)}\right]>0 \tag{23}$$

hold. Thus, there exist $t_2\in\mathbb{R}$ and positive constants α,ε $(\alpha>\varepsilon)$ such that (20) is satisfied and

$$\begin{aligned}&\left(b(t)-\frac{a_1(t)\bar{M}_2}{(\bar{m}_1+k_1(t))^2}\right)\cdot\frac{(\bar{m}_1+k_2(t))^2}{a_2(t)\bar{M}_2}>\alpha+\varepsilon,\\ &\alpha-\varepsilon>\frac{a_1(t)(X^0(t)+\beta+k_2(t))}{a_2(t)(\bar{m}_1+k_1(t))},\ \text{for all } t\le t_2.\end{aligned} \tag{24}$$

Hence, for $t\le t_2$

$$\begin{aligned}-b(t)+\frac{a_1(t)\bar{M}_2}{(\bar{m}_1+k_1(t))^2}+\alpha\frac{a_2(t)\bar{M}_2}{(\bar{m}_1+k_2(t))^2}&<-\varepsilon\frac{a_2(t)\bar{M}_2}{(\bar{m}_1+k_2(t))^2},\\ \frac{a_1(t)}{\bar{m}_1+k_1(t)}-\alpha\frac{a_2(t)}{X^0(t)+\beta+k_2(t)}&<-\varepsilon\frac{a_2(t)}{X^0(t)+\beta+k_2(t)}.\end{aligned} \tag{25}$$

Since $X^0(t)$, $a_2(t)$ are bounded above and below by positive constants, $k_2(t)$ is nonnegative and bounded, there exists $\gamma>0$ such that

$$\inf_{t\in\mathbb{R}}\frac{a_2(t)\bar{M}_2}{(\bar{m}_1+k_2(t))^2}>\gamma,\ \inf_{t\in\mathbb{R}}\frac{a_2(t)}{X^0(t)+\beta+k_2(t)}>\gamma. \tag{26}$$

Let $(x^*(t), y^*(t))$ be a positively bounded solution of (3) with $(x^*(t), y^*(t))\in\mathcal{A}_0^-(t)$, for all $t\le t_2$, whose existence is given in the proof of Theorem 3.2.
Let $(\bar{x}(t),\bar{y}(t))$ be another positively bounded solution of (3). In order to get a contradiction, we will show that $(\bar{x}(t),\bar{y}(t))\notin \operatorname{int}\mathbb{R}_+^2$, for all $t\le t_2$. For each $t\le t_2$, let us denote

$$\mathcal{A}^1(t) = \{(x,y) : x > X^0(t), y > 0\},$$
$$\mathcal{A}^2(t) = \{(x,y) : 0 < x \le X^0(t), y > \bar{M}_2\},$$
$$\mathcal{A}^3(t) = \{(x,y) : 0 < x < \bar{m}_1, 0 < y \le \bar{M}_2\},$$
$$\mathcal{A}^4(t) = \{(x,y) : \bar{m}_1 \le x \le X^0(t), 0 < y < \bar{m}_2\}.$$

Clearly that $\text{int}\mathbb{R}^2_+ = \mathcal{A}_0^-(t) \cup \mathcal{A}^1(t) \cup \mathcal{A}^2(t) \cup \mathcal{A}^3(t) \cup \mathcal{A}^4(t)$, for $t \le t_2$.
Claim 1. $(\bar{x}(t), \bar{y}(t)) \notin \mathcal{A}^1(t)$ for all $t \le t_2$.
Suppose that there exists $\bar{t} \in (-\infty, t_2]$ such that $(\bar{x}(\bar{t}), \bar{y}(\bar{t})) \in \mathcal{A}^1(\bar{t})$. Let $\tilde{x}(t)$ be the solution of (5) with $\tilde{x}(\bar{t}) = \bar{x}(\bar{t})$. Lemma 2.3 implies that $\bar{x}(t) > \tilde{x}(t)$ for all $t < \bar{t}$ belonging to the domain of $\tilde{x}(t)$. By the uniqueness of the initial value problem for (5), $\tilde{x}(t) > X^0(t)$ for all t belonging to the domain of $\tilde{x}(t)$. It then follows from the uniqueness of the positively bounded solution $X^0(t)$ of (5) that there exists $\omega \in [-\infty, t_2]$ such that $\limsup_{t\to\omega^+} \tilde{x}(t) = +\infty$. Thus, $\limsup_{t\to\omega^+} \bar{x}(t) = +\infty$, which contradicts the boundedness of $\bar{x}(t)$. The claim is proved.
Claim 2. $(\bar{x}(t), \bar{y}(t)) \notin \mathcal{A}^2(t)$ for all $t \le t_2$.
Suppose that there exists $\bar{t} \in (-\infty, t_2]$ such that $(\bar{x}(\bar{t}), \bar{y}(\bar{t})) \in \mathcal{A}^2(\bar{t})$. By Claim 1, $\bar{x}(t) \le X^0(t)$ for all $t \le \bar{t}$. Thus, whenever $y > \bar{M}_2$ and $t \le \bar{t}$ we have

$$\begin{aligned}
\dot{\bar{y}}(t) &= \bar{y}(t)[r_2(t) - \frac{a_2(t)\bar{y}(t)}{\bar{x}(t) + k_2(t)}] \\
&\le \bar{y}(t)[r_2(t) - \frac{a_2(t)\bar{M}_2}{X^0(t) + k_2(t)}] \\
&\le \bar{y}(t)[r_2(t) - \frac{a_2(t)\bar{M}_2}{\bar{M}_1 + k_2(t)}] \\
&< 0 \quad \text{(by (20))}.
\end{aligned}$$

This implies that $\bar{y}(t) > \bar{M}_2$ for all $t \le \bar{t}$. By (19), there exist $\mu > 0$ and $t_3 \le \bar{t}$ such that

$$r_2(t) - \frac{a_2(t)\bar{M}_2}{\bar{M}_1 + k_2(t)} \le -\mu, \text{ for all } t \le t_3.$$

Thus,

$$\bar{y}(t) \ge \bar{y}(t_3)\exp\{\mu(t_3 - t)\} \to +\infty, \text{ as } t \to -\infty,$$

which contradicts the boundedness of $\bar{y}(t)$. The claim is proved.
Claim 3. $(\bar{x}(t), \bar{y}(t)) \notin \mathcal{A}^3(t)$ for all $t \le t_2$.
Suppose that there exists $\bar{t} \in (-\infty, t_2]$ such that $(\bar{x}(\bar{t}), \bar{y}(\bar{t})) \in \mathcal{A}^3(\bar{t})$. By Claims 1 and 2, $\bar{y}(t) < \bar{M}_2$ for all $t \le \bar{t}$. Thus, whenever $\bar{x}(t) < \bar{m}_1$ and $t \le \bar{t}$, we have

$$\begin{aligned}
\dot{\bar{x}}(t) &= \bar{x}(t)[r_1(t) - b(t)\bar{x}(t) - \frac{a_1(t)\bar{y}(t)}{\bar{x}(t) + k_1(t)}] \\
&\ge \bar{x}(t)[r_1(t) - b(t)\bar{m}_1 - \frac{a_1(t)\bar{M}_2}{k_1(t)}] > 0, \quad \text{(by (20))}.
\end{aligned}$$

This implies that $\bar{x}(t) < \bar{m}_1$ for all $t \le \bar{t}$. It follows from (19) that there exist $\mu > 0$ and $t_4 \le \bar{t}$ such that

$$r_1(t) - b(t)\bar{m}_1 - \frac{a_1(t)\bar{M}_2}{k_1(t)} > \mu, \text{ for all } t \le t_4.$$

Thus,

$$\bar{x}(t) \le \bar{x}(t_4)\exp\{\mu(t-t_4)\} \to 0, \text{ as } t \to -\infty,$$

which contradicts $\inf_{t\in\mathbb{R}} \bar{x}(t) > 0$. The claim is proved.

Claim 4. $(\bar{x}(t), \bar{y}(t)) \notin \mathcal{A}^4(t)$ for all $t \le t_2$.

Suppose that there exists $\bar{t} \in (-\infty, t_2]$ such that $(\bar{x}(\bar{t}), \bar{y}(\bar{t})) \in \mathcal{A}^4(\bar{t})$. By Claims 2 and 3, $\bar{x}(t) \ge \bar{m}_1$ for all $t \le \bar{t}$. Thus, whenever $\bar{y}(t) < \bar{m}_2$ and $t \le \bar{t}$ we have

$$\dot{\bar{y}}(t) = \bar{y}(t)[r_2(t) - \frac{a_2(t)\bar{y(t)}}{\bar{x}(t)+k_2(t)}] \ge \bar{y}(t)[r_2(t) - \frac{a_2(t)\bar{m}_2}{\bar{m_1}+k_2(t)}] > 0, \text{ by(20)}.$$

This implies that $\bar{y}(t) < \bar{m}_2$ for all $t < \bar{t}$. By (19), there exist $\mu > 0$ and $t_3 \le \bar{t}$ such that

$$r_2(t) - \frac{a_2(t)\bar{m}_2}{\bar{m_1}+k_2(t)} \ge \mu, \text{ forall } t \le t_5.$$

Thus,

$$\bar{y}(t) \le \bar{y}(t_5)\exp\{\mu(t-t_5)\} \to 0, \text{ as } t \to -\infty,$$

which contradicts $\inf_{t\in\mathbb{R}} \bar{y}(t) > 0$. The claim is proved.

Claim 5. $(\bar{x}(t), \bar{y}(t)) \notin \mathcal{A}_0^-(t)$ for all $t \le t_2$.

Suppose that there exists $\bar{t} \in (-\infty, t_2]$ such that $(\bar{x}(\bar{t}), \bar{y}(\bar{t})) \in \mathcal{A}_0^-(\bar{t})$. By Claims 1 - 4, we have $(\bar{x}(t), \bar{y}(t)) \in \mathcal{A}_0^-(t)$ for all $t \le t_2$. Denote

$$V(t) = |\ln \frac{x^*(t)}{\bar{x}(t)}| + \alpha|\ln \frac{y^*(t)}{\bar{y}(t)}|, \ t \le t_2.$$

Thus, $V(t_2)$ is bounded on $(-\infty; t_2]$. By the same argument as in the proof of Theorem 2.5, we have

$$D^+V(t) \le -\varepsilon\gamma\delta V(t), \quad \forall t \in (-\infty, t_2],$$

where $\delta = \min\{\bar{m}_1, \frac{\bar{m}_2}{\alpha}\}$. Thus,

$$V(t_2) \le V(t)\exp\{-\varepsilon\gamma\delta(t_2 - t)\} \to 0, \text{ as } t \to -\infty.$$

Hence, $V(t_2) = 0$, which contradicts $V(t_2) > 0$. The claim is proved.

The theorem now follows from Claims 1- 5. ■

Consider the case of r_i, a_i, k_i, $(i = 1, 2)$ and b being almost periodic. We now recall Bochner's criterion for the almost periodicity: *A continuous function $\varphi(t)$, $\varphi : \mathbb{R} \to \mathbb{R}^d$ is almost periodic if and only if for every sequence of real* *numbers* $\{\tau_n\}_{n=1}^{\infty}$*, there exists a subsequence* $\{\tau_{n_k}\}_{k=1}^{\infty}$ *such that the sequence* $\{\varphi(t+\tau_{n_k})\}_{k=1}^{\infty}$ *uniformly converges on* $\mathbb{R}$ (see, for example, [3]).

Theorem 3.4. *Let* $r_i(t), a_i(t)$ $(i=1,2)$*,* $k_1(t)$ *and* $b(t)$ *be almost periodic and bounded below by positive constants. Let* $k_2(t)$ *be almost periodic and nonnegative. If the conditions* (18) *and* (22) *hold, then system* (3) *has a unique positively bounded solution* $(x^*(t), y^*(t))$*. Moreover, that solution is almost periodic and* $\lim_{t\to+\infty}(|x(t)-x^*(t)|+|y(t)-y^*(t)|)=0$*, for any solution* $(x(t),y(t))$ *of* (3) *with initial condition* $x(t_0)>0$*,* $y(t_0)>0$*.*

Proof. It is easy to see that for an almost periodic function $g(t)$, $(g:\mathbb{R}\to\mathbb{R})$ we have

$$\liminf_{t\to+\infty} g(t) = \liminf_{t\to-\infty} g(t) = \inf_{t\in\mathbb{R}} g(t),\ \limsup_{t\to+\infty} g(t) = \limsup_{t\to-\infty} g(t) = \sup_{t\in\mathbb{R}} g(t).$$

Thus, conditions (18) and (22) become respectively

$$\inf_{t\in\mathbb{R}}\left[r_1(t) - \frac{a_1(t)M_2^-}{k_1(t)}\right] > 0, \tag{27}$$

$$\inf_{t\in\mathbb{R}}\left[\left(b(t) - \frac{a_1(t)M_2^-}{(m_1^-+k_1(t))^2}\right)\frac{(m_1^-+k_2(t))^2}{M_2^-} - \frac{a_1(t)(X^0(t)+k_2(t))}{m_1^-+k_1(t)}\right] > 0. \tag{28}$$

Thus, by Theorems 3.3 and 2.5, (3) has a unique positively bounded solution $(x^*(t),y^*(t))$. Moreover $\lim_{t\to+\infty}|x(t)-x^*(t)| = \lim_{t\to+\infty}|y(t)-y^*(t)| = 0$ for any solution $(x(t),y(t))$ of (3) with initial condition $x(t_0)>0$, $y(t_0)>0$. We now show the almost periodicity of $(x^*(t),y^*(t))$. By the almost periodicity, we have

$$M_1^- := \sup_{t\in\mathbb{R}} X^0(t), \qquad M_2^- := \sup_{t\in\mathbb{R}} \frac{r_2(t)(M_1^-+k_2(t))}{a_2(t)},$$

$$m_1^- := \inf_{t\in\mathbb{R}} \frac{1}{b(t)}[r_1(t) - \frac{a_1(t)M_2^-}{k_1(t))}], \qquad m_2^- := \inf_{t\in\mathbb{R}} \frac{r_2(t)(m_1^-+k_2(t)}{a_2(t)}.$$

From (27), it follows that $m_1^->0$ and $m_2^->0$. Thus, there exist positive numbers m_1, M_1, m_2, M_2 with $m_1<m_1^-$, $m_2<m_2^-$, $M_1>M_1^-$, $M_2>M_2^-$ such that

$$\sup_{t\in\mathbb{R}} \frac{r_2(t)[M_1+k_2(t)]}{a_2(t)} < M_2,$$

$$\inf_{t\in\mathbb{R}} \frac{1}{b(t)}[r_1(t) - \frac{a_1(t)M_2}{k_1(t))}] > m_1, \ \inf_{t\in\mathbb{R}} \frac{r_2(t)(m_1+k_2(t)}{a_2(t)} > m_2.$$

Claim 1. $(x^*(t),y^*(t)) \in [m_1,M_1]\times[m_2,M_2]$, for all $t\in\mathbb{R}$.
Proof of Claim 1: For $t\in\mathbb{R}$, let us put

$$\mathcal{A}_0(t) = \{(x,y):\ m_1 \le x \le X^0(t), m_2 \le y \le M_2\}.$$

By the same argument as in the proof of Theorem 2.4(i), we can see that if $(x(t), y(t))$ is a solution of (3) with $(x(t_0), y(t_0)) \in \mathcal{A}_0(t_0)$ for some $t_0 \in \mathbb{R}$, then $(x(t), y(t)) \in \mathcal{A}_0(t)$ for all $t \geq t_0$. By the same argument as in the proof of Theorem 3.2, for any $t_2 \in \mathbb{R}$, there exists a positive bounded solution $(\bar{x}(t), \bar{y}(t))$ of (3) such that $(\bar{x}(t), \bar{y}(t)) \in \mathcal{A}_0(t)$ for all $t \leq t_2$. Since t_2 is arbitrary, the uniqueness of a positively bounded solution of (3) implies that $(\bar{x}(t), \bar{y}(t)) = (x^*(t), y^*(t)) \in \mathcal{A}_0(t)$ for all $t \in \mathbb{R}$. Thus, $(x^*(t), y^*(t)) \in [m_1, M_1] \times [m_2, M_2]$ for all $t \in \mathbb{R}$. The Claim is proved.

Let $\{\tau_n\}_{n=1}^{\infty}$ be an arbitrary sequence of numbers, we wish to show that there exists a subsequence $\{\tau_{n_k}\}_{k=1}^{\infty}$ such that the sequence $\{(x^*(t+\tau_{n_k}), y^*(t+\tau_{n_k}))\}_{k=1}^{\infty}$ uniformly converges on $\mathbb{R}$. Since $r_i(t), a_i(t)$, $k_i(t)$ $(i = 1, 2)$, $b(t)$ and $X^0(t)$ are almost periodic, it follows from Bochner's criterion that there exists a subsequence $\{\tau_{n_k}\}_{k=1}^{\infty}$ of $\{\tau_n\}_{n=1}^{\infty}$ such that $r_i(t+\tau_{n_k}), a_i(t+\tau_{n_k}), k_i(t+\tau_{n_k})$ $(i = 1, 2)$, $b(t + \tau_{n_k})$, $X^0(t + \tau_{n_k})$ uniformly converge to $r_i^*(t)$, $a_i^*(t)$, $k_i^*(t)$ $(i = 1, 2)$, $b^*(t)$, $X^{0^*}(t)$, respectively, on $\mathbb{R}$ as $k \to \infty$. Thus, $X^{0^*}(t)$ is a unique positively bounded solution of the logistic equation

$$\dot{x} = x[r_1^*(t) - b^*(t)x], \tag{29}$$

and so the system

$$\begin{aligned} \dot{x} &= x\Big[r_1^*(t) - b^*(t)x - \frac{a_1^*(t)}{x + k_1^*(t)}y\Big], \\ \dot{y} &= y\Big[r_2^*(t) - \frac{a_2^*(t)}{x + k_2^*(t)}y\Big] \end{aligned} \tag{30}$$

satisfies the inequalities (27) and (28). By Theorem 3.3 and Claim 1, (30) has a unique positive solution $(\bar{x}(t), \bar{y}(t))$ which is defined on $\mathbb{R}$ and belongs to $[m_1, M_1] \times [m_2, M_2]$, for all $t \in \mathbb{R}$.

Claim 2. $\{(x^*(t + \tau_{n_k}), y^*(t + \tau_{n_k}))\}_{k=1}^{\infty}$ uniformly converges to $(\bar{x}(t), \bar{y}(t))$ on $\mathbb{R}$ as $k \to \infty$.

Proof of Claim 2. Suppose that it is false. Then there exist a subsequence $\{\tau_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{\tau_{n_k}\}_{k=1}^{\infty}$, a sequence of numbers $\{s_j\}_{j=1}^{\infty}$ and a number $\alpha > 0$ such that

$$\|(x^*(s_j + \tau_{n_{k_j}}), y^*(s_j + \tau_{n_{k_j}})) - (\bar{x}(s_j), \bar{y}(s_j))\| \geq \alpha, \text{ for all } j. \tag{31}$$

By the almost periodicity, we may assume, without loss of generality, that $r_i(t+s_j + \tau_{n_{k_j}}) \to \hat{r}_i(t)$, $a_i(t + s_j + \tau_{n_{k_j}}) \to \hat{a}_i(t)$, $k_i(t + s_j + \tau_{n_{k_j}}) \to \hat{k}_i(t)$, $b(t + s_j + \tau_{n_{k_j}}) \to \hat{b}(t)$, $X^0(t + s_j + \tau_{n_{k_j}}) \to \hat{X}^0(t)$ as $j \to \infty$ $(i = 1, 2)$, uniformly with respect to t. Thus, $r_i^*(t + s_j) \to \hat{r}_i(t)$, $a_i^*(t + s_j) \to \hat{a}_i(t)$, $k_i^*(t + s_j) \to \hat{k}_i(t)$, $b^*(t + s_j) \to \hat{b}(t)$, $X^{0^*}(t + s_j) \to \hat{X}^0(t)$ $(i = 1, 2)$ as $j \to \infty$, uniformly with respect to t. Since $(x^*(t), y^*(t)) \in [m_1, M_1] \times [m_2, M_2]$ for all $t \in \mathbb{R}$, we can assume, without loss of generality, that $(x^*(s_j + \tau_{n_{k_j}}), y^*(s_j + \tau_{n_{k_j}})) \to (\xi^*, \eta^*)$, as $j \to \infty$. Similarly, we may assume that $(\bar{x}(s_j), \bar{y}(s_j)) \to (\bar{\xi}, \bar{\eta})$. By (31), it

follows that $\|(\xi^*,\eta^*)-(\bar{\xi},\bar{\eta})\| \geq \alpha$.
For each $j=1,2,\ldots$, $(x^*(t+s_j+\tau_{n_{k_j}}), y^*(t+s_j+\tau_{n_{k_j}}))$ is a solution of the system

$$\begin{aligned}\dot{x} &= x\Big[r_1(t+s_j+\tau_{n_{k_j}}) - b(t+s_j+\tau_{n_{k_j}})x - \frac{a_1(t+s_j+\tau_{n_{k_j}})}{x+k_1(t+s_j+\tau_{n_{k_j}})}y\Big],\\ \dot{y} &= y\Big[r_2(t+s_j+\tau_{n_{k_j}}) - \frac{a_2(t+s_j+\tau_{n_{k_j}})}{x+k_2(t+s_j+\tau_{n_{k_j}})}y\Big].\end{aligned} \tag{32}$$

Let $(\hat{x}(t),\hat{y}(t))$ be the solution with the initial condition $(\hat{x}(t_0),\hat{y}(t_0))=(\bar{\eta},\bar{\xi})$ of the following system

$$\begin{aligned}\dot{x} &= x\Big[\hat{r}_1(t) - \hat{b}(t)x - \frac{\hat{a}_1(t)}{x+\hat{k}_1(t)}y\Big],\\ \dot{y} &= y\Big[\hat{r}_2(t) - \frac{\hat{a}_2(t)}{x+\hat{k}_2(t)}y\Big].\end{aligned} \tag{33}$$

Since the right hand side of (32) uniformly converges to the right hand side of (33) as $j\to\infty$, on any set of the form $\mathbb{R}\times K$, where K is a compact subset of $\mathbb{R}_+$, we have $(x^*(t+s_j+\tau_{n_{k_j}}), y^*(t+s_j+\tau_{n_{k_j}})) \to (\hat{x}(t),\hat{y}(t))$ as $j\to\infty$ uniformly on any compact subset of the domain of $(\hat{x}(t),\hat{y}(t))$. Since $(x^*(t+s_j+\tau_{n_{k_j}}), y^*(t+s_j+\tau_{n_{k_j}})) \in [m_1,M_1]\times[m_2,M_2]$ for all $t\in\mathbb{R}$, it follows that $(\hat{x}(t),\hat{y}(t))$ is defined on $\mathbb{R}$ and $(\hat{x}(t),\hat{y}(t))\in[m_1,M_1]\times[m_2,M_2]$ for all $t\in\mathbb{R}$.

Now recall that $(\bar{x}(t),\bar{y}(t))$ is the unique solution of (30) with $(\bar{x}(t),\bar{y}(t))\in[m_1,M_1]\times[m_2,M_2]$ for all $t\in\mathbb{R}$. Thus, for each $j=1,2,\ldots$, $(\bar{x}(t+s_j),\bar{y}(t+s_j))$ is a solution of

$$\begin{aligned}\dot{x} &= x\Big[r_1^*(t+s_j) - b^*(t+s_j)x - \frac{a_1^*(t+s_j)}{x+k_1^*(t+s_j)}y\Big],\\ \dot{y} &= y\Big[r_2^*(t+s_j) - \frac{a_2^*(t+s_j)}{x+k_2^*(t+s_j)}y\Big],\end{aligned} \tag{34}$$

with $(\bar{x}(s_j),\bar{y}(s_j)) = (\bar{\xi},\bar{\eta})$. Let $(\bar{x}^*(t),\bar{y}^*(t))$ be the solution of (33) with $(\bar{x}^*(0),\bar{y}^*(0))=(\bar{\xi},\bar{\eta})$. Since the right hand side of (34) uniformly converges to the right and side of (33) as $j\to\infty$ on any set of the form $\mathbb{R}\times K$, where K is a compact subset of $\mathbb{R}_+$, it follows that $(\bar{x}(t+s_j),\bar{y}(t+s_j))\to(\bar{x}^*(t),\bar{y}^*(t))$ as $j\to\infty$ uniformly on any compact subset of the domain of $(\bar{x}^*(t),\bar{y}^*(t))$. Since $(\bar{x}(t+s_j),\bar{y}(t+s_j))\in[m_1,M_1]\times[m_2,M_2]$ for all $t\in\mathbb{R}$, we have that $(\bar{x}^*(t),\bar{y}^*(t))$ is defined on $\mathbb{R}$ and $(\bar{x}^*(t),\bar{y}^*(t))\in[m_1,M_1]\times[m_2,M_2]$ for all $t\in\mathbb{R}$. By the uniqueness of the positively bounded solution of (33), we have $(\bar{x}^*(t),\bar{y}^*(t))\equiv(\hat{x}(t),\hat{y}(t))$, which contradicts the fact that $(\bar{x}^*(0),\bar{y}^*(0)) = (\bar{\xi},\bar{\eta})\neq(\xi^*,\eta^*)=(\hat{x}(0),\hat{y}(0))$. The claim is proved.

Now the theorem follows from Claim 2 and Bochner's criterion. ■

Corollary 3.5. *Let $r_i(t), a_i(t)$ $(i = 1, 2)$, $k_1(t)$ and $b(t)$ be positive T-periodic. Let $k_2(t)$ be T- periodic and nonnegative. If the conditions* (18) *and* (22) *hold, then system* (3) *has a unique positive T- periodic solution $(x^*(t), y^*(t))$. Moreover,*

$$\lim_{t\to+\infty} |x(t) - x^*(t)| = \lim_{t\to+\infty} |y(t) - y^*(t)| = 0$$

for any solution $(x(t), y(t))$ of (3) *with initial condition $x(t_0) > 0$, $y(t_0) > 0$.*

Proof. By Theorem 3.4, system (3) has a unique positive bounded solution $(x^*(t), y^*(t))$ and $\lim_{t\to+\infty} |x(t) - x^*(t)| = \lim_{t\to+\infty} |y(t) - y^*(t)| = 0$. Clearly that $(x^*(t), y^*(t))$ and $(x^*(t + T), y^*(t + T))$ are solutions of the following system

$$\begin{aligned}\dot{x} &= x[r_1(t+T) - b(t+T)x - \frac{a_1(t+T)y}{x + k_1(t+T)}],\\ \dot{y} &= y[r_2(t+T) - \frac{a_2(t+T)y}{x + k_2(t+T)}].\end{aligned} \tag{35}$$

By the uniqueness, we have $(x^*(t), y^*(t)) = (x^*(t + T), y^*(t + T))$, $\forall t \in \mathbb{R}$, i.e., $(x^*(t), y^*(t))$ is T-periodic. The proof is complete. ■

Remark 3.6. In [14], it was showed that (Theorem 3.1 [14]): *If*

$$\liminf_{t\to+\infty}[r_1(t) - \frac{a_1(t)}{k_1(t)} M_2^*] > 0, \tag{A1}$$

$$\liminf_{t\to+\infty} \Big[b(t) - \Big(\frac{a_1(t)}{(m_1^* + k_1(t))^2} + \frac{a_2(t)}{(m_1^* + k_2(t))^2}\Big) M_2^*\Big] > 0, \tag{A2}$$

$$\liminf_{t\to+\infty} \Big[\frac{a_2(t)}{M_1^* + k_2(t)} - \frac{a_1(t)}{m_1^* + k_1(t)}\Big] > 0, \tag{A3}$$

where

$$M_1^* = \limsup_{t\to+\infty} \frac{r_1(t)}{b_1(t)}, \quad M_2^* = \limsup_{t\to+\infty} \frac{r_2(t)(M_1^* + k_2(t))}{a_2(t)},$$

$$m_1^* = \liminf_{t\to+\infty} \frac{1}{b_1(t)}[r_1(t) - \frac{a_1(t)}{k_1(t)} M_2^*], \quad m_2^* = \liminf_{t\to+\infty} \frac{r_2(t)(m_1^* + k_2(t))}{a_2(t)},$$

then system (3) *is globally asymptotically stable.*

Since $M_1^* \geq \limsup_{t\to+\infty} X^0(t) = M_1^+$, the condition (A1) implies the condition (6). The following example shows that (6) is more general than (A1).

Example 3.7.

$$\dot{x} = x\left[1 - (\gamma + \cos t)x - \frac{y}{x+1}\right], \ \dot{y} = y\left[\frac{1}{4.5} - \frac{y}{x+1}\right].$$

Let $M > 3.5$, $\gamma = 1 + \dfrac{1}{M}$. Then $M_1^* = M$ and $M_2^* = \dfrac{M+1}{4.5} > 1$. Thus, (A1) is not satisfied. On the other hand, $X^0(t) = \dfrac{2}{2\gamma + \cos t + \sin t}$, $M_1^+ = \dfrac{2}{2\gamma - \sqrt{2}} <$ 3.5 and $M_2^+ = \dfrac{M_1^+ + 1}{4.5} < 1$. Thus, (6) is satisfied.

We now show that (A2) and (A3) imply (13). Indeed, from (A2) and (A3) there exist $\varepsilon > 0,\ t_1 \in \mathbb{R}$ such that for all $t \geq t_1$ we have

$$\left(b(t) - \frac{a_1(t)M_2^*}{(m_1^* + k_1(t))^2}\right)\frac{(m_1^* + k_2(t))^2}{M_2^*} > a_2(t) > \frac{a_1(t)[M_1^* + k_2(t)]}{m_1^* + k_1(t)} + \varepsilon.$$

Thus,

$$\left(b(t) - \frac{a_1(t)M_2^*}{(m_1^* + k_1(t))^2}\right)\frac{(m_1^* + k_2(t))^2}{M_2^*} - \frac{a_1(t)[M_1^* + k_2(t)]}{m_1^* + k_1(t)} > \varepsilon, \text{ for all } t \geq t_1.$$

Since $m_1^+ \geq m_1^*,\ M_2^* \geq M_2^+$ and $M_1^* \geq \limsup\limits_{t\to+\infty} X^0(t)$, it follows that

$$\liminf_{t\to+\infty}\left[\left(b(t) - \frac{a_1(t)M_2^+}{(m_1^+ + k_1(t))^2}\right)\frac{(m_1^+ + k_2(t))^2}{M_2^+} - \frac{a_1(t)[X^0(t) + k_2(t)]}{m_1^+ + k_1(t)}\right] > 0,$$

i. e., (13) is satisfied if (A2) and (A3) hold.

The following example shows that (13) does not imply both (A2) and (A3).

Example 3.8.

$$\dot{x} = x\left[1 - x - \frac{y}{x+1}\right],\ \dot{y} = y\left[1 - \frac{4y}{x+1}\right].$$

Clearly that $M_1^* = M_1^+ = 1$, $M_2^* = M_2^+ = m_1^* = m_1^+ = \dfrac{1}{2}$. Thus, (6), (13), (A1) and (A3) are satisfied, but (A2) is not satisfied. Thus, our work improves and extends the main results in [14].

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