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Monomials Basis of the Araki-Kudo-Dyer-Lashof Algebra*

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Abstract. The Araki-Kudo-Dyer-Lashof algebra R, which is an algebra of operations acting on the homology of infinite loop space, is isomorphic to the algebra of Dickson coinvariants. In this paper, we give a new basis for the Araki-Kudo-Dyer-Lashof algebra and discuss its relationship with other known bases.

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1. Introduction and Statement of Results

Let $\mathcal{F}$ be the free graded associative algebra with unit over $\mathbb{F}_2$ generated by the symbols $Q^0, Q^1, \ldots, Q^i, \ldots,$ where $\deg Q^i = i$. For any string of non-negative integers $I = (i_{k-1}, \ldots, i_0)$, define $Q^I = Q^{i_{k-1}} \cdots Q^{i_0}$. We call that Q^I (or I) is *admissible* if $i_s \leq 2i_{s-1}$, for $1 \leq s \leq k-1$, and define the *excess* of Q^I (or I) to be

$$e(Q^I) = i_{k-1} - \sum_{j=0}^{k-2} i_j.$$

The *length* of Q^I, $\ell(Q^I)$ is the number of integers in I, i.e. $\ell(Q^I) = \ell(I) = k$ if $I = (i_{k-1}, \ldots, i_0)$. The *degree* of Q^I is $i_{k-1} + \cdots + i_0$.

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Let $\mathcal{J}$ be the two-sided ideal of $\mathcal{F}$ generated by the elements of one of the following forms:

(i) $Q^aQ^b + \sum_i \binom{i-b-1}{2i-a} Q^{a+b-i}Q^i,\ a > 2b.$

(ii) Q^I, with $e(Q^I) < 0$.

The quotient algebra $R = \mathcal{F}/\mathcal{J}$ is called the Araki-Kudo-Dyer-Lashof algebra. It was used to describe the mod 2 homology of the infinite loop space QS^0 ([1], [3]), namely,

$$H_*(QS^0) = P[Q^I[1] | I \quad \text{admissible},\ e(Q^I) \geq 0] \otimes \mathbb{F}_2[\mathbb{Z}],$$

where $[1] \in H_*(QS^0)$ is the image of the non-base point generator of $H_0S^0 = \mathbb{F}_2 \otimes \mathbb{F}_2$ under the canonical inclusion $S^0 \hookrightarrow QS^0$.

The relations (i) are usually called Adem relations because of their similarity with the usual Adem relations in the Steenrod algebra. Together with (ii), it is well-known that the set of all admissible monomials of non-negative excess forms an additive basis of R. This basis is called the admissible basis.

Let $R[k]$ be the subspace of R spanned by the elements Q^I of length k. In fact, $R[k]$ is a sub-coalgebra of R.

In this paper, we provide a new additive basis for the Araki-Kudo-Dyer-Lashof algebra, and discuss the relationship between this basis and the known bases.

The following is one of our main results.

Theorem 1.1. *The set of all monomials $Q^{j_{k-1}} \cdots Q^{j_0}$, where $j_n \geq 2j_{n-1}$, for $1 \leq n \leq k-1$, and j_n is divisible by 2^n, is an additive basis of $R[k]$.*

For example, the set $\{Q^{12}Q^0, Q^{10}Q^2, Q^8Q^4\}$ is an additive basis for $R[2]$ in degree 12.

Let Q^I and Q^J be monomials of length k, we call $Q^I \leq Q^J$ (resp. $Q^I \leq_R Q^J$) if $I \leq J$ in the lexicographic ordering from the left (resp. from the right).

Let A_{Adm} be the admissible basis for R, and let A_C be the basis in Theorem 1.1. We choose the order $\leq$ for A_{Adm} and the order $\leq_R$ for A_C. Then, using Lemma 2.2 and Theorem 1.1, we obtain the following result.

Corollary 1.2. *The change of basis matrix between A_{Adm} and A_C is upper triangular with respect to the order chosen for each basis.*

In order to find the change of basis matrix we first describe the basis A_C and then use Adem relations to convert them to the admissible basis. For example, in degree 12,

$$A_C = \{Q^{12}Q^0, Q^{10}Q^2, Q^8Q^4\} \quad \text{and} \quad A_{Adm} = \{Q^6Q^6, Q^7Q^5, Q^8Q^4\}.$$

On the other hand, by direct inspection, we have

$$Q^{12}Q^0 = Q^6Q^6; Q^{10}Q^2 = Q^7Q^5 + Q^6Q^6; Q^8Q^4 = Q^8Q^4.$$

So the change of basis matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any linear ordering on the set of monomials in R, a monomial is called maximal (minimal) with respect to the given ordering if it cannot be expressed as combination of larger (smaller) monomials. An easy but crucial observation, which was used in great effect in Arnon's work on the construction of new bases for the Steenrod algebra (see [2]), is that the set of maximal (minimal) monomials with respect to any given linear ordering forms a vector space basis for R.

The following theorem, which claims that our basis is the basis of maximal monomials with respect to the order $\leq$, is the second main result in this paper.

Theorem 1.3. *A_C is the basis consisting of all maximal monomials of $R[k]$ with respect to the order $\leq$.*

Remark 1.4. It is straightforward to verify that the admissible basis is the basis consisting of all minimal monomials with respect to the left lexicographic ordering.

The Araki-Kudo-Dyer-Lashof algebra is closely related to the Dickson-Mùi algebra. To be more precise, define a sequence I_{ik}, each of length k, as follows:

$$I_{ik} = \begin{cases} (2^{k-i-1}(2^i-1), \dots, 2(2^i-1), 2^i-1, 2^{i-1}, \dots, 2, 1), & 1 \leq i < k; \\ (2^{k-1}, \dots, 2, 1), & i = k. \end{cases}$$

It is clear that for each pair $k < i$, $Q^{I_{ik}}$ is an admissible monomial of non-negative excess. Let ξ_{ik} be the dual of $Q^{I_{ik}}$ in $R[k]^*$. That is, ξ_{ik} is such that $\langle \xi_{ik}, Q^{I_{ik}} \rangle = 1$ and $\langle \xi_{ik}, Q^I \rangle = 0$ for all other admissible sequence $I \neq I_{ik}$. In [6], Madsen computed the dual $R[k]^*$ and proved that it is isomorphic to the polynomial algebra on the generators ξ_{ik}:

$$R[k]^* \cong \mathbb{F}_2[\xi_{1k}, \dots, \xi_{kk}],$$

where ξ_{ik} is in degree $2^{k-i}(2^i-1)$. This was soon extended to odd primary cases by May [7]. Later, it was recognized that this dual algebra is exactly the Dickson algebra of invariant elements of the polynomial rings $P_k = \mathbb{F}_2[x_1, \dots, x_k]$ under the usual action of the general linear group GL_k of $k \times k$ invertible matrices over the field $\mathbb{F}_2$.

To describe our next results, we need to introduce several definitions. Let E_k be a k-dimensional vector space over $\mathbb{F}_2$. It is well-known that the cohomology of BE_k is the polynomial algebra $P_k := \mathbb{F}_2[x_1, \dots, x_k]$ where each x_i is in degree 1. The homology of BE_k, $H_*(BE_k) = \Gamma(a_1, \dots, a_k)$, is the divided power algebra generated by $a_1, \dots, a_k$, each of degree 1, where a_i is the dual of $x_i \in H^1(BE_1)$. The general linear group $GL_k = GL(E_k)$ acts regularly on E_k and therefore on the homology and cohomology of BE_k. The Dickson algebra [4], which is the algebra of all GL_k-invariants has the following well-known description:

$$D_k := H^*(BE_k)^{GL_k} \cong \mathbb{F}_2[x_1, \cdots, x_k]^{GL_k} = \mathbb{F}_2[Q_{k,0}, Q_{k,1}, \cdots, Q_{k,k-1}],$$

where $Q_{k,i}$ denotes the Dickson invariant of degree $2^k - 2^i$ (see Section 4).

In [10], Mùi provided an explicit isomorphism $R[k] \cong D_k^*$ as coalgebra over the Steenrod algebra. We note in passing that a description of the dual of the Araki-Kudo-Dyer-Lashof algebra was worked out in [7] by May in odd primary cases, but the situation is much more complicated. Kechagias in [5] provided a similar correspondence between $R[k]^*$ and the invariant rings $[H^*(BE_k)]^{GL_k}$ - which was completely identified by Mùi [9]. However, $R[k]^*$ is not isomorphic to the entire invariant ring. In [11], Turner introduced an additive basis for the dual of the Dickson algebra, which we will call the Turner basis.

Theorem 1.5. ([11]) *The set* $\{[a_1^{[t_1]} a_2^{[2(t_1+t_2)]} \cdots a_k^{[2^{k-1}(t_1+\cdots+t_k)]}] | t_i \geq 0\}$ *forms a basis for the dual* D_k^* *of the Dickson algebra.*

Under Mùi's isomorphism, we automatically obtain a basis for the Araki-Kudo-Dyer-Lashof $R[k]$, which is also called the Turner basis. Order elements of this basis lexicographically. The following theorem, which claims that the relation between the admissible basis and the Turner's basis is upper triangular, is the final result in this paper.

Theorem 1.6. *The change of basis matrix between the admissible basis and Turner's one is triangular with respect to the order chosen for each basis.*

Combining Theorem 1.6 and Corollary 1.2, we have the following result.

Corollary 1.7. *The change of basis matrix between our basis and Turner's basis is upper triangular with respect to the order chosen for each basis.*

2. Proof of Theorem 1.1

For a fixed positive integer k, we define two sets

$$\mathcal{S} = \{I = (i_{k-1}, \dots, i_0) : 0 \leq i_n \leq 2i_{n-1}, e(I) \geq 0\},$$
$$\mathcal{S}' = \{J = (j_{k-1}, \dots, j_0) : j_n \geq 2j_{n-1} \geq 0, 2^n \big| j_n\}.$$

Let $\Delta\colon \mathcal{S} \to \mathcal{S}'$ be a function such that $\Delta(i_{k-1}, \dots, i_0) = (j_{k-1}, \dots, j_0)$, where

$$j_{k-1} = 2^{k-1} i_0, j_n = 2^n (i_{k-1-n} - \sum_{s=0}^{k-n-2} i_s) \text{ if } 0 \leq n \leq k-2. \qquad (1)$$

Lemma 2.1. *The function Δ is a bijection.*

Proof. It is clear from (1) that j_n is divisible by 2^n and

$$j_0 = i_{k-1} - i_{k-2} - \cdots - i_0 = e(I) \geq 0.$$

By direct inspection, we have $j_n = 2^{n+1} i_{k-1-n} - 2^n i_{k-n} + 2j_{n-1}, \forall\, 0 < n \leq k-1$.

Hence $j_n - 2j_{n-1} = 2^n(2i_{k-1-n} - i_{k-n}) \geq 0$. Moreover, $j_0 \geq 0$ and it follows that j_n are also non-negative for all $0 \leq n \leq k-1$. Thus, Δ is well defined.

We now define an inverse of Δ. Let $\Phi\colon \mathcal{S}' \to \mathcal{S}$ be such that

$$\Phi(j_{k-1}, \ldots, j_0) = (i_{k-1}, \ldots, i_0),$$

where

$$i_0 = \frac{j_{k-1}}{2^{k-1}},\ i_s = \frac{2j_{k-1-s} + j_{k-s} + \cdots + j_{k-1}}{2^{k-s}},\ 0 < s \leq k-1.$$

It is straightforward to show that Φ and Δ are indeed inverse functions of one another. ∎

Let $Q^a Q^b$ be an admissible non-trivial monomial. Thus, we must have $b \leq a \leq 2b$. The Adem relation shows that

$$Q^{2b} Q^{a-b} = \sum_{t \geq b} \binom{t-(a-b)-1}{2t-2b} Q^{a+b-t} Q^t = Q^a Q^b + \sum_{t>b} M_t,$$

where for each $t > b$, M_t is admissible and strictly less than $Q^a Q^b$.

So, we have

$$Q^a Q^b = Q^{2b} Q^{a-b} + \sum_t M_t, \tag{2}$$

where M_t is admissible and strictly less than $Q^a Q^b$. More generally, we have the following.

Lemma 2.2. *Let $Q^I = Q^{i_{k-1}} \cdots Q^{i_0}$ be an admissible non-trivial monomial. Then*

$$Q^I = Q^{\Delta(I)} + \sum_t M_t,$$

where M_t is admissible and strictly less than Q^I.

Example 2.3. If $I = (22, 12, 8)$, then $\Delta(I) = (32, 8, 2)$. Thus

$$Q^{(22,12,8)} = Q^{(32,8,2)} + \text{other terms}.$$

In fact, we have

$$Q^{22} Q^{12} Q^8 = Q^{32} Q^8 Q^2 + Q^{21} Q^{13} Q^8 + Q^{21} Q^{12} Q^9.$$

Proof of Lemma 2.2. Clearly, the assertion of the lemma is true for $k = 1$. Suppose Q^I is an admissible monomial of non-negative excess. Then $Q^{i_{k-1}} \cdots Q^{i_1}$

is also admissible, having non-negative excess. By induction, we may write

$$Q^{i_{k-1}}\cdots Q^{i_1} = Q^{2^{k-2}i_1}\cdots Q^{i_{k-1}-\cdots-i_1} + \sum_t P_t,$$

where P_t are admissible and strictly less than $Q^{i_{k-1}}\cdots Q^{i_1}$. It follows that

$$\begin{aligned} Q^I &= Q^{2^{k-2}i_1}\cdots Q^{i_{k-1}-\cdots-i_1}Q^{i_0} + \sum_t P_t Q^{i_0} \\ &= Q^{2^{k-2}i_1}\cdots Q^{i_{k-1}-\cdots-i_1}Q^{i_0} + \sum_t P'_t, \end{aligned} \tag{3}$$

where P'_t is admissible and strictly less than Q^I. Because I is admissible, $i_{k-1} - \cdots - i_1 \leq 2i_0$, and by hypothesis, $e(Q^I) = i_{k-1} - i_{k-2} - \cdots - i_1 - i_0 \geq 0$.

Applying (2) for $Q^{i_{k-1}-\cdots-i_1}Q^{i_0}$, we get

$$Q^{2^{k-2}i_1}\cdots Q^{i_{k-1}-\cdots-i_1}Q^{i_0} = Q^{2^{k-2}i_1}\cdots Q^{2(i_{k-2}-\cdots-i_1)}Q^{2i_0}Q^{i_{k-1}-\cdots-i_0} + \sum N_s,$$

where $N_s = Q^{2^{k-2}i_1}\cdots Q^{2(i_{k-2}-\cdots-i_1)}Q^{u_s}Q^{u'_s}$, for $u_s < i_{k-1} - i_{k-2} - \cdots - i_1$ and $u'_s > i_0$.

Now apply again the Adem relation for $Q^{2(i_{k-2}-\cdots-i_1)}Q^{u_s}$ we obtain

$$N_s = \sum_l Q^{2^{k-2}i_1}\cdots Q^{2^2(i_{k-3}-\cdots-i_1)}Q^{u_{sl}}Q^{v_l}Q^{u'_s},$$

where $u_{sl} < i_{k-1} - 2(i_{k-3} + \cdots + i_1) < 2(i_{k-2} - \cdots - i_1)$.

Repeatedly applying Adem relation, finally we have

$$N_s = \sum Q^{a_{k-1}}\cdots Q^{a_1}Q^{u'_s},$$

where $a_{k-1} < i_{k-1}$. Thus,

$$Q^{2^{k-2}i_1}\cdots Q^{i_{k-1}-\cdots-i_1}Q^{i_0} = Q^{2^{k-2}i_1}\cdots Q^{2(i_{k-2}-\cdots-i_1)}Q^{2i_0}Q^{i_{k-1}-\cdots-i_0} + \sum L_s,$$

where L_s is admissible and strictly less than Q^I.

By induction, we have

$$Q^{2^{k-2}i_1}\cdots Q^{i_{k-1}-\cdots-i_1}Q^{i_0} = Q^{2^{k-1}i_0}\cdots Q^{i_{k-1}-\cdots-i_0} + \sum N'_r,$$

where N'_r is admissible and strictly less than Q^I.

Thus, (3) can be rewritten as,

$$Q^I = Q^{\Delta(I)} + \sum_t M_t,$$

where M_t is admissible and strictly less than Q^I. ∎

Proof of Theorem 1.1. Put

$$A_k := \{Q^{j_{k-1}}Q^{j_{k-2}}\cdots Q^{j_0} : j_s \geq 2j_{s-1} \geq 0, 2^s \big| j_s, 1 \leq s \leq k-1\}.$$

Lemma 2.1 shows that in each degree, the number of elements of A_k is equal to the dimension of $R[k]$. Therefore, it suffices to prove that A_k is a generating set for $R[k]$. Now we make use of Lemma 2.2. For any admissible monomial Q^I, we can write Q^I as a sum of $Q^{\Delta(I)}$ and some other monomials M_t which are also admissible and strictly less than Q^I. By induction on the order of monomials, we have the assertion. ■

Example 2.4. We have

$$Q^{21}Q^{13}Q^8 = Q^{32}Q^{10}Q^0 + Q^{21}Q^{12}Q^9 + Q^{21}Q^{11}Q^{10},$$
$$Q^{21}Q^{11}Q^{10} = Q^{40}Q^2Q^0.$$

Thus, from Example 2.3, we obtain

$$Q^{22}Q^{12}Q^8 = Q^{32}Q^8Q^2 + Q^{32}Q^{10}Q^0 + Q^{40}Q^2Q^0.$$

where all monomials on the right hand side are in A_3.

3. Proof of Theorem 1.3

It is sufficient to prove that if Q^I is not of the form described in A_C, then it is not maximal. A monomial $Q^I = Q^{j_{k-1}}\cdots Q^{j_0}$ is not in A_C if and only if at least one of the following is satisfied:

1. $j_s < 2j_{s-1}$ for some s, or
2. j_s is not divisible by 2^s for some s.

In the first case, if $j_s < j_{s-1}$ then $Q^{j_s}Q^{j_{s-1}} = 0$ and $Q^I = 0$ as well. Otherwise, we can apply the Adem relation:

$$Q^I = Q^{j_{k-1}}\cdots Q^{2j_{s-1}}Q^{j_s - j_{s-1}}\cdots Q^{j_0} + \sum_t M_t,$$

where $Q^{j_{k-1}}\cdots Q^{2j_{s-1}}Q^{j_s-j_{s-1}}\cdots Q^{j_0} > Q^I$.

We now consider the second case. Let s be such that Q^I contains a factor $Q^{j_s}Q^{j_{s-1}}\cdots Q^{j_0}$, where j_r is divisible by 2^r for all $0 \leq r \leq s-1$ and $j_s = 2^m u$, with $m \leq s-1$ and u odd. Moreover, we can assume that $j_s > 2j_{s-1}, j_{s-1} \geq 2j_{s-1}, \ldots, j_1 \geq 2j_0$.

We consider two separate cases.

Case 1. If $m = 0$, then $j_s = u$ is odd. Since

$$Q^{j_s}Q^{j_{s-1}} = \sum_{2t\geq j_s} \binom{t-j_{s-1}-1}{2t-j_s} Q^{j_s+j_{s-1}-t}Q^t,$$

it follows that $Q^{j_s}Q^{j_{s-1}} \neq 0$ if and only if there exists some $t, j_s < 2t \leq j_s+j_{s-1}$, such that $\binom{t-j_{s-1}-1}{2t-j_s}$ is odd. In that case, we have

$$Q^{j_s}Q^{j_{s-1}} = Q^{j_s+j_{s-1}-t}Q^t + \text{other terms}.$$

Apply relation (2), we have

$$Q^{j_s}Q^{j_{s-1}} = Q^{2t}Q^{j_s+j_{s-1}-2t} + \text{other terms}.$$

Therefore, Q^I can be expressed as

$$Q^I = Q^{j_{k-1}} \cdots Q^{2t}Q^{j_s+j_{s-1}-2t} \cdots Q^{j_0} + \text{other terms},$$

where $2t > j_s$, so $Q^{j_{k-1}} \cdots Q^{2t}Q^{j_s+j_{s-1}-2t} \cdots Q^{j_0} > Q^I$.
Case 2. If $m > 0$, the Adem relation for $Q^{j_s}Q^{j_{s-1}}$ has the form:

$$Q^{j_s}Q^{j_{s-1}} = Q^{j_s/2+j_{s-1}}Q^{j_s/2} + \sum_{t>j_s/2} \binom{t-j_{s-1}-1}{2t-j_s} Q^{j_s+j_{s-1}-t}Q^t.$$

If there exists $t, \frac{j_s}{2} < t \leq \frac{j_s+j_{s-1}}{2}$, such that $\binom{t-j_{s-1}-1}{2t-j_s}$ is odd, then we are back to the Case 1.

If no such t exist, then $Q^{j_s}Q^{j_{s-1}} = Q^{j_s/2+j_{s-1}}Q^{j_s/2}$. Since $j_s/2 > 2j_{s-2}$, we can then apply the Adem relation for $Q^{j_s/2}Q^{j_{s-2}}$. Repeat this process at most m step, we obtain either $Q^{j_s} \cdots Q^{j_0} = 0$, therefore $Q^I = 0$ or $Q^{j_s} \cdots Q^{j_0}$ can be expressed as

$$Q^{j_s} \cdots Q^{j_0} = Q^{j_s/2+j_{s-1}} \cdots Q^{j_s/2^r+j_{s-r}-t}Q^tQ^{j_{s-r-1}} \cdots Q^{j_0} + \text{other terms}, \quad (4)$$

where $r \leq m$ and $j_s/2^r + j_{s-r} \geq 2t > j_s/2^r$.

Applying relation (2), we have

$$Q^{j_s/2^r+j_{s-r}-t}Q^t = Q^{2t}Q^{j_s/2^r+j_{s-r}-2t} + \text{other terms}.$$

Then, (4) can be rewritten as

$$Q^{j_s} \cdots Q^{j_0} = Q^{j_s/2+j_{s-1}} \cdots Q^{2t}Q^{j_s/2^r+j_{s-r}-2t} \cdots Q^{j_0} + \text{other terms}.$$

Note that

$$2t > j_s/2^r > j_s/2^{r-1} + j_{s-r+1} \text{ and } 2^2t \leq j_s/2^{r-1} + 2j_{s-r} \leq j_s/2^{r-1} + j_{s-r+1}.$$

Applying (2), and repeating this process, finally (4) can be expressed by

$$Q^{j_s}\cdots Q^{j_0} = Q^{2^{r+1}t}Q^{j_s/2+j_{s-1}-2^rt}\cdots Q^{j_s/2^r+j_{s-r}-2t}\cdots Q^{j_0} + \text{other terms.}$$

Hence,

$$Q^I = Q^{j_{k-1}}\cdots Q^{2^{r+1}t}\cdots Q^{j_0} + \text{other terms},$$

where $Q^I < Q^{j_{k-1}}\cdots Q^{2^{r+1}t}\cdots Q^{j_0}$.

The proof of Theorem 1.3 is complete.

Using a similar method, we can prove that A_{Adm} is the basis consisting of all maximal monomials, and A_C is the basis consisting of all minimal monomials of $R[k]$ with respect to the right lexicography order $\leq_R$.

4. Proof of Theorem 1.6

We use the notations for $H^*(BE_k) = \mathbb{F}_2[x_1,\dots,x_k]$ and $H_*(BE_k) = \Gamma(a_1,\dots,a_k)$ as in the introduction.

Let GL_k and T_k be the general linear group and the group of upper triangular matrices with 1's on the main diagonal. These groups act canonically on E_k and therefore on the homology and cohomology of BE_k. The invariant ring of T_k and GL_k are determined by Mùi [9] and Dickson [4] as follows:

$$\mathbb{F}_2[x_1,\dots,x_k]^{T_k} = \mathbb{F}_2[V_1,\dots,V_k],$$

$$\mathbb{F}_2[x_1,\dots,x_k]^{GL_k} = \mathbb{F}_2[Q_{k,0},\dots,Q_{k,k-1}],$$

where

$$V_n = \prod_{\lambda_i\in\mathbb{F}_2}(\lambda_1x_1+\cdots+\lambda_{n-1}x_{n-1}+x_n),$$

and the generators V_i and $Q_{k,j}$ are related by the following recursive formula:

$$Q_{k,n} = Q_{k-1,n-1}^2 + V_kQ_{k-1,n},\ 0\leq n<k.$$

By convention, let $Q_{k,j}=0$ if $j<0$ or $j>k$, and $Q_{n,n}=1$.

For a string of non-negative integer $I=(t_1,\dots,t_k)$, we denote $v(I) = v(t_1,\dots,t_k)$ the dual of $V_1^{t_1}\cdots V_k^{t_k}$ with respect to the additive basis $V_1^{h_1}\cdots V_k^{h_k}$ of $\mathbb{F}_2[V_1,\dots,V_k]$. Similarly, let $q(I)=q(t_1,\dots,t_k)$ be the dual of $Q_{k,0}^{t_1}\cdots Q_{k,k-1}^{t_k}$ with respect to the additive basis $Q_{k,0}^{h_1}\cdots Q_{k,k-1}^{h_k}$ of $\mathbb{F}_2[Q_{k,0},\dots,Q_{k,k-1}]$.

In [10], Mùi described explicitly an isomorphism as coalgebra over the Steenrod algebra between $R[k]$ and the dual of the Dickson algebra D_k^*:

$$Q^{i_1}\cdots Q^{i_k}\mapsto [v(i_1-\cdots-i_k, i_2-\cdots-i_k,\dots,i_k)].$$

In this section, we will use this isomorphism to find the relationship between our new basis and Turner's basis in [11].

We order the k-tuples $I=(i_1,\dots,i_k)$ lexicographically from the left.

Proof of Theorem 1.6. It is easy to see that for each k-tuple $I=(i_1,\dots,i_k)$,

$$V^I=V_1^{i_1}\cdots V_k^{i_k}=x_1^{i_1}x_2^{2i_2}\cdots x_k^{2^{k-1}i_k}+\text{greater monomials.}$$

Moreover, it is clear that if $x_1^{i_1}x_2^{2i_2}\cdots x_k^{2^{k-1}i_k}$ occurs as a nontrivial summand in a V^J, then J must be greater than I. Therefore, we can construct a representation of $v(I)$ as follows.

Put

$$\theta(I)=a_1^{[i_1]}a_2^{[2i_2]}\cdots a_k^{[2^{k-1}i_k]}+\sum_{\ell=1}^{s}\mu_\ell a_1^{[u_1^\ell]}a_2^{[2u_2^\ell]}\cdots a_k^{[2^{k-1}u_k^\ell]},$$

where $I>(u_1^1,\dots,u_k^1)>\cdots>(u_1^s,\dots,u_k^s)$, $\sum_{s=1}^k 2^{s-1}u_s^\ell=\sum_{s=1}^k 2^{s-1}i_s$ for all ℓ; and μ_ℓ is defined inductively by

$$\mu_\ell=\left\langle a_1^{[i_1]}a_2^{[2i_2]}\cdots a_k^{[2^{k-1}i_k]}+\sum_{j=1}^{\ell-1}\mu_j a_1^{[u_1^j]}a_2^{[2u_2^j]}\cdots a_k^{[2^{k-1}u_k^j]},V_1^{u_1^\ell}\cdots V_k^{u_k^\ell}\right\rangle.$$

It is easy to check that

$$\langle\theta(I),V^J\rangle=\begin{cases}1 \text{ if } I=J,\\ 0 \text{ if } I\neq J.\end{cases}$$

Thus, $\theta(I)$ is a representation of $v(I)$. So that, under Mùi's isomorphism,

$$\begin{aligned}Q^{i_1}\cdots Q^{i_k}&\mapsto[\theta(j_1,\dots,j_k)]\\&=[a_1^{[j_1]}a_2^{[2j_2]}\cdots a_k^{[2^{k-1}j_k]}]+\text{smaller terms.}\end{aligned}$$

Since Q^I is admissible, $[a_1^{[j_1]}a_2^{[2j_2]}\cdots a_k^{[2^{k-1}j_k]}]$ is an element in the Turner basis. The proof is complete. ■

Example 4.1. In degree 12, the admissible basis and Turner's basis for $R[2]$ are

$$\{Q^8Q^8,Q^9Q^7,Q^{10}Q^6\},\quad\text{and}\quad\{[a_1^{[0]}a_2^{[16]}],[a_1^{[2]}a_2^{[14]}],[a_1^{[4]}a_2^{[12]}]\}.$$

Under the Mùi's isomorphism and the above analysis, we obtain

$$\begin{aligned}Q^8Q^8&\mapsto[v(0,8)]=[\theta(0,8)]=[a_1^{[0]}a_2^{[16]}];\\Q^9Q^7&\mapsto[v(2,7)]=[\theta(2,7)]=[a_1^{[2]}a_2^{[14]}];\\Q^{10}Q^6&\mapsto[v(4,6)]=[\theta(4,6)]=[a_1^{[4]}a_2^{[12]}]+[a_1^{[2]}a_2^{[14]}].\end{aligned}$$

Thus, the change of basis matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Final remark. We believe that our basis can be used to describe the structure of the mod 2 homology of QS^0 considered as an $E(1)$-module, and then the structure of $Ext_{E(1)}(\mathbb{F}_2, H^*(QS^0))$, which is the E_2-term of Adams spectral sequence converging to $ku^*(QS^0)$ (see [12]).

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