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On the Asymptotic Behavior of Solutions for a Third-Order Nonlinear Differential Equation *

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Abstract. This paper considers the third-order nonlinear differential equation. Some sufficient conditions are established to show the convergence to zero of all solutions and their first and second derivatives, which are new and complement previously known results.

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1. Introduction

Consider the following third-order nonlinear differential equation

$$x'''(t) + a(t)x''(t) + b(t)x'(t) + g(t, x(t)) = p(t), \tag{1}$$

where $t \in \mathbb{R}^+ = [0, +\infty]$, g is a continuous function on $\mathbb{R}^+ \times \mathbb{R}$, $\mathbb{R} = (-\infty, +\infty)$, a, b and p are continuous functions on $\mathbb{R}$.

In applied science some practical problems are associated with Equation (1), such as nonlinear oscillations ([1],[2],[3]), electronic theory [4], biological model

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and other models [5],[6]. Just as above, in the past few decades, the study for third order differential equations has been paid attention by many scholars, and many results relative to the stability, instability of solutions, boundedness of solutions, convergence of solutions and existence of periodic solutions for Equation (1) and its analogous equations have been obtained (see [7],[8] and the references therein). However, as pointed out in [8], due to the difficulty of constructing proper Lyapunov functions for higher-order nonlinear differential equations, the results about the stability of nonlinear differential equations whose orders are more than two are relatively scarce.

A primary purpose of this paper is to study the problem of convergence behavior for all solutions of Eq.(1).We will establish some sufficient conditions to show the convergence to zero of all solutions and their first and second derivatives, which are new and complement previously known results. In particular, an example is also provided to illustrate the effectiveness of the new results.

Let d_1 and d_2 be constants. Define

$$y(t) = \frac{dx(t)}{dt} + d_1 x(t), \quad z(t) = \frac{dy(t)}{dt} + d_2 y(t), \tag{2}$$

then we can transform (1) into the following equivalent system

$$\begin{cases} \dfrac{dx(t)}{dt} = -d_1 x(t) + y(t), \\ \dfrac{dy(t)}{dt} = -d_2 y(t) + z(t), \\ \dfrac{dz(t)}{dt} = -(a(t) - d_1 - d_2) z(t) + (-(a(t) - d_1) d_1^2 + b(t) d_1) x(t) \\ \qquad -g(t, x(t)) + ((a(t) - d_1)(d_1 + d_2) - b(t) - d_2^2) y(t) + p(t). \end{cases} \tag{3}$$

It is known in [9],[10],[11] that for a, b, g and p being continuous, given an initial data $(x(0), y(0), z(0)) = (x_0, y_0, z_0) \in \mathbb{R}^3$, then there exists a solution of (3) on an interval $[0, T)$ satisfying the initial condition and satisfying (3) on $[0, T)$. Since we will prove that all solutions of (3) remain bounded, then $T = +\infty$. We denote such a solution by

$$(x(t), y(t), z(t)) = (x(t, x_0, y_0, z_0), y(t, x_0, y_0, z_0), z(t, x_0, y_0, z_0)).$$

We also assume that the following conditions (C_1) and (C_2) hold.

(C_1) There exist constants $L \geq 0, K \geq 0, d_1 > 1, d_2 > 1$ and $d_3 > 0$ such that

$$|((-(a(t) - d_1) d_1^2 + b(t) d_1) u - g(t, u))| \leq L|u|, \text{ for all } u \in \mathbb{R}, \text{and } t \geq K,$$

and

$$d_3 = \inf_{t \geq K} (a(t) - d_1 - d_2) - (L + \sup_{t \geq K} |(a(t) - d_1)(d_1 + d_2) - b(t) - d_2^2|).$$

(C_2) $\lim\limits_{t \to +\infty} p(t) = 0.$

2. Main Results

Theorem 2.1. *Let* (C_1) *and* (C_2) *hold. Moreover, assume that* $(x(t), y(t), z(t))$ *is a solution of system* (3) *with initial data* $(x(0), y(0), z(0)) = (x_0, y_0, z_0) \in \mathbb{R}^3$. *Then, there holds*

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} z(t) = 0.$$

Proof. Since $\min\{d_1 - 1, d_2 - 1, d_3\} > 0$, it follows that there exist constants $\lambda > 0$ and $\gamma > 0$ such that

$$\gamma = \min\{(d_1 - 1) - \lambda, (d_2 - 1) - \lambda, d_3 - \lambda\} > 0. \tag{4}$$

From (C_2), for any $\varepsilon > 0$, we can choose a sufficient large constant $K_1 \geq K$ such that

$$|p(t)| \leq \frac{1}{2}\gamma\varepsilon, t \geq K_1. \tag{5}$$

Calculating the upper right derivatives of $e^{\lambda s}|x(s)|$, $e^{\lambda s}|y(s)|$ and $e^{\lambda s}|z(s)|$, in view of (4), (5) and (C_1), for $t \geq K_1$, we have

$$\begin{aligned} D^+(e^{\lambda s}|x(s)|)|_{s=t} &= \lambda e^{\lambda t}|x(t)| + e^{\lambda t}\mathrm{sgn}(x(t))\{-d_1 x(t) + y(t)\} \\ &< e^{\lambda t}\{(\lambda - d_1)|x(t)| + |y(t)|\} + \frac{1}{2}\gamma\varepsilon e^{\lambda t}, \end{aligned} \tag{6}$$

$$\begin{aligned} D^+(e^{\lambda s}|y(s)|)|_{s=t} &= \lambda e^{\lambda t}|y(t)| + e^{\lambda t}\mathrm{sgn}(y(t))\{-d_2 y(t) + z(t)\} \\ &< e^{\lambda t}\{(\lambda - d_2)|y(t)| + |z(t)|\} + \frac{1}{2}\gamma\varepsilon e^{\lambda t}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} D^+(e^{\lambda s}|z(s)|)|_{s=t} &= \lambda e^{\lambda t}|z(t)| + e^{\lambda t}\mathrm{sgn}(z(t))\{-(a(t) - d_1 - d_2)z(t) \\ &\quad + (-(a(t) - d_1)d_1^2 + b(t)d_1)x(t) - g(t, x(t)) \\ &\quad + ((a(t) - d_1)(d_1 + d_2) - b(t) - d_2^2)y(t) + p(t)\} \\ &\leq e^{\lambda t}\{(\lambda - \inf_{t\geq K}(a(t) - d_1 - d_2))|z(t)| + L|x(t)| \\ &\quad + \sup_{t\geq K}|(a(t) - d_1)(d_1 + d_2) - b(t) - d_2^2||y(t)|\} + \frac{1}{2}\gamma\varepsilon e^{\lambda t}. \end{aligned} \tag{8}$$

Let

$$M(t) = \max_{0\leq s\leq t}\{e^{\lambda s}\max\{|x(s)|, |y(s)|, |z(s)|\}\}. \tag{9}$$

It is obvious that $e^{\lambda t}\max\{|x(t)|, |y(t)|, |z(t)|\} \leq M(t)$, and $M(t)$ is nondecreasing.

Now, we consider two cases.

Case 1. There exists a constant $K_1 > K$ such that

$$M(t) > e^{\lambda t}\max\{|x(t)|,|y(t)|,|z(t)|\} \quad \text{for all} \quad t \geq K_1. \tag{10}$$

Then we claim that

$$M(t) \equiv M(K_1) \text{ is a constant for all } t \geq K_1. \tag{11}$$

Assume on the contrary, that (11) does not hold. Then there exists $t_1 > K_1$ such that $M(t_1) > M(K_1)$. Since

$$e^{\lambda t}\max\{|x(t)|,|y(t)|,|z(t)|\} \leq M(K_1) \text{ for all } 0 \leq t \leq K_1,$$

then there exists $\beta \in (K_1, t_1)$ such that

$$e^{\lambda\beta}\max\{|x(\beta)|,|y(\beta)|,|z(\beta)|\} = M(t_1) \geq M(\beta),$$

which contradicts (10). This contradiction implies that (11) holds. It follows that there exists $t_2 \geq K_1$ such that

$$\max\{|x(t)|,|y(t)|,|z(t)|\} < e^{-\lambda t}M(t) = e^{-\lambda t}M(K_1) < \varepsilon \text{ for all } t \geq t_2. \tag{12}$$

Case 2. There is such a point $t_0 \geq K$ that $M(t_0) = e^{\lambda t_0}\max\{|x(t_0)|,|y(t_0)|,|z(t_0)|\}$. Let $M(t_0) = e^{\lambda t_0}\max\{|x(t_0)|,|y(t_0)|,|z(t_0)|\} = e^{\lambda t_0}|x(t_0)|$, in view of (6), we get

$$\begin{aligned} D^+(e^{\lambda s}|x(s)|)|_{s=t_0} &\leq (\lambda - d_1)|x(t_0)|e^{\lambda t_0} + |y(t_0)|e^{\lambda t_0} + \frac{1}{2}\gamma\varepsilon e^{\lambda t_0} \\ &\leq (\lambda - (d_1 - 1))M(t_0) + \frac{1}{2}\gamma\varepsilon e^{\lambda t_0} \\ &< -\gamma M(t_0) + \gamma\varepsilon e^{\lambda t_0}. \end{aligned} \tag{13}$$

Let $M(t_0) = e^{\lambda t_0}\max\{|x(t_0)|,|y(t_0)|,|z(t_0)|\} = e^{\lambda t_0}|y(t_0)|$, in view of (7), we get

$$\begin{aligned} D^+(e^{\lambda s}|y(s)|)|_{s=t_0} &\leq (\lambda - d_2)|y(t_0)|e^{\lambda t_0} + |z(t_0)|e^{\lambda t_0} + \frac{1}{2}\gamma\epsilon e^{\lambda t_0} \\ &\leq (\lambda - (d_2 - 1))M(t_0) + \frac{1}{2}\gamma\varepsilon e^{\lambda t_0} \\ &< -\gamma M(t_0) + \gamma\varepsilon e^{\lambda t_0}. \end{aligned} \tag{14}$$

Let $M(t_0) = e^{\lambda t_0}\max\{|x(t_0)|,|y(t_0)|,|z(t_0)|\} = e^{\lambda t_0}|z(t_0)|$, in view of (8), we obtain

$$\begin{aligned} D^+(e^{\lambda s}|z(s)|)|_{s=t_0} \leq\ & e^{\lambda t_0}\{(\lambda - \inf_{t\geq K}(a(t) - d_1 - d_2))|z(t_0)| + L|x(t_0)| \\ & + \sup_{t\geq K}|(a(t) - d_1)(d_1 + d_2) - b(t) - d_2^2||y(t_0)|\} + \frac{1}{2}\gamma\varepsilon e^{\lambda t_0} \\ \leq\ & (\lambda - d_3)M(t_0) + \frac{1}{2}\gamma\varepsilon e^{\lambda t_0} \\ <\ & -\gamma M(t_0) + \gamma\varepsilon e^{\lambda t_0}. \end{aligned} \tag{15}$$

In addition, if $M(t_0) \geq \varepsilon e^{\lambda t_0}$, (13),(14),(15) imply that $M(t)$ is strictly decreasing in a small neighborhood $(t_0, t_0 + \delta_0)$. This contradicts that $M(t)$ is non - decreasing. Hence

$$M(t_0) < \varepsilon e^{\lambda t_0}, \text{ and } \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|\} < \varepsilon. \tag{16}$$

For all $t > t_0$, by the same approach used in the proof of (16), we have

$$\max\{|x(t)|, |y(t)|, |z(t)|\} < \varepsilon, \text{ if } M(t) = e^{\lambda t}\max\{|x(t)|, |y(t)|, |z(t)|\}. \tag{17}$$

On the other hand, if $M(t) > e^{\lambda t}\max\{|x(t)|, |y(t)|, |z(t)|\}, t > t_0$, we can choose $t_0 \leq t_3 < t$ such that

$$M(t_3) = e^{\lambda t_3}\max\{|x(t_3)|, |y(t_3)|, |z(t_3)|\} < \varepsilon e^{\lambda t_3}$$

and

$$M(s) > e^{\lambda s}\max\{|x(s)|, |y(s)|, |z(s)|\}, \forall s \in (t_3, t].$$

By using a similar argument as in the proof of *Case 1*, we can show that

$$M(s) \equiv M(t_3) \text{ is a constant for all } s \in (t_3, t], \tag{18}$$

which implies that

$$\begin{aligned}\max\{|x(t)|, |y(t)|, |z(t)|\} &< e^{-\lambda t}M(t)\\ &= e^{-\lambda t}M(t_3)\\ &= \max\{|x(t_3)|, |y(t_3)|, |z(t_3)|\}e^{-\lambda(t-t_3)}\\ &< \varepsilon.\end{aligned}$$

In summary, there must exist $N > 0$ such that $\max\{|x(t)|, |y(t)|, |z(t)|\} \leq \varepsilon$ holds for all $t > N$. The proof of Theorem 2.1 is now complete. ■

Remark 2.2. Since

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} z(t) = 0,$$

it follows from (3) that

$$\lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty}(-d_1x(t) + y(t)) = 0,$$

and

$$\lim_{t\to+\infty} x''(t) = \lim_{t\to+\infty}(-d_1x'(t) + y'(t)) = \lim_{t\to+\infty}(-d_1x'(t) + (-d_2y(t) + z(t))) = 0.$$

3. An Example

Example 3.1. All solutions and their first and second derivatives of the third - order nonlinear differential equation

$$x'''(t) + (9 - \frac{1}{t^2+1})x''(t) + (\frac{-4}{t^2+1} + 23)x'(t) + x(t)e^{\cos x(t)} + (18 + \frac{-4}{t^2+1})x(t) = \frac{t}{1+t^2}, \tag{16}$$

converge to zero.

Proof. Set

$$y(t) = \frac{dx(t)}{dt} + 2x(t), \quad z(t) = \frac{dy(t)}{dt} + 2y(t), \tag{17}$$

then we can transform (16) into the following equivalent system

$$\begin{cases} \frac{dx(t)}{dt} = -2x(t) + y(t), \\ \frac{dy(t)}{dt} = -2y(t) + z(t), \\ \frac{dz(t)}{dt} = -(5 - \frac{1}{t^2+1})z(t) - x(t)e^{\cos x(t)} + y(t) + \frac{t}{1+t^2}. \end{cases} \tag{18}$$

It is straight forward to check that all assumptions needed in Theorem 2.1 are satisfied. It follows that all solutions of system (18) converge to zero. Hence, all solutions and their first and second derivatives of equation (16) converge to zero. ■

Remark 3.2. Since $p(t) = \dfrac{t}{1+t^2}$ and $\int_0^{+\infty} \frac{t}{1+t^2} dt = +\infty$, it is clear that the conditions (4) in [8] is not satisfied. Therefore, the results obtained in [8] and the references cited therein can not be applicable to prove that all solutions and their first and second derivatives of equation (16) converge to zero. This implies that the results of this paper are essentially new.

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