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Strong Convergence Theorems for a Finite Family of Relatively Nonexpansive Mappings

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Abstract. In this paper, I proved a strong convergence theorem for a finite family of relatively nonexpansive mappings in a Banach space.The results presented in this work improve the corresponding one announced by many others.

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1. Introduction and Preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$.

Halpern[4] introduced a classical iteration process as follows: take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n u + (1 - t_n)Tx_n, n \geq 0, \quad (1)$$

where $u \in C$ is an arbitrary element, $\{t_n\}_{n=1}^{\infty}$ is a sequence in the interval $[0, 1]$. He proved that the sequence $\{x_n\}$ converges to a fixed point of a nonexpansive mapping. Acedo and Xu [1] studied the following cyclic algorithm. Let $x_0 \in C$ and $\{\alpha_n\}$ be a sequence in $(0, 1)$. $\{x_n\}_{n=1}^{\infty}$ is generated in the following way:

$$
\begin{aligned}
&x_1 = \alpha_0 x_0 + (1-\alpha_0) T_0 x_0,\\
&x_2 = \alpha_1 x_1 + (1-\alpha_1) T_1 x_1,\\
&\cdots\\
&x_N = \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1}) T_{N-1} x_{N-1},\\
&x_{N+1} = \alpha_N x_N + (1-\alpha_N) T_0 x_N,\\
&\cdots .
\end{aligned}
$$

In general, x_{n+1} is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_n x_n, \ \forall n \geq 0, \tag{2}$$

where $T_n = T_{n(\text{mod} N)}$ (here the mod$N$ function takes values in $\{1, 2, \ldots, N-1\}$).They proved weak and strong convergence theorems in Hilbert spaces by cyclic algorithm (2).

Very recently, Qin and Su [7] proposed the following iteration for a relatively nonexpansive mapping in a Banach space. More precisely, they proved the following:

Theorem 1.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm*

$$
\begin{cases}
x_0 = x \in C,\\
y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J T x_n),\\
C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n)\},\\
Q_n = \{v \in C : \langle J x_0 - J x_n, x_n - v \rangle \geq 0\},\\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0,
\end{cases} \tag{3}
$$

where J is the single-valued duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)} x_0$.

In this paper, motivated and inspired by the above results, we consider an algorithm to modify the iterative process (3) to have strong convergence for a finite family of relatively nonexpansive mappings in the framework of Banach spaces.

$$
\begin{cases}
x_0 = x \in C,\\
y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J T_n x_n),\\
C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n)\},\\
Q_n = \{v \in C : \langle J x_0 - J x_n, x_n - v \rangle \geq 0\},\\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0,
\end{cases} \tag{4}
$$

where $T_n = T_{n(\text{mod} N)}$

Throughout the paper, let E be a real Banach space, E^* the dual space of E. We denote by J the normalized duality mapping defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single valued and if E is uniformly smooth then J is uniformly continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single valued, one-to-one and surjective.

As we know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 , \tag{5}$$

for all $x, y \in E$. Observe that, in a Hilbert space H, (5) reduces to $\phi(x, y) = \|x - y\|^2 , x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x)$. The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the functional ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(y, x) \le (\|y\|^2 + \|x\|^2), \tag{6}$$

for all $x, y \in E$.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} (Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called relatively nonexpansive (see, e.g.,[5]) if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings was studied in [3].

For the proof of our main results we need the following lemmas.

Lemma 1.2. ([5]) *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 1.3. ([2]) *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0 \rangle \ge 0$ for all $y \in C$.*

Lemma 1.4. ([2]) *Let E be a reflexive, strictly convex and smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x)$ for all $y \in C$.*

Lemma 1.5. ([6]) *Let E be a strictly convex and smooth real Banach space, let C be a closed convex subset of E, and let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

2. Main Result

Theorem 2.1. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E and $\{T_1, T_2, \ldots, T_N\}$ be a finite family of relatively nonexpansive mappings from C into itself with $F \neq \emptyset$, where $F = \cap_{i=1}^{N} F(T_i)$. Assume that T_i is uniformly continuous for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JT_n x_n), \\ C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{7}$$

where $T_n = T_{n(\text{mod}N)}$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ converges to $\Pi_F x_0$.

Proof. We first show that C_n and Q_n are closed and convex for each $n \geq 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n are closed and convex for each $n \geq 0$. We prove that C_n is convex. Since

$$\phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n)$$

is equivalent to

$$2\alpha_n\langle v, Jx_0\rangle + 2(1-\alpha_n)\langle v, Jx_n\rangle - 2\langle v, Jy_n\rangle \leq \alpha_n \|x_0\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2$$

we obtain C_n is convex. Next, we show that $F \subset C_n$ for all n. Indeed, we have, for each $p \in F$

$$\begin{aligned} \phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JT_n x_n)) \\ &= \|p\|^2 - 2\langle p, \alpha_n Jx_0 + (1-\alpha_n)JT_n x_n\rangle \\ &\quad + \|\alpha_n Jx_0 + (1-\alpha_n)JT_n x_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n\langle p, Jx_0\rangle + 2(1-\alpha_n)\langle p, JT_n x_n\rangle \\ &\quad + \alpha_n \|x_0\|^2 + (1-\alpha_n)\|T_n x_n\|^2 \\ &\leq \alpha_n\phi(p, x_0) + (1-\alpha_n)\phi(p, T_n x_n) \\ &\leq \alpha_n\phi(p, x_0) + (1-\alpha_n)\phi(p, x_n). \end{aligned}$$

So $p \in C_n$ for all n and $F \subset C_n$. Next we show that $F \subset Q_n$ for all n. We prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_n$. Since

x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 1.3, we have

$$\langle Jx_0 - Jx_{n+1}, x_{n+1} - z\rangle \geq 0, \forall z \in C_n \cap Q_n.$$

As $F \subset C_n \cap Q_n$ by the induction assumptions, for all $z \in F$, $\langle Jx_0 - Jx_{n+1}, x_{n+1} - z\rangle \geq 0$ holds. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Hence $F \subset Q_n$ for all n. This implies $\{x_n\}$ is well defined. It follows from the definition of Q_n that $x_n = \Pi_{Q_n} x_0$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0).$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows from $x_n = \Pi_{Q_n} x_0$ and Lemma 1.4 that

$$\phi(x_n, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0),$$

for each $p \in F \subset Q_n$ and for each $n \geq 0$. Therefore, $\phi(x_n, x_0)$ is bounded. Moreover, from (6), we have that $\{x_n\}$ is bounded. So, we obtain that the limit of $\{\phi(x_n, x_0)\}$ exists. From Lemma 1.4, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{Q_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0),\end{aligned}$$

for each $n \geq 0$. This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{8}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, from the definition of C_n, we also have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, x_0) + (1 - \alpha_n)\phi(x_{n+1}, x_n).$$

It follows from $\lim_{n\to\infty} \alpha_n = 0$ and (8) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{9}$$

By using Lemma 1.2, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

as well as

$$\lim_{n\to\infty} \|x_{n+l} - x_n\| = 0, \tag{10}$$

for all $l \in \{1, 2, \ldots, N\}$.
So

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{11}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0.$$

Note that

$$\begin{aligned} \|JT_nx_n - Jy_n\| &= \|JT_nx_n - (\alpha_n Jx_0 + (1-\alpha_n)JT_nx_n)\| \\ &= \alpha_n \|Jx_0 - JT_nx_n\|. \end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} \|JT_nx_n - Jy_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets , we have

$$\lim_{n\to\infty} \|T_nx_n - y_n\| = 0. \tag{12}$$

This implies that

$$\|x_n - T_nx_n\| \le \|x_n - y_n\| + \|y_n - T_nx_n\|.$$

It follows from (11) and (12) that

$$\lim_{n\to\infty} \|T_nx_n - x_n\| = 0. \tag{13}$$

Hence

$$\|x_n - T_{n+l}x_n\| \le \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\|,$$

for all $l \in \{1, 2, \ldots, N\}$. From the assumption on T_l, we know that T_l is uniformly continuous. On the other hand, it follows from (10) and (13) that

$$\lim_{n\to\infty} \|x_n - T_{n+l}x_n\| = 0,$$

for all $l \in \{1, 2, \ldots, N\}$. Thus $\lim_{n\to\infty} \|x_n - T_lx_n\| = 0$, for all $l \in \{1, 2, \ldots, N\}$.

Finally, we prove that $x_n \to \Pi_F x_0$. Assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x} \in C$, then $\tilde{x} \in \hat{F} = F$. Next we show that $\tilde{x} = \Pi_F x_0$ and the convergence is strong. Put $\bar{x} = \Pi_F x_0$. From $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $\bar{x} \in F \subset C_n \cap Q_n$, we have $\phi(x_{n+1}, x_0) \le \phi(\bar{x}, x_0)$. On the other hand, from weak lower semicontinuity of the norm, we obtain

$$\begin{aligned} \phi(\tilde{x}, x_0) &= \|\tilde{x}\|^2 - 2\langle \tilde{x}, Jx_0\rangle + \|x_0\|^2 \\ &\le \liminf_{i\to\infty} (\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0\rangle + \|x_0\|^2) \\ &\le \liminf_{i\to\infty} \phi(x_{n_i}, x_0) \\ &\le \limsup_{i\to\infty} \phi(x_{n_i}, x_0) \\ &\le \phi(\bar{x}, x_0). \end{aligned}$$

It follows from the definition of $\Pi_F x_0$ that we have $\tilde{x} = \bar{x}$ and hence $\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(\bar{x}, x_0)$. So, we have $\lim_{i\to\infty} \|x_{n_i}\| = \|\bar{x}\|$. By using the Kadec-Klee property of E, we obtain that $\{x_{n_i}\}$ converges strongly to $\Pi_F x_0$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $\Pi_F x_0$. This completes the proof. ∎

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