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Meromorphic Functions That Share a Nonzero Polynomial with Finite Weight

Pulak Sahoo

Department of Mathematics, Netaji Subhas Open University, 1, Woodburn Park, Kolkata 700020, West Bengal, India.

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Abstract. In the paper, we study the uniqueness theorems of meromorphic functions concerning differential polynomials with weighted value sharing method and obtain two theorems which improve and generalize the recent results due to X.M. Li and L. Gao.

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1. Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7], [15] and [16]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a(z)(\not\equiv \infty)$ is called a small function with respect to f, provided that $T(r,a) = S(r,f)$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share

0 CM, and we say that f and g share ∞ IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (see[16]). Throughout this paper, we need the following definition.

$$\Theta(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane.

In 1959, W.K. Hayman (see [6], Corollary of Theorem 9) proved the following theorem:

Theorem 1.1. *Let f be a transcendental meromorphic function and $n(\geq 3)$ be an integer. Then $f^n f' = 1$ has infinitely many solutions.*

In 1997, C.C. Yang and X.H. Hua [14] proved the following result which corresponded to Theorem 1.1.

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2000, M.L. Fang [4] proved the following result:

Theorem 1.3. *Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' - z = 0$ has infinitely many solutions.*

Corresponding to Theorem 1.3, M.L. Fang and H.L. Qiu [5] proved the following result.

Theorem 1.4. *Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

In 2003, W. Bergweiler and X.C. Pang [3] proved the following result:

Theorem 1.5. *Let f be a transcendental meromorphic function, and let $R \not\equiv 0$ be a rational function. If all zeros and poles of f are multiple, except possibly finitely many, then $f' - R = 0$ has infinitely many solutions.*

A natural question arises:

Question 1. Similarly to Theorem 1.2 and Theorem 1.4, is there a unicity theorem corresponding to Theorem 1.5?

Recently X.M. Li and L. Gao [12] proved the following uniqueness theorems which deal with Question 1.

Theorem 1.6. *Let f and g be two transcendental meromorphic functions, let $n \geq 11$ be a positive integer, and let $P \not\equiv 0$ be a polynomial with its degree*

$\gamma_P \leq 11$. *If* $f^n f' - P$ *and* $g^n g' - P$ *share* 0 *CM, then either* $f = tg$ *for a complex number* t *satisfying* $t^{n+1} = 1$, *or* $f = c_1 e^{cQ}$ *and* $g = c_2 e^{-cQ}$, *where* c_1, c_2 *and* c *are three nonzero complex numbers satisfying* $(c_1 c_2)^{n+1} c^2 = -1$, Q *is a polynomial satisfying* $Q = \int_0^z P(\eta) d\eta$.

Theorem 1.7. *Let* f *and* g *be two transcendental meromorphic functions, let* $n \geq 15$ *be a positive integer, and let* $P \not\equiv 0$ *be a polynomial. If* $(f^n(f-1))' - P$ *and* $(g^n(g-1))' - P$ *share* 0 *CM and* $\Theta(\infty, f) > 2/n$, *then* $f = g$.

Naturally one may ask the following question which is the motivation of the paper.
Question 2. Is it really possible in any way to relax the nature of sharing the value 0 in Theorem 1.6 and Theorem 1.7 without changing the lower bound of n ?

In the paper, we will prove two theorems the first one of which will improve Theorem 1.6. Our second theorem will improve and generalize Theorem 1.7. To state the main results we need the following definition known as weighted sharing of values introduced by I. Lahiri [8, 9] which measure how close a shared value is to being shared CM or to being shared IM.

Definition 1.8. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

The following theorems are the main results of the paper.

Theorem 1.9. *Let* f *and* g *be two transcendental meromorphic functions, let* $n(\geq 11)$ *be a positive integer, and let* $P \not\equiv 0$ *be a polynomial with its degree* $\gamma_P \leq 11$. *If* $f^n f' - P$ *and* $g^n g' - P$ *share* $(0, 2)$, *then either* $f = tg$ *for a complex number* t *satisfying* $t^{n+1} = 1$, *or* $f = c_1 e^{cQ}$ *and* $g = c_2 e^{-cQ}$, *where* c_1, c_2 *and* c *are three nonzero complex numbers satisfying* $(c_1 c_2)^{n+1} c^2 = -1$, Q *is a polynomial satisfying* $Q = \int_0^z P(\eta) d\eta$.

Theorem 1.10. *Let* f *and* g *be two transcendental meromorphic functions, let* n, m *be two positive integer, and let* $P \not\equiv 0$ *be a polynomial. If* $(f^n(f-1)^m)' - P$ *and* $(g^n(g-1)^m)' - P$ *share* $(0, 2)$, *then each of the following holds:*

(i) *when* $m = 1$, $n \geq 15$ *and* $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, *then* $f = g$;

(ii) *when* $m \geq 2$ *and* $n \geq m + 14$, *then either* $f = g$ *or* f *and* g *satisfy the algebraic equation* $R(f, g) = 0$, *where* $R(w_1, w_2) = w_1^n(w_1 - 1)^m - w_2^n(w_2 - 1)^m$.

Remark 1.11. Obviously Theorem 1.9 is an improvement of Theorem 1.6.

Remark 1.12. Theorem 1.10 improves and generalizes Theorem 1.7.

We now explain some definitions and notations which are used in the paper.

Definition 1.13. ([10]) For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting functions of simple a-points of f. For a positive integer p we denote by $N(r, a; f \mid\leq p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.

Definition 1.14. ([9]) Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.15. Let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$\delta_k(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N_k(r, a; f)}{T(r, f)}.$$

Remark 1.16. From the definitions of $\delta_k(a, f)$ and $\Theta(a, f)$, it is clear that

$$0 \leq \delta_k(a, f) \leq \delta_{k-1}(a, f) \leq \delta_1(a, f) \leq \Theta(a, f) \leq 1.$$

Definition 1.17. ([1, 2]) Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and also a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the reduced counting function of those 1-points of f and g, where $p > q$, by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of f and g, where $p = q = 1$, by $\overline{N}_E^{(2}(r, 1; f)$ the reduced counting function of those 1-points of f and g, where $p = q \geq 2$. In the same manner we can define $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$ and $\overline{N}_E^{(2}(r, 1; g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. ([13]) *Let f be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in f of the form*

$$P_n(f) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + ... + a_1 f(z) + a_0,$$

where $a_n(\neq 0)$, a_{n-1}, ... , a_1, a_0 are complex numbers. Then

$$T(r, P_n(f)) = nT(r, f) + O(1).$$

Lemma 2.2. ([7]) *Let f be a nonconstant meromorphic function, k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\infty;f) + N(r,0;f) + N(r,c;f^{(k)}) - N(r,0;f^{(k+1)}) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + N_{k+1}(r,0;f) + \overline{N}(r,c;f^{(k)}) - N_0(r,0;f^{(k+1)}) + S(r,f), \end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 2.3. ([17]) *Let f and g be two nonconstant meromorphic functions, and let p, k be two positive integers. Then*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.4. ([7, 15]) *Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r, a_i(z)) = S(r,f)$, i=1,2. Then*

$$T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

Lemma 2.5. *Let f and g be two transcendental meromorphic functions such that $f^{(k)} - P$ and $g^{(k)} - P$ share $(0, 2)$, where k is a positive integer, $P \not\equiv 0$ is a polynomial. If*

$$\begin{aligned} \Delta_1 =&(k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) \\ &+ \delta_{k+1}(0,f) + \delta_{k+1}(0,g) > k+7 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \Delta_2 =&(k+2)\Theta(\infty,g) + 2\Theta(\infty,f) + \Theta(0,g) + \Theta(0,f) \\ &+ \delta_{k+1}(0,g) + \delta_{k+1}(0,f) > k+7, \end{aligned} \tag{2}$$

then either $f^{(k)}g^{(k)} = P^2$ or $f = g$.

Proof. Since f and g are two transcendental meromorphic functions, $f^{(k)}$ and $g^{(k)}$ are also two transcendental meromorphic functions. Let

$$F = \frac{f^{(k)}}{P}, G = \frac{g^{(k)}}{P},$$

and let

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{3}$$

Let $z_0 \notin \{z : P(z) = 0\}$ be a common simple zero of $f^{(k)} - P$ and $g^{(k)} - P$. Then z_0 is a common simple zero of $F - 1$ and $G - 1$. Substituting their Taylor series at z_0 into (3), we see that z_0 is a zero of H. Thus we have

$$\begin{aligned} N_E^{1)}(r, 1; F) &\le N(r, 0; H) \le T(r, H) + O(1) \\ &\le N(r, \infty; H) + S(r, F) + S(r, G). \end{aligned} \tag{4}$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a pole of H. Then from (3) we can see that H have poles only at the zeros of F' and G', 1-points of F whose multiplicities are not equal to the multiplicities of the corresponding 1-points of G, and poles of f and g. Hence we have

$$\begin{aligned} N(r, \infty; H) \le{}& \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}_L(r, 1; F) \\ &+ \overline{N}_L(r, 1; G) + N_0(r, 0; F') + N_0(r, 0; G') + O(\log r), \end{aligned} \tag{5}$$

where $N_0(r, 0; F')$ denotes the counting function of those zeros of F' which are not the zeros of $f(F - 1)$, $N_0(r, 0; G')$ is similarly defined. Since f is a transcendental meromorphic function we have

$$T(r, P) = o\{T(r, f)\}. \tag{6}$$

By Lemma 2.2, we have

$$T(r, f) \le \overline{N}(r, \infty; f) + N_{k+1}(r, 0; f) + \overline{N}(r, 1; F) - N_0(r, 0; F') + S(r, f). \tag{7}$$

Similarly

$$T(r, g) \le \overline{N}(r, \infty; g) + N_{k+1}(r, 0; g) + \overline{N}(r, 1; G) - N_0(r, 0; G') + S(r, g). \tag{8}$$

Since $f^{(k)} - P$ and $g^{(k)} - P$ share 0 IM, therefore using (4) and (5) we obtain

$$\begin{aligned} \overline{N}(r, 1; F) + \overline{N}(r, 1; G) &= 2N_E^{1)}(r, 1; F) + 2\overline{N}_L(r, 1; F) \\ &\quad + 2\overline{N}_L(r, 1; G) + 2\overline{N}_E^{(2}(r, 1; F) \\ &\le N_E^{1)}(r, 1; F) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; f) \\ &\quad + \overline{N}(r, 0; g) + 3\overline{N}_L(r, 1; F) + 3\overline{N}_L(r, 1; G) + N_0(r, 0; F') \\ &\quad + N_0(r, 0; G') + 2\overline{N}_E^{(2}(r, 1; F) + S(r, f) + S(r, g). \end{aligned} \tag{9}$$

Noting that $f^{(k)} - P$ and $g^{(k)} - P$ share $(0, 2)$ we have

$$\begin{aligned}&N_E^{1)}(r,1;F)+2\overline{N}_E^{(2}(r,1;F)+3\overline{N}_L(r,1;F)+3\overline{N}_L(r,1;G)\\&\le N(r,1;G)+S(r,f)+S(r,g)\\&\le T(r,G)+S(r,g)\\&\le T(r,g)+k\overline{N}(r,\infty;g)+S(r,g).\end{aligned}\tag{10}$$

From (7) - (10), we obtain

$$\begin{aligned}T(r,f)\le\ &2\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)\\&+N_{k+1}(r,0;f)+N_{k+1}(r,0;g)+S(r,f)+S(r,g).\end{aligned}\tag{11}$$

Similarly

$$\begin{aligned}T(r,g)\le\ &2\overline{N}(r,\infty;g)+(k+2)\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,0;f)\\&+N_{k+1}(r,0;g)+N_{k+1}(r,0;f)+S(r,f)+S(r,g).\end{aligned}\tag{12}$$

Suppose that there exists a subset $I\subseteq R^+$ satisfying $mesI=\infty$ such that $T(r,g)\le T(r,f)$, $r\in I$. Hence from (11) we have

$$\begin{aligned}\Delta_2=\ &(k+2)\Theta(\infty,g)+2\Theta(\infty,f)+\Theta(0,g)+\Theta(0,f)\\&+\delta_{k+1}(0,g)+\delta_{k+1}(0,f)\le k+7,\end{aligned}$$

contradicting (2). Similarly if there exists a subset $I\subseteq R^+$ satisfying $mesI=\infty$ such that $T(r,f)\le T(r,g)$, $r\in I$, from (12) we can obtain

$$\begin{aligned}\Delta_1=\ &(k+2)\Theta(\infty,f)+2\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)\\&+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\le k+7,\end{aligned}$$

contradicting (1). We now assume that $H=0$. That is

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B,\tag{13}$$

where $A(\neq 0)$ and B are finite complex constants. We now discuss the following three cases.

Case 1. Let $B\neq 0$ and $A=B$. If $B=-1$, we obtain from (13) $FG=1$, i.e., $f^{(k)}g^{(k)}=P^2$.

If $B\neq -1$, from (13) we get

$$\frac{1}{F}=\frac{BG}{(1+B)G-1}\ \textit{and}\ G=\frac{-1}{b(F-\frac{1+B}{B})}.$$

So by Lemma 2.2 we obtain

$$\overline{N}\left(r, \frac{1}{1+B}; G\right) \leq \overline{N}(r,0;F) \leq N_{k+1}(r,0;f) + k\overline{N}(r,\infty;f) + O(\log r) + S(r,f) \tag{14}$$

and

$$\overline{N}\left(r, \frac{1+B}{B}; F\right) \leq \overline{N}(r,\infty;g) + O(\log r). \tag{15}$$

Using Lemma 2.2, (14) and (15) we obtain

$$\begin{aligned} T(r,g) \leq & N_{k+1}(r,0;g) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;g) \\ & - N_0(r,0;G') + S(r,g) \\ \leq & \ N_{k+1}(r,0;g) + N_{k+1}(r,0;f) + k\overline{N}(r,\infty;f) \\ & + \overline{N}(r,\infty;g) + S(r,f) + S(r,g) \end{aligned} \tag{16}$$

and

$$\begin{aligned} T(r,f) \leq & \ N_{k+1}(r,0;f) + \overline{N}\left(r, \frac{1+B}{B}; F\right) + \overline{N}(r,\infty;f) \\ & - N_0(r,0;F') + S(r,f) \\ \leq & \ N_{k+1}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f). \end{aligned} \tag{17}$$

Suppose that there exists a subset $I \subseteq R^+$ satisfying $mesI = \infty$ such that $T(r,f) \leq T(r,g)$, $r \in I$. So from (16) we obtain

$$k\Theta(\infty,f) + \Theta(\infty,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) \leq k+2,$$

which by (1) gives

$$2\Theta(\infty,f) + \Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) > 5,$$

a contradiction by Remark 1.16. If there exists a subset $I \subseteq R^+$ satisfying $mesI = \infty$ such that $T(r,g) \leq T(r,f)$, $r \in I$, by the same argument we obtain a contradiction from (1) and (17).

Case 2. Let $B \neq 0$ and $A \neq B$. If $B = -1$, from (13) we obtain $F = \frac{A}{-(G-(a+1))}$. If $B \neq -1$, from (13) we obtain $F - \frac{1+B}{B} = \frac{-A}{B^2(G+\frac{A-B}{B})}$. Using the same argument as in case 1 we obtain a contradiction in both cases.

Case 3. Let $B = 0$. Then from (13) we get

$$g = Af + (1-A)P_1, \tag{18}$$

where P_1 is a polynomial of degree $\gamma_{P_1} \geq k$. If $A \neq 1$, by Lemma 2.4 and (18) we get

$$\begin{aligned} T(r,g) \leq\ & \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,(1-A)P_1;g) + S(r,g) \\ \leq\ & \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + S(r,g). \end{aligned} \tag{19}$$

Since f and g are transcendental meromorphic functions from (18) we have

$$T(r,f) = T(r,g) + O(\log r).$$

So from (19), we obtain

$$\Theta(0,f) + \Theta(0,g) + \Theta(\infty,g) \leq 2,$$

which gives by (1)

$$(k+2)\Theta(\infty,f) + \Theta(\infty,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) > k+5,$$

a contradiction by Remark 1.16. Thus $A = 1$ and so $f = g$. This completes the proof of the lemma. ■

Lemma 2.6. [12] *Let f and g be two transcendental meromorphic functions, let $n \geq 2$ be a positive integer, and let P be a nonconstant polynomial with its degree $\gamma_P \leq n$. If $f^n f' g^n g' = P^2$, then f and g are expressed as $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$ respectively, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^{n+1} c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta) d\eta$.*

Lemma 2.7. *Let f and g be two transcendental meromorphic functions, let n, m be two positive integers with $n \geq m+3$, and let P be a nonconstant polynomial. Then*

$$(f^n(f-1)^m)'(g^n(g-1)^m)' \neq P^2.$$

Proof. If possible, let

$$(f^n(f-1)^m)'(g^n(g-1)^m)' = P^2.$$

That is,

$$f^{n-1}(f-1)^{m-1}(cf-d)f'g^{n-1}(g-1)^{m-1}(cg-d)g' = P^2, \tag{20}$$

where $c = n+m$ and $d = n$.

Let $z_0 \notin \{z : P(z) = 0\}$ be a 1-point of f with multiplicity $p_0(\geq 1)$. Then from (20) it follows that z_0 is a pole of g. Suppose that z_0 is a pole of g of order $q_0(\geq 1)$. Then we have $mp_0 - 1 = (n+m)q_0 + 1$, i.e., $mp_0 = (n+m)q_0 + 2 \geq n+m+2$, and so

$$p_0 \geq \frac{n+m+2}{m}.$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a zero of $cf - d$ with multiplicity $p_1(\geq 1)$. Then from (20) it follows that z_1 is a pole of g. Suppose that z_1 is a pole of g of order $q_1(\geq 1)$. Then we have $2p_1 - 1 = (n+m)q_1 + 1$, and so

$$p_1 \geq \frac{n+m+2}{2}.$$

Let $z_2 \notin \{z : P(z) = 0\}$ be a zero of f with multiplicity $p_2(\geq 1)$. Then it follows from (20) that z_2 is a pole of g. Suppose that z_2 is a pole of g of order $q_2(\geq 1)$. Then we have

$$np_2 - 1 = (n+m)q_2 + 1. \tag{21}$$

From (21) we get $mq_2 + 2 = n(p_2 - q_2) \geq n$, i.e., $q_2 \geq \frac{n-2}{m}$. Thus from (21) we obtain $np_2 = (n+m)q_2 + 2 \geq \frac{(n+m)(n-2)}{m} + 2$, and so

$$p_2 \geq \frac{n+m-2}{m}.$$

Let $z_3 \notin \{z : P(z) = 0\}$ be a pole of f. Then it follows from (20) that z_3 is a zero of $g(g-1)(cg-d)$ or a zero of g'. So we have

$$\begin{aligned}\overline{N}(r,\infty;f) \leq\ & \overline{N}(r,0;g) + \overline{N}(r,1;g) + \overline{N}\left(r,\frac{d}{c};g\right) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g)\\ \leq\ & \left(\frac{m+2}{n+m+2} + \frac{m}{n+m-2}\right)T(r,g) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)(cg-d)$.
By the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}2T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}\left(r,\frac{d}{c};f\right) + \overline{N}(r,\infty;f) - \overline{N}_0(r,0;f') + S(r,f)\\ &\leq \left(\frac{m+2}{n+m+2} + \frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\} - \overline{N}_0(r,0;f')\\ &\quad + \overline{N}_0(r,0;g') + S(r,f) + S(r,g).\end{aligned} \tag{22}$$

Similarly

$$\begin{aligned}2T(r,g) \leq & \left(\frac{m+2}{n+m+2} + \frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\} + \overline{N}_0(r,0;f')\\ & - \overline{N}_0(r,0;g') + S(r,f) + S(r,g).\end{aligned} \tag{23}$$

Adding (22) and (23) we obtain

$$\left(1 - \frac{m+2}{n+m+2} - \frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

contradicting the fact that $n \geq m+3$. This proves the lemma. ■

Lemma 2.8. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty,f) + \Theta(\infty,g) > \frac{4}{n},$$

where $n(\geq 3)$ *is an integer. Then*

$$f^n(af+b) \equiv g^n(ag+b)$$

implies $f \equiv g$, *where* a, b *are two nonzero constants.*

Proof. We omit the proof since it can be carried out in the line of Lemma 2.6 [11]. ■

3. Proof of the Theorem

Proof of Theorem 1.9. We consider $F_1(z) = \frac{f^{n+1}}{n+1}$ and $G_1(z) = \frac{g^{n+1}}{n+1}$. Then we see that $F_1' - P$ and $G_1' - P$ share the value 0 with weight two. Using Lemma 2.1, we have

$$\begin{aligned} \Theta(0, F_1) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; F_1)}{T(r, F_1)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; f)}{(n+1)T(r, f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{(n+1)T(r, f)} \\ &\geq \frac{n}{n+1}. \end{aligned} \tag{24}$$

Similarly,

$$\Theta(0, G_1) \geq \frac{n}{n+1}. \tag{25}$$

$$\begin{aligned} \Theta(\infty, F_1) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; F_1)}{T(r, F_1)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; f)}{(n+1)T(r, f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{(n+1)T(r, f)} \\ &\geq \frac{n}{n+1}. \end{aligned} \tag{26}$$

Similarly,

$$\Theta(\infty, G_1) \geq \frac{n}{n+1}. \tag{27}$$

$$\begin{aligned}\delta_2(0,F_1) &= 1-\limsup_{r\to\infty}\frac{N_2(r,0;F_1)}{T(r,F_1)}\\ &= 1-\limsup_{r\to\infty}\frac{N_2(r,0;f^n)}{(n+1)T(r,f)}\\ &\geq 1-\limsup_{r\to\infty}\frac{2T(r,f)}{(n+1)T(r,f)}\\ &\geq \frac{n-1}{n+1}.\end{aligned} \tag{28}$$

Similarly,

$$\delta_2(0,G_1) \geq \frac{n-1}{n+1}. \tag{29}$$

Using (1), (2) and (24)-(29) we obtain

$$\varDelta_1 \geq \frac{9n-2}{n+1} \text{ and } \varDelta_2 \geq \frac{9n-2}{n+1}.$$

Since $n \geq 11$, we get $\varDelta_1 > 8$ and $\varDelta_2 > 8$. So by Lemma 2.5 we obtain either $F_1'G_1' = P^2$ or $F_1 = G_1$. Suppose that $F_1'G_1' = P^2$, i.e., $f^n f' g^n g' = P^2$. Hence by Lemma 2.6 we obtain $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1c_2)^{n+1}c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta)d\eta$.

If $F_1 = G_1$, then $f = tg$ for a complex number t such that $t^{n+1} = 1$. This completes the proof of Theorem 1.9. ■

Proof of Theorem 1.10. Let $F_2(z) = f^n(f-1)^m$ and $G_2(z) = g^n(g-1)^m$. Then $F_2' - P$ and $G_2' - P$ share the value 0 with weight two. Using Lemma 2.1, we obtain

$$\begin{aligned}\varTheta(0,F_2) &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,0;F_2)}{T(r,F_2)}\\ &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,0;f^n(f-1)^m)}{(n+m)T(r,f)}\\ &\geq 1-\limsup_{r\to\infty}\frac{2T(r,f)}{(n+m)T(r,f)}\\ &\geq \frac{n+m-2}{n+m}.\end{aligned} \tag{30}$$

Similarly,

$$\varTheta(0,G_2) \geq \frac{n+m-2}{n+m}. \tag{31}$$

$$\begin{aligned}\Theta(\infty, F_2) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\infty;F_2)}{T(r,F_2)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\infty;f)}{(n+m)T(r,f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{T(r,f)}{(n+m)T(r,f)} \\ &\geq \frac{n+m-1}{n+m}.\end{aligned} \tag{32}$$

Similarly,

$$\Theta(\infty, G_2) \geq \frac{n+m-1}{n+m}. \tag{33}$$

$$\begin{aligned}\delta_2(0, F_2) &= 1 - \limsup_{r\to\infty} \frac{N_2(r,0;F_2)}{T(r,F_2)} \\ &= 1 - \limsup_{r\to\infty} \frac{N_2(r,0;f^n(f-1)^m)}{(n+m)T(r,f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{(m+2)T(r,f)}{(n+m)T(r,f)} \\ &\geq \frac{n-2}{n+m}.\end{aligned} \tag{34}$$

Similarly,

$$\delta_2(0, G_2) \geq \frac{n-2}{n+m}. \tag{35}$$

Using (1), (2) and (30)-(35) we obtain

$$\Delta_1 \geq \frac{9n+7m-13}{n+m} \text{ and } \Delta_2 \geq \frac{9n+7m-13}{n+m}.$$

Since $n \geq m+14$, we get $\Delta_1 > 8$ and $\Delta_2 > 8$. So by Lemma 2.5 and Lemma 2.7 we can conclude that $F_2 = G_2$, i.e.,

$$f^n(f-1)^m = g^n(g-1)^m. \tag{36}$$

Let $m = 1$. Then from (36) we have

$$f^n(f-1) = g^n(g-1)$$

and by Lemma 2.8 we obtain $f = g$.
Let $m \geq 2$. Then from (36) we obtain

$$\begin{aligned}&f^n[f^m + ... + (-1)^i \ {}^mC_i f^{m-i} + ... + (-1)^m] \\ &\quad = g^n[g^m + \cdots + (-1)^i \ {}^mC_i g^{m-i} + ... + (-1)^m].\end{aligned} \tag{37}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (37) we obtain

$$g^{n+m}(h^{n+m}-1)+...+(-1)^i\ {}^mC_i g^{n+m-i}(h^{n+m-i}-1)+\cdots+(-1)^m g^n(h^n-1)=0,$$

which imply $h = 1$. Hence $f = g$.

If h is not a constant, then from (37) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = w_1^n(w_1 - 1)^m - w_2^n(w_2 - 1)^m.$$

This completes the proof of Theorem 1.10. ■

References

1. T.C. Alzahary and H.X. Yi, Weighted value sharing and a question of I. Lahiri, *Complex Var. Theory Appl.* **49**, No. 15 (2004), 1063-1078.
2. A. Banerjee, Meromorphic functions sharing one value, *Int. J. Math. Math. Sci.* **22** (2005), 3587-3598.
3. W. Bergweiler and X.C. Pang, On the derivative of meromorphic functions with multiple zeros, *J. Math. Anal. Appl.* **278**, No. 2(2003), 285-292.
4. M.L. Fang, A note on a problem of Hayman, *Analysis* (*Munich*), **20**(2000), No. 1, 45-49.
5. M.L. Fang and H.L. Qiu, Meromorphic functions that share fixed points, *J. Math. Anal. Appl.* **268**(2002), 426-439.
6. W.K. Hayman, Picard values of meromorphic Functions and their derivatives, *Ann. of Math.* **70**(1959), 9-42.
7. W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.
8. I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math. J.* **161** (2001), 193-206.
9. I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, *Complex Var. Theory Appl.* **46** (2001), 241-253.
10. I. Lahiri, Value distribution of certain differential polynomials, *Int. J. Math. Math. Sci.* **28**, No.2 (2001), 83-91.
11. I. Lahiri, On a question of Hong Xun Yi, *Arch Math.* (*Brno*), **38**, No.2 (2002), 119-128.
12. X.M. Li and L. Gao, Meromorphic functions sharing a nonzero polynomial CM, *Bull. Korean Math. Soc.* **47**, (2010), No. 2, 319-339.
13. C.C. Yang, On deficiencies of differential polynomials II, *Math. Z.* **125** (1972), 107-112.
14. C.C. Yang and X.H. Hua, Uniqueness and value sharing of meromorphic functions, *Ann. Acad. Sci. Fenn. Math.* **22**, No. 2(1997), 395-406.
15. L.Yang, *Value Distribution Theory*, Springer- Verlag, Berlin, 1993.
16. H.X. Yi and C.C. Yang, *Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, 1995.
17. Q.C. Zhang, Meromorphic function that shares one small function with its derivative, *J. Inequal. Pure Appl. Math.* **6**, No. 4(2005), Art. 116.