Vietnam Journal of Mathematics **39**:1(2011) 79-89

Vietnam Journal
of
MATHEMATICS

A Characterization of Partition Systems of $\mathbb{R}^n$ and Application

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Received December 26, 2010

Abstract. An *m-partition* of $\mathbb{R}^n$ is, by definition, a system of $m+n$ vectors

$$U = \{u^{1,0}, u^{1,1}, u^{2,0}, u^{2,1}, \dots, u^{m,0}, u^{m,1}, u^{m+1}, \dots, u^n\} \subseteq \mathbb{R}^n,$$

such that, for every $x \in \mathbb{R}^n$ there exists a unique vector λ sastisfying

$$\begin{aligned} \lambda = \quad & (\lambda^{1,0}, \lambda^{1,1}, \dots, \lambda^{m,0}, \lambda^{m,1}, \lambda^{m+1}, \dots, \lambda^n)^T \in \mathbb{R}^{m+n}, \\ & \lambda^{i,s} \geq 0, \quad (i,s) \in I \times S, \\ & \lambda^{i,0}.\lambda^{i,1} = 0, \quad i \in I, \\ x = & \sum_{(i,s)\in I\times S} \lambda^{i,s} u^{i,s} + \sum_{j\in J} \lambda^j u^j, \end{aligned}$$

where $I := \{1, 2, \dots, m\}$, $J := \{m+1, \dots, n\}$ and $S := \{0, 1\}$.

Systems of this type are usually encountered in linear complementarity problems. By studying them we expect to provide some strong tools for investigating theory of complementarity problems. Specifically, in this paper we shall prove a basic characterization of partition systems in $\mathbb{R}^n$ and then derive some direct applications to linear complementarity problems.

2000 Mathematics Subject Classification: 15A15, 90C33.

Key words: m-partition, complete partition, linear complementarity problem.

1. Introduction

Let U be a system of $m+n$ vectors in the space $\mathbb{R}^n$:

$$U = \{u^{1,0}, u^{1,1}, u^{2,0}, u^{2,1}, \dots, u^{m,0}, u^{m,1}, u^{m+1}, \dots, u^n\}. \tag{1}$$

For convenience we also consider U as an $n \times (m+n)$ real matrix. U is called an *m-partition* of the space if for every $x \in \mathbb{R}^n$ there exists a unique vector λ sastisfying

$$\begin{cases} \lambda = (\lambda^{1,0}, \lambda^{1,1}, \cdots, \lambda^{m,0}, \lambda^{m,1}, \lambda^{m+1}, \cdots, \lambda^n)^T \in \mathbb{R}^{m+n}, \\ \lambda^{i,s} \geq 0, \quad (i,s) \in I \times S, \\ \lambda^{i,0}.\lambda^{i,1} = 0, \quad i \in I, \end{cases} \tag{2}$$

$$x = U\lambda = \sum_{(i,s)\in I\times S} \lambda^{i,s} u^{i,s} + \sum_{j\in J} \lambda^j u^j, \tag{3}$$

where $I := \{1, 2, \dots, m\}$, $J := \{m+1, \dots, n\}$ and $S := \{0, 1\}$. By (3) we say x is expressed as a *complementarity combination* of U. Therefore, U is an m-partition of $\mathbb{R}^n$ if any vector $x \in \mathbb{R}^n$ can be expressed uniquely as a complementarity combination of U.

In the special case $m = n$; i.e. $I = \{1, \dots, n\}$, $J = \emptyset$, the system U will be called a *complete partition*. In that case, $U = \{u^{1,0}, u^{1,1}, u^{2,0}, u^{2,1}, \dots, u^{n,0}, u^{n,1}\}$ and for every $x \in \mathbb{R}^n$, there exists uniquely $\lambda \in \mathbb{R}^{2n}$ such that

$$\begin{cases} \lambda = (\lambda^{1,0}, \lambda^{1,1}, \lambda^{2,0}, \lambda^{2,1}, \dots, \lambda^{n,0}, \lambda^{n,1})^T \geq 0, \\ \lambda^{i,0}\lambda^{i,1} = 0, \quad i \in I, \end{cases} \tag{4}$$

$$x = U\lambda = \sum_{(i,s)\in I\times S} \lambda^{i,s} u^{i,s}. \tag{5}$$

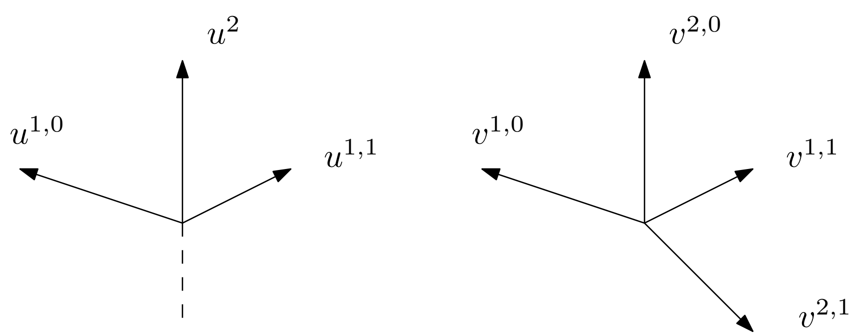

Fig. 1 $U = \{u^{1,0}, u^{1,1}, u^2\}$ is an 1-*partition* of $\mathbb{R}^2$ while $V = \{v^{1,0}, v^{1,1}, v^{2,0}, v^{2,1}\}$ is a *complete partition*

Let U be given in (1). For each $\alpha \subseteq I$ we establish a matrix $U(\alpha) \in \mathbb{R}^{n\times n}$, called *the complementarity matrix* of U associated with α, whose i^{th} column is defined by

$$U(\alpha)^i = \begin{cases} u^{i,0}, & i \in \alpha, \\ u^{i,1}, & i \in I\backslash\alpha, \\ u^i, & i \in J. \end{cases} \tag{6}$$

Denote $\mathfrak{M}(U) = \{U(\alpha) \mid \alpha \subseteq I\}$. Clearly, $|\mathfrak{M}(U)| = 2^m$. Two subsets α and β of I are said to be *adjacent* if $(\alpha \cup \beta) \setminus (\alpha \cap \beta) = \{r\}$, for some $r \in I$. In this case we also say $U(\alpha)$ and $U(\beta)$ to be adjacent at r^{th} column. Obviously,

$$U(\alpha)^i = U(\beta)^i, \quad i \in I \cup J \setminus \{r\},$$

$$\{U(\alpha)^r, U(\beta)^r\} = \{u^{r,0}, u^{r,1}\}.$$

The main result of the paper is stated as follows.

Theorem 1.1. *U is an m-partition of $\mathbb{R}^n$ if and only if*

$$\det(U(\alpha)).\det(U(\beta)) < 0 \ \ \textit{for any pair of adjacent matrices } U(\alpha),\ U(\beta). \tag{7}$$

The next section deals with the case of complete partition, the proof of Theorem 1.1 is given in Section 3 and the last section is devoted to study on the number of solutions of linear complementarity problems.

2. Case of Complete Partitions

In this section we always consider U as a system of $2n$ vectors

$$U = \{u^{1,0}, u^{1,1}, u^{2,0}, u^{2,1}, \ldots, u^{n,0}, u^{n,1}\} \subseteq \mathbb{R}^n.$$

Hence, $I = \{1, 2, \ldots, n\}$, $S = \{0, 1\}$ and $\mathfrak{M}(U) = \{U(\alpha) \mid \alpha \subseteq I\}$ has 2^n members.

The following result yields some characterizations of complete partitions:

Theorem 2.1. *The following are equivalent*

(i) *U is a complete partition of $\mathbb{R}^n$;*

(ii) *For every pair of adjacent complementarity matrices $U(\alpha), U(\beta)$, we have*

$$\det(U(\alpha)).\det(U(\beta)) < 0; \tag{8}$$

(iii) *For any $\lambda = (\lambda^{1,0}, \lambda^{1,1}, \lambda^{2,0}, \lambda^{2,1}, \ldots, \lambda^{n,0}, \lambda^{n,1})^T \in \mathbb{R}^{2n}$ satisfying*

$$\lambda^{i,0}.\lambda^{i,1} \geq 0, \quad \lambda^{i,0} + \lambda^{i,1} \neq 0, \quad i \in I, \tag{9}$$

the system

$$\{\lambda^{i,0}.u^{i,0} - \lambda^{i,1}.u^{i,1} \mid i \in I\} \tag{10}$$

is linearly independent.

Remark 2.2. Let $\alpha \subseteq I$, we denote by $U^{-\alpha}$ the matrix obtained from U by replacing $u^{i,0}$, $u^{i,1}$ with $-u^{i,0}$, $-u^{i,1}$ respectively, for all $i \in \alpha$. It is clear that if U satisfies the statement (iii) in Theorem 2.1 then so does $U^{-\alpha}$. Since the three statements are equivalent, it implies that $U^{-\alpha}$ is a complete partition if so is U.

Remark 2.3. Note that if any of the three statements of the theorem is satisfied then $U(\alpha)$ is nonsingular for every $\alpha \subseteq I$.

Proof of Theorem 2.1.

(i) $\Rightarrow$ (ii). Without loss of generality we can assume that $\alpha = \{1, 2, \cdots, r-1\}$ and $\beta = \alpha \cup \{r\}$. Hence

$$U(\alpha) = [u^{1,0}, u^{2,0}, \dots, u^{(r-1),0}, u^{r,1}, \dots, u^{n,1}];$$

$$U(\beta) = [u^{1,0}, u^{2,0}, \dots, u^{r,0}, u^{(r+1),1}, \dots, u^{n,1}].$$

Setting $t := U(\beta)^{-1}u^{r,1}$ one has

$$u^{r,1} = U(\beta)t = \sum_{i=1}^{r} t_i u^{i,0} + \sum_{j=r+1}^{n} t_j u^{j,1}.$$

Substituting the right-hand side into the r^{th} column of $U(\alpha)$ and computing the determinant of the matrix we obtain

$$\det(U(\alpha)) = t_r \det(U(\beta)).$$

Since $U(\alpha)$ is nonsingular, t_r is nonzero, and (8) now is equivalent to say that $t_r < 0$. On the contrary, suppose that $t_r > 0$. Then we set

$$x^\varepsilon = \sum_{i=1}^{r-1} u^{i,0} + \varepsilon u^{r,1} + \sum_{j=r+1}^{n} u^{j,1} \tag{11}$$

$$= \sum_{i=1}^{r-1} (1 + \varepsilon t_i) u^{i,0} + \varepsilon t_r u^{r,0} + \sum_{j=r+1}^{n} (1 + \varepsilon t_j) u^{j,1}. \tag{12}$$

For $\varepsilon > 0$ small enough we have $1 + \varepsilon t_i > 0$, for all $i \neq r$. But (11) and (12) now give us two different representations of x^ε as a complementarity combination of U, contradicting the assumption that U is a complete partition.

(ii) $\Rightarrow$ (iii). From the hypothesis and by induction on $|\alpha|$ we can derive that

$$(-1)^{|\alpha|}.\det(U(\alpha)).\det(U(\emptyset)) > 0, \quad \alpha \subseteq I.$$

Consequently,

$$(-1)^{|I\setminus\alpha|}.\det(U(\alpha)).(-1)^{|I\setminus\beta|}.\det(U(\beta)) > 0, \quad \alpha,\beta \subseteq I. \tag{13}$$

From the well known properties of determinants it is easy to check that

$$\det\{\lambda^{1,0}u^{1,0} - \lambda^{1,1}u^{1,1}, \lambda^{2,0}u^{2,0} - \lambda^{2,1}u^{2,1}, \cdots, \lambda^{n,0}u^{n,0} - \lambda^{n,1}u^{n,1}\}$$
$$= \sum_{\alpha\subseteq I}\Big(\prod_{i\in\alpha}\lambda^{i,0}\Big)\Big(\prod_{j\in I\setminus\alpha}\lambda^{j,1}\Big)(-1)^{|I\setminus\alpha|}\det(U(\alpha)).$$

Combining (9) and (13) we may assert that all terms of the sum have the same sign and, furthermore, at least one of them is nonzero. So, the determinant is nonzero too and the system (10) is thus linear independent.

Throughout the rest of this section we suppose that the statement (iii) holds. The proof of (i) will be divided into a sequence of lemmas.

Lemma 2.4. *For every* $\lambda = (\lambda^{i,s} : (i,s) \in I\times S) \in \mathbb{R}^{2n}$*, satisfying*

$$\sum_{i=1}^{n}(\lambda^{i,0}u^{i,0} - \lambda^{i,1}u^{i,1}) = 0 \quad \textit{and} \quad \lambda^{i,0}.\lambda^{i,1} \geq 0,\ i\in I, \tag{14}$$

we have $\lambda = 0$.

Proof. Since $\lambda^{i,0}.\lambda^{i,1} \geq 0$, for all $i \in I$, it suffices to show that the set $\gamma := \{i \mid \lambda^{i,0} + \lambda^{i,1} \neq 0\}$ is empty. Indeed, if $\gamma \neq \emptyset$ then from (14) we have

$$\sum_{i\in\gamma}(\lambda^{i,0}u^{i,0} - \lambda^{i,1}u^{i,1}) = 0,$$

a contradiction, since $\{\lambda^{i,0}.u^{i,0} - \lambda^{i,1}.u^{i,1} \mid i \in \gamma\}$ is a nonempty subsystem of (10). ■

Lemma 2.5. *For every* $\lambda = (\lambda^{i,s} : (i,s) \in I\times S)$, $\mu = (\mu^{i,s} : (i,s) \in I\times S) \in \mathbb{R}^{2n}$ *satisfying*

$$\lambda^{i,0}.\lambda^{i,1} = \mu^{i,0}.\mu^{i,1} = 0,\ \lambda^{i,s} \geq 0, \mu^{i,s} \geq 0; \quad i\in I,\ s\in S$$

and

$$\sum_{(i,s)\in I\times S}\lambda^{i,s}u^{i,s} = \sum_{(i,s)\in I\times S}\mu^{i,s}u^{i,s},$$

we have $\lambda = \mu$.

Proof. By the hypothesis one gets

$$\sum_{i=1}^{n}[(\lambda^{i,0} - \mu^{i,0})u^{i,0} - (\mu^{i,1} - \lambda^{i,1})u^{i,1}] = 0,$$

and, furthermore,

$$(\lambda^{i,0} - \mu^{i,0}).(\mu^{i,1} - \lambda^{i,1}) = \lambda^{i,0}.\mu^{i,1} + \lambda^{i,1}.\mu^{i,0} \geq 0, \quad i \in I.$$

It follows from Lemma 2.4 that $\lambda^{i,0} - \mu^{i,0} = \lambda^{i,1} - \mu^{i,1} = 0$ for all $i \in I$, i.e. $\lambda = \mu$. ∎

Now let a fixed $x \in \mathbb{R}^n$. Since (iii) holds, $U(\alpha)$ is nonsingular for every $\alpha \subseteq I$. Corresponding to such a subset we set $y^\alpha := U(\alpha)^{-1}x$, that is,

$$x = U(\alpha)y^\alpha = \sum_{i\in\alpha} y_i^\alpha u^{i,0} + \sum_{i\in I\setminus\alpha} y_i^\alpha u^{i,1}.$$

We define $t^\alpha = \mathrm{sgn}(\mathrm{y}^\alpha)$, the *signal vector* of y^α, as follows

$$t_i^\alpha := \begin{cases} 1, & \text{if } (y_i^\alpha > 0) \text{ or } (y_i^\alpha = 0 \text{ and } i \in \alpha), \\ -1, & \text{if } (y_i^\alpha < 0) \text{ or } (y_i^\alpha = 0 \text{ and } i \in I \setminus \alpha). \end{cases}$$

Lemma 2.6. *The mapping $\Phi : \mathcal{P}(I) \to \{-1, 1\}^n$ defined by $\Phi(\alpha) = t^\alpha$ is bijective, where $\mathcal{P}(I)$ denotes the set of all subsets of I.*

Proof. Since $|\mathcal{P}(I)| = |\{-1, 1\}^n| = 2^n$, it suffices to prove that Φ is injective.

Suppose conversely that there exist different sets $\alpha, \beta \subseteq I$ such that $t^\alpha = t^\beta$. By the definition of y^α and y^β we have

$$\sum_{i\in\alpha} y_i^\alpha u^{i,0} + \sum_{i\in I\setminus\alpha} y_i^\alpha u^{i,1} = x = \sum_{i\in\beta} y_i^\beta u^{i,0} + \sum_{i\in I\setminus\beta} y_i^\beta u^{i,1}.$$

It follows that

$$\sum_{i\in\alpha\cap\beta} (y_i^\alpha - y_i^\beta)u^{i,0} + \sum_{i\in\alpha\setminus\beta} (y_i^\alpha u^{i,0} - y_i^\beta u^{i,1}) + \sum_{i\in\beta\setminus\alpha} (-y_i^\beta u^{i,0} + y_i^\alpha u^{i,1})$$
$$+ \sum_{i\in I\setminus(\alpha\cup\beta)} (y_i^\alpha - y_i^\beta)u^{i,1} = 0.$$

Besides, from $t^\alpha = t^\beta$ we have $y_i^\alpha y_i^\beta \geq 0$ for all $i \in I$. So, by virtue of Lemma 2.4, one has $y_i^\alpha - y_i^\beta = 0$ for all $i \in (\alpha \cap \beta) \cup (I \setminus (\alpha \cup \beta))$ and $y_i^\alpha = y_i^\beta = 0$ for all $i \in (\alpha \setminus \beta) \cup (\beta \setminus \alpha)$. Since $\alpha \neq \beta$ there exists, for instance, $r \in \alpha \setminus \beta$. By the definition of signal vectors we have $t_r^\alpha = 1$ and $t_r^\beta = -1$, a contradiction. ∎

We are now in a position to prove the assertion (i). For every $x \in \mathbb{R}^n$, we show that there exists a unique vector λ satisfying

$$x = \sum_{(i,s)} \lambda^{i,s} u^{i,s} \quad \text{with} \quad \lambda^{i,s} \geq 0, \quad \lambda^{i,0}.\lambda^{i,1} = 0,\ (i,s) \in I \times S. \tag{15}$$

Indeed, by virtue of Lemma 2.6 there exists $\alpha \subseteq I$ such that

$$t^\alpha = \mathrm{sgn}(\mathrm{y}^\alpha) = (1, 1, \ldots, 1).$$

So $y_i^\alpha \geq 0$ for all $i \in I$ and the vector λ, defined by

$$\lambda^{i,0} = \begin{cases} y_i^\alpha, & \text{if } i \in \alpha, \\ 0, & \text{if } i \in I \setminus \alpha, \end{cases} \qquad \lambda^{i,1} = \begin{cases} 0, & \text{if } i \in \alpha, \\ y_i^\alpha, & \text{if } i \in I \setminus \alpha, \end{cases}$$

satisfies (15). The uniqueness of λ derives from Lemma 2.5 and the proof is complete. ■

3. Proof of Theorem 1.1

Now we turn to the general case where $I = \{1, 2, \dots, m\}$, $J = \{m+1, \dots, n\}$ and $U = \{u^{1,0}, u^{1,1}, \dots, u^{m,0}, u^{m,1}, u^{m+1}, \dots, u^n\}$. To this case we consider the generalized system

$$\overline{U} = \{u^{1,0}, u^{1,1}, \dots, u^{m,0}, u^{m,1}, u^{m+1,0}, u^{m+1,1}, \dots, u^{n,0}, u^{n,1}\},$$

with

$$u^{i,0} := -u^i, \quad u^{i,1} := u^i, \quad i \in J.$$

For each $\bar{\alpha} \subseteq I \cup J$ we define $\overline{U}(\bar{\alpha})$, *the complementarity matrix* of $\overline{U}$ associated with $\overline{\alpha}$, as follows

$$\overline{U}(\overline{\alpha})^i = \begin{cases} u^{i,0}, & i \in \overline{\alpha}, \\ u^{i,1}, & i \in (I \cup J) \setminus \bar{\alpha}. \end{cases}$$

The scheme for proving Theorem 1.1 might be described as follows: Firstly, we prove that U is an $m-$partition if and only if $\overline{U}$ is a complete partition; secondly, we prove that the class $\{U(\alpha) \mid \alpha \subseteq I\}$ satisfies (7) if and only if so does the class $\{\overline{U}(\overline{\alpha}) \mid \overline{\alpha} \subseteq I \cup J\}$. These together with Theorem 2.1 derive Theorem 1.1.

Let us denote the set of all $\lambda = (\lambda^{1,0}, \lambda^{1,1}, \dots, \lambda^{m,0}, \lambda^{m,1}, \lambda^{m+1}, \dots, \lambda^n) \in \mathbb{R}^{m+n}$ satisfying (2) and that of all $\overline{\lambda} = (\overline{\lambda}^{1,0}, \overline{\lambda}^{1,1}, \dots, \overline{\lambda}^{n,0}, \overline{\lambda}^{n,1}) \in \mathbb{R}^{2n}$ satisfying (4) by Λ and $\overline{\Lambda}$ respectively. Besides, we establish a map H from Λ into $\overline{\Lambda}$ defined by

$$\Lambda \ni \lambda \mapsto H(\lambda) = \overline{\lambda} \in \overline{\Lambda},$$

with

$$\overline{\lambda}^{i,s} = \lambda^{i,s} \quad (i, s) \in I \times S,$$

$$\overline{\lambda}^{i,0} = \begin{cases} -\lambda^i, & \lambda^i < 0, \\ 0, & \lambda^i \geq 0, \end{cases} \quad \overline{\lambda}^{i,1} = \begin{cases} 0, & \lambda^i < 0, \\ \lambda^i, & \lambda^i \geq 0, \end{cases} \quad i \in J.$$

The next result is obvious but useful.

Lemma 3.1.

(i) *H is a bijection from Λ onto $\overline{\Lambda}$;*

(ii) *$U\lambda = \overline{U}\overline{\lambda}$ for every $\lambda \in \Lambda, \overline{\lambda} = H(\lambda)$.*

Corollary 3.2. *U is an m-partition if and only if $\overline{U}$ is a complete partition.*

Lemma 3.3.

(i) $\overline{U}(\overline{\alpha}) = U(\overline{\alpha})$ *if* $\overline{\alpha} \subseteq I$*;*

(ii) $\det(\overline{U}(\overline{\alpha})). \det(\overline{U}(\overline{\beta})) < 0$ *if* $\overline{\alpha}, \overline{\beta} \subseteq I \cup J$ *are adjacent at a certain column* $r \in J$*;*

(iii) *If* $\overline{\alpha}$ *and* $\overline{\beta}$ *are adjacent at a column* $r \in I$ *then so are* $\alpha = \overline{\alpha} \cap I$ *and* $\beta = \overline{\beta} \cap I$*. Furthermore,*

$$\det(\overline{U}(\overline{\alpha})). \det(\overline{U}(\overline{\beta})) = \det(U(\alpha)). \det(U(\beta)). \tag{16}$$

Proof. (i) By definition one has

$$\overline{U}(\bar{\alpha})^i = \begin{cases} u^{i,0} = U(\bar{\alpha})^i & \text{if } i \in \bar{\alpha}, \\ u^{i,1} = U(\bar{\alpha})^i & \text{if } i \in I \setminus \bar{\alpha}, \\ u^{i,1} = u^i = U(\bar{\alpha})^i & \text{if } i \in J \text{ (and hence } i \notin \bar{\alpha}), \end{cases}$$

which implies $\overline{U}(\bar{\alpha}) = U(\bar{\alpha})$.

(ii) By the hypothesis one has $\overline{U}(\bar{\alpha})^i = \overline{U}(\bar{\beta})^i$ for all $i \neq r$ and $\overline{U}(\bar{\alpha})^r = -\overline{U}(\bar{\beta})^r$, because $u^{r,0} = -u^{r,1}$. Therefore $\det(\overline{U}(\bar{\alpha})). \det(\overline{U}(\bar{\beta})) < 0$.

(iii) The fact that α and β are adjacent at r is trivial. Thus we need only verify (16). By the hypothesis it is clear that

$$\det(\overline{U}(\bar{\alpha})) = (-1)^{|\bar{\alpha}\setminus\alpha|} \det(U(\alpha)), \quad \det(\overline{U}(\bar{\beta})) = (-1)^{|\bar{\beta}\setminus\beta|} \det(U(\beta)). \tag{17}$$

Since $\bar{\alpha} \setminus \alpha = \bar{\alpha} \setminus I = \bar{\beta} \setminus I = \bar{\beta} \setminus \beta$, (16) follows immediately from (17). ■

Lemma 3.4. *The following are equivalent:*

(i) $\det(U(\alpha)). \det(U(\beta)) < 0$ *for every pair of adjacent subsets* α*,* $\beta \subseteq I$*;*

(ii) $\det(\overline{U}(\bar{\alpha})). \det(\overline{U}(\bar{\beta})) < 0$ *for every pair of adjacent subsets* $\bar{\alpha}, \bar{\beta} \subseteq I \cup J$*.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from the last two statements in Lemma 3.3, while the converse one follows from the first statement of the mentioned lemma. ■

Proof of Theorem 1.1. Since, according to Theorems 2.1, $\overline{U}$ is a complete partition if and only if the class $\{\overline{U}(\bar{\alpha}) \mid \bar{\alpha} \subseteq I \cup J\}$ satisfies (7), the assertion of Theorem 1.1 now follows from Corollary 3.2 and Lemma 3.4. ■

4. Applications

In this section we shall employ the characterizations of partition systems developed in Theorem 1.1 and Theorem 2.1 to obtain some results on the number of solutions of linear complementarity problems. Recall that for given matrix $M \in \mathbb{R}^{n\times n}$ and vector $q \in \mathbb{R}^n$, the Linear Complementarity Problem LCP(M, q) is to find $(x, z) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\begin{cases} x \geq 0,\ z \geq 0, \\ x^T.z = 0, \\ q = -Mx + z. \end{cases} \tag{18}$$

The advantage of studying such a problem is well documented in the literature. See, for instance [4], [1] and [2]. By SOL(M, q) we denote the set of all solutions of LCP(M, q), i.e. the set of all (x, z) satisfying (18). It is worth noting that, if we denote $U[M] = \{-M^1, e^1, \ldots, -M^n, e^n\}$ with M^i and e^i stand for the i^{th} column of M and of the unit matrix E of order n, respectively, then LCP(M, q) is the one of finding

$$\lambda = (x_1, z_1, x_2, z_2, \ldots, x_n, z_n)^T \geq 0$$

such that

$$x_i z_i = 0,\ i \in I = \{1, 2, \ldots, n\} \quad \text{and} \quad q = U\lambda. \tag{19}$$

For each subset $\emptyset \neq \alpha \subseteq I$ we define M_α to be the submatrix of M obtained by omitting the rows and the columns corresponding to indices not belong to α. The determinants of these matrices are called principal minors of M. M is said to be a $P-$matrix, and denoted by $M \in \mathcal{P}$, if $\det(M_\alpha) > 0$ for every $\alpha \neq \emptyset$. As a generalization of this concept we call M a $P^{(k)}-$matrix, and write $M \in \mathcal{P}^{(k)}$, if there exists a subset $\gamma \subseteq I$ such that $|\gamma| = k$ and the matrix $M^{-\gamma}$, obtained from M by replacing the column M^i with $-M^i$ for every $i \in \gamma$, is a $P-$matrix. For example, the following matrix:

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

is a $P^{(2)}-$matrix since

$$M^{-\{2,3\}} = \begin{pmatrix} 2 & -1 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in \mathcal{P}.$$

Evidently, the class of $P^{(0)}-$matrices coincides with the one of $P-$matrices.

It is well known that, if $M \in \mathcal{P}$ then $|\text{SOL}(M, q)| \leq 1$ for any $q \in \mathbb{R}^n$. In fact, this is a characterization of $P-$matrices [3]. We now generalize this result to LCP$(M, q)'$s with underlying $P^{(k)}-$matrices.

Lemma 4.1. *$M \in \mathcal{P}$ if and only if $U[M]$ is a complete partition of $\mathbb{R}^n$.*

Proof. Note that, $U[M](\alpha)$, the complementarity matrix of $U[M]$ associated with $\alpha \subseteq I$, has the columns defined by

$$U[M](\alpha)^i = \begin{cases} -M^i, & i \in \alpha, \\ e^i, & i \in I \setminus \alpha. \end{cases}$$

So it is easy to verify that $\det(U[M](\emptyset)) = \det(E) = 1$ and

$$\det(U[M](\alpha)) = (-1)^{|\alpha|} \det(M_\alpha) \tag{20}$$

for every nonempty subset $\alpha \subseteq I$.

By virtue of Theorem 2.1 it suffices to prove that M is a $P-$ matrix if and only if the class $\{U[M](\alpha) \mid \alpha \subseteq I\}$ satisfies (8).

Assume that M is a $P-$matrix and $\alpha, \beta \subseteq I$ are adjacent. By definition one has, for instance, $|\beta| = |\alpha| + 1$. If $\alpha = \emptyset$ so that $|\beta| = 1$, then

$$\det(U[M](\beta)) \det(U[M](\alpha)) = \det(U[M](\beta)) = -\det(M_\beta) < 0.$$

If $\alpha \neq \emptyset$ we have

$$\det(U[M](\beta)) \det(U[M](\alpha)) = (-1)^{|\alpha|+|\beta|} \det(M_\beta) \det(M_\alpha) < 0.$$

Consequently, $\det(U[M](\beta)) \det(U[M](\alpha)) < 0$ for any pair of adjacent matrices.

Now assume $\{U[M](\alpha) \mid \alpha \subseteq I\}$ satisfies (8) we shall prove $\det(M_\alpha) > 0$, for all nonempty $\alpha \subseteq I$, by induction on $|\alpha|$. If $|\alpha| = 1$ then α is adjacent with $\emptyset$ and

$$\det(M_\alpha) = -\det(U[M](\alpha)) = -\det(U[M](\alpha)) \det(U[M](\emptyset)) > 0.$$

Assume $\det(M_\beta) > 0$ for all $|\beta| = k \geq 1$ and $\alpha \subseteq I$ such that $|\alpha| = k + 1$. Take an arbitrary $r \in \alpha$ then α is adjacent with $\beta = \alpha \setminus \{r\}$. We have

$$\det(M_\alpha) \det(M_\beta) = (-1)^{2k+1} \det(U[M](\alpha)) \det(U[M](\beta)) > 0.$$

Since $|\beta| = k$, so that $\det(M_\beta) > 0$, $\det(M_\alpha) > 0$. Thus, $\det(M_\alpha) > 0$ for every $\emptyset \neq \alpha \subseteq I$, i.e. $M \in \mathcal{P}$. ■

Corollary 4.2. *$M \in \mathcal{P}$ if and only if $|\mathrm{SOL}(M, q)| = 1$ for any $q \in \mathbb{R}^n$.*

Proof. It is worth noting that $U[M]$ is a complete partition if and only if for every $q \in \mathbb{R}^n$ there exists a unique vector $\lambda = (x_1, z_1, \cdots, x_n, z_n) \geq 0$ satisfying (19), i.e. $\mathrm{LCP}(M, q)$ has a unique solution. The corollary thus follows immediately from the preceding lemma. ■

Proposition 4.3. *If $M \in \mathcal{P}^{(k)}$ then $|\mathrm{SOL}(M, q)| \leq 2^k$ for all $q \in \mathbb{R}^n$.*

Proof. Assume conversely that $|\mathrm{SOL}(M, q)| > 2^k$. Denote by γ the subset of I such that $|\gamma| = k$ and $M^{-\gamma} \in \mathcal{P}$. For each $(x, z) \in \mathrm{SOL}(M, q)$ we denote by $\alpha(x, z)$ the set of indices $i \in \gamma$ such that $x_i > 0$ (and hence, $z_i = 0$):

$$\alpha(x, z) := \{i \in \gamma \mid x_i > 0 = z_i\} \subseteq \gamma.$$

Since $|\mathrm{SOL}(M,q)| > 2^k =$ the number of subsets of γ, there exist (x,z), $(x',z') \in \mathrm{SOL}(M,q)$ such that $(x,z) \neq (x',z')$ and $\alpha(x,z) = \alpha(x',z') =: \alpha$. Without loss of generality we may assume that $\alpha = \{1,2,\ldots,l\} \subseteq \gamma = \{1,\ldots,l,l+1,\ldots,k\}$. Since $M^{-\gamma} \in \mathcal{P}$, it follows from Lemma 4.1 that

$$U[M^{-\gamma}] = \{M^1, e^1, \ldots, M^l, e^l, M^{l+1}, e^{l+1}, \ldots, M^k, e^k, -M^{k+1}, e^{k+1}, \ldots, -M^n, e^n\}$$

is a complete partition. Therefore, by Remark 2.2, the system

$$V = \{-M^1, -e^1, \ldots, -M^l, -e^l, M^{l+1}, e^{l+1}, \ldots, M^k, e^k, -M^{k+1}, e^{k+1}, \ldots, -M^n, e^n\}$$

is also a complete partition. On the other hand, by the definition of α one has

$$q = \sum_{i=1}^{l} x_i(-M^i) + \sum_{i=l+1}^{k} z_i e^i + \sum_{i=k+1}^{n} (x_i(-M^i) + z_i e^i), \tag{21}$$

$$q = \sum_{i=1}^{l} x'_i(-M^i) + \sum_{i=l+1}^{k} z'_i e^i + \sum_{i=k+1}^{n} (x'_i(-M^i) + z'_i e^i). \tag{22}$$

But (21) and (22) now give us two different representations of q as a complementarity combination of V, contradicting the fact that V is a complete partition.

■

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