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Totally Contact Umbilical GCR-Lightlike Submanifolds Indefinite Sasakian Manifolds

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Abstract. In the present paper we find the conditions for the integrability of the distributions $\bar{D}$ and $D \oplus \{V\}$ of a GCR-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$. We find the conditions for the distributions $\bar{D}$ and $D \oplus \{V\}$ to define totally geodesic foliation in M and for the characterization of the connection ∇ to be a metric connection on the lightlike submanifold M. Finally, we study totally contact umbilical GCR-lightlike submanifolds and prove that a totally contact umbilical GCR-lightlike submanifold is a totally contact geodesic and a totally geodesic GCR-lightlike submanifold.

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1. Introduction

In 1978, Bejancu [1] introduced CR-submanifolds of Kaehler manifolds as a result in the process of generalization of invariant and anti-invariant submanifolds. Contact CR-submanifolds of Sasakian manifolds were introduced by Bejancu et al. [3] in 1981. Since contact geometry has vital role in the theory of differential equations, optics and phase spaces of a dynamical system, therefore contact geometry with definite and indefinite metric becomes the topic of main discussion.

Theory of contact CR-lightlike and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds was introduced by Duggal and Sahin [11], but there does not exist any inclusion relation between invariant and screen real submanifolds so new class of submanifolds called, Generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Sasakian manifolds (which is an umbrella of invariant, screen real, contact CR lightlike submanifolds) were derived by Duggal and Sahin [10].

The objective of this paper is to further elaborate the existing theory of GCR- lightlike submanifold of indefinite Sasakian manifolds. In section 3, we prove conditions for the integrability of the distributions, for the distributions to define totally geodesic foliation in submanifold and find a condition for the induced connection to be a metric connection. In section 4, we study totally contact umbilical GCR-lightlike submanifolds and prove that a totally contact umbilical GCR-lightlike submanifold is a totally contact geodesic and a totally geodesic GCR-lightlike submanifold.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [7] by Duggal and Bejancu.

Let $(\overline{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1, 1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\overline{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\{u \in T_x\overline{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\}, \tag{1}$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$RadTM : x \in M \longrightarrow RadT_xM, \tag{2}$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\overline{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is,

$$TM = RadTM \perp S(TM), \tag{3}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to

TM in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \quad (4)$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \quad (5)$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as

$$\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\},$$

where $\{\xi_1, ..., \xi_r\}, \{N_1, ..., N_r\}$ are local lightlike bases of

$$\{\Gamma(RadTM\mid_u), \Gamma(ltr(TM)\mid_u)\}$$

and

$$\{W_{r+1}, ..., W_n\}, \{X_{r+1}, ..., X_m\}$$

are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have

Theorem 2.1. ([7]) *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_{\mathrm{u}})$ consisting of smooth sections $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathrm{u}}$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \textit{for any} \quad i, j \in \{1, 2, .., r\}, \quad (6)$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (5), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall \quad X, Y \in \Gamma(TM), \quad (7)$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \quad \forall \quad X \in \Gamma(TM), U \in \Gamma(tr(TM)), \quad (8)$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear operator on M, known as shape operator.

According to (4), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then (7) and (8) give

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad (9)$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \quad (10)$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{12}$$

where $X \in \Gamma(TM)$, $\quad N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. By using (4)-(5) and (9)-(12), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{13}$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \tag{14}$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0, \tag{15}$$

for any $\xi \in \Gamma(RadTM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(ltr(TM))$.

Let P be the projection morphism of TM on $S(TM)$. Then using (3), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X, Y), \tag{16}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{17}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X, Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. h^* and A^* are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as the second fundamental forms of distributions $S(TM)$ and $RadTM$, respectively.

Next, we recall some basic definitions and results of indefinite Sasakian manifolds. An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an ϵ-contact metric manifold, if there are a $(1, 1)$ tensor field ϕ, a vector field V, called characteristic vector field and a 1-form η such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon\eta(X)\eta(Y), \quad \bar{g}(V, V) = \epsilon, \tag{18}$$

$$\phi^2(X) = -X + \eta(X)V, \quad \bar{g}(X, V) = \epsilon\eta(X), \tag{19}$$

$$d\eta(X, Y) = \bar{g}(X, \phi Y), \tag{20}$$

for any $X, Y \in \Gamma(TM)$, where $\epsilon = \pm 1$ then it follows that

$$\phi V = 0, \tag{21}$$

$$\eta \circ \phi = 0, \quad \eta(V) = 1. \tag{22}$$

Then $(\phi, V, \eta, \bar{g})$ is called an ϵ-contact metric structure of $\bar{M}$. We say that $\bar{M}$ has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is Nijenhuis tensor field of ϕ. A normal ϵ -contact metric manifold is called an ϵ-Sasakian manifold and for this we have

$$\bar{\nabla}_X V = \phi X, \tag{23}$$

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(X, Y)V + \epsilon\eta(Y)X. \tag{24}$$

3. Genralized Cauchy-Riemann (GCR)-Lightlike Submanifold

Calin[6], proved that if the characteristic vector field V is tangent to $(M, g, S(TM))$ then it belongs to $S(TM)$. We assume the characteristic vector field V is tangent to M throughout this paper.

Definition 3.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Sasakian manifold $(\overline{M}, \bar{g})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM). \tag{25}$$

(B) There exist two subbundles D_0 and $\bar{D}$ of $S(TM)$ such that

$$S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V, \quad \phi(\bar{D}) = L \perp S, \tag{26}$$

where D_0 is an invariant non degenerate distribution on M, $\{V\}$ is the one dimensional distribution spanned by V and L, S are vector subbundles of $ltr(TM)$ and $S(TM)^{\perp}$, respectively.

Then the tangent bundle TM of M is decomposed as

$$TM = \{D \oplus \bar{D} \oplus \{V\}\}, \quad D = Rad(TM) \oplus D_0 \oplus \phi(D_2). \tag{27}$$

Let Q, P_1, P_2 be the projection morphisms on D, ϕS, ϕL respectively, therefore

$$X = QX + V + P_1X + P_2X, \tag{28}$$

for $X \in \Gamma(TM)$. Applying ϕ to (28), we obtain

$$\phi X = fX + \omega P_1 X + \omega P_2 X, \tag{29}$$

where $fX \in \Gamma(D)$, $\omega P_1 X \in \Gamma(S)$ and $\omega P_2 X \in \Gamma(L)$, or, we can write (29), as

$$\phi X = fX + \omega X, \tag{30}$$

where fX and ωX are the tangential and transversal components of ϕX, respectively.
Similarly,

$$\phi U = BU + CU, \quad U \in \Gamma(tr(TM)), \tag{31}$$

where BU and CU are the sections of TM and $tr(TM)$, respectively.

Differentiating (29) and using (11)-(14) and (31), we have

$$D^l(X, \omega P_1 Y) = -\nabla^l_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^l(X, fY) + Ch^l(X, Y), \quad (32)$$

$$D^s(X, \omega P_2 Y) = -\nabla^s_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^s(X, fY) + Ch^s(X, Y), \quad (33)$$

for all $X, Y \in \Gamma(TM)$. By using Sasakian property of $\bar{\nabla}$ with (9) and (10), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then we have*

$$(\nabla_X f)Y = A_{\omega Y} X + Bh(X, Y) - g(X, Y)V + \epsilon\eta(Y)X, \quad (34)$$

and

$$(\nabla^t_X \omega)Y = Ch(X, Y) - h(X, fY), \quad (35)$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \quad (36)$$

$$(\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y. \quad (37)$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then we have*

$$(\nabla_X B)U = A_{CU} X - fA_U X, \quad (38)$$

and

$$(\nabla^t_X C)U = -\omega A_U X - h(X, BU), \quad (39)$$

where $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$ and

$$(\nabla_X B)U = \nabla_X BU - B\nabla^t_X U, \quad (40)$$

$$(\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U. \quad (41)$$

Lemma 3.4. *For $Y \in \Gamma(\overline{D})$ and $Z \in \Gamma(D)$, we have $g(\nabla_X Y, Z) = g(fA_{\omega Y} X, Z)$.*

Proof. Using (34), for any $Y \in \Gamma(\bar{D})$, we have $-f\nabla_X Y = A_{\omega Y} X + Bh(X, Y) - g(X, Y)V$. Let $Z \in \Gamma(D)$ then $\phi Z \in \Gamma(D)$, therefore $g(f\nabla_X Y, \phi Z) = -g(A_{\omega Y} X, \phi Z)$. Hence using (18) the assertion follows.

Particularly, let $Z \in \Gamma(D_0)$, then the non degeneracy of the distribution D_0 implies that $\nabla_X Y = fA_{\omega Y} X$, for any $Y \in \Gamma(\bar{D})$. ■

Theorem 3.5. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ then*

(A) *The distribution $D \oplus \{V\}$ is integrable, if and only if*

$$h(X, fY) = h(Y, fX), \quad \forall X, Y \in \Gamma(D \oplus \{V\}). \tag{42}$$

(B) *The distribution $\overline{D}$ is integrable, if and only if*

$$A_{\phi Z}U = A_{\phi U}Z, \quad \forall Z, U \in \Gamma(\overline{D}). \tag{43}$$

Proof. Using (32) and (33), we have $\omega P \nabla_X Y = h(X, fY) - Ch(X, Y)$, for any $X, Y \in \Gamma(D \oplus \{V\}$. Here replacing X by Y and subtracting the resulting equation from this equation, we get $\omega[X, Y] = h(X, fY) - h(Y, fX)$, which proves (A).

Next, from (34) and (36), we have

$$-f(\nabla_Z U) = A_{\omega U}Z - g(Z, U)V + \epsilon\eta(U)Z + Bh(Z, U), \tag{44}$$

for all $Z, U \in \Gamma(\bar{D})$. Then, similarly as above, we have

$$f[Z, U] = A_{\phi Z}U - A_{\phi U}Z, \tag{45}$$

which completes the proof of (B). ■

Theorem 3.6. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if, $Bh(X, \phi Y) = 0$, for any $X, Y \in D \oplus \{V\}$.*

Proof. Since $\overline{D} = \phi(L \perp S)$, therefore $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if

$$g(\nabla_X Y, \phi\xi) = g(\nabla_X Y, \phi W) = 0, \tag{46}$$

for any $X, Y \in \Gamma(D \oplus \{V\})$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(S)$.
Using (9) and (24), we have

$$g(\nabla_X Y, \phi\xi) = -\bar{g}(\overline{\nabla}_X \phi Y, \xi) = -\bar{g}(h^l(X, fY), \xi), \tag{47}$$

$$g(\nabla_X Y, \phi W) = -\bar{g}(\overline{\nabla}_X \phi Y, W) = -\bar{g}(h^s(X, fY), W). \tag{48}$$

Hence, from (47) and (48) the assertion follows. ■

Theorem 3.7. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the distribution $\overline{D}$ defines a totally geodesic foliation in M, if and only if, $A_N X$ has no component in $\phi S \perp \phi D_2$ and $A_{\omega Y}X$ has no component in $D_2 \perp D_0$, for any $X, Y \in \Gamma(\overline{D})$ and $N \in \Gamma(ltr(TM))$.*

Proof. We know that $\overline{D}$ defines a totally geodesic foliation in M, if and only if

$$g(\nabla_X Y, N) = g(\nabla_X Y, \phi N_1) = g(\nabla_X Y, V) = g(\nabla_X Y, \phi Z) = 0, \tag{49}$$

for $X, Y \in \Gamma(\overline{D})$, $N \in \Gamma(ltr(TM))$, $Z \in \Gamma(D_0)$ and $N_1 \in \Gamma(L)$. Using (9) and (11), we have

$$g(\nabla_X Y, N) = \bar{g}(\overline{\nabla}_X Y, N) = -\bar{g}(Y, \overline{\nabla}_X N) = g(Y, A_N X). \tag{50}$$

Using (9) , (10) and (24), we obtain

$$g(\nabla_X Y, \phi N_1) = -g(\phi\overline{\nabla}_X Y, N_1) = -g(\overline{\nabla}_X \omega Y, N_1) = g(A_{\omega Y} X, N_1), \tag{51}$$

and

$$g(\nabla_X Y, \phi Z) = -g(\phi\overline{\nabla}_X Y, Z) = -g(\overline{\nabla}_X \omega Y, Z) = g(A_{\omega Y} X, Z), \tag{52}$$

also

$$g(\nabla_X Y, V) = g(\overline{\nabla}_X Y, V) = -g(Y, \overline{\nabla}_X V) = g(Y, \phi X) = 0. \tag{53}$$

Thus from (50)-(53), the result follows. ■

Theorem 3.8. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the induced connection ∇ is a metric connection, if and only if*

$$A^*_{\phi\xi} X - \nabla^{*t}_X \phi\xi \in \Gamma(\phi D_2 \perp D_1), \quad \text{for} \quad \xi \in \Gamma(D_1),$$
$$\nabla^*_X \phi\xi + h^*(X, \phi\xi) \in \Gamma(\phi D_2 \perp D_1), \quad \text{for} \quad \xi \in \Gamma(D_2),$$
$$h(X, \phi\xi) \in \Gamma(L \perp S)^\perp \quad \text{and} \quad A^*_\xi X \in \Gamma(\bar{D} \perp D_0 \perp \phi D_2),$$

for $\xi \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM)$.

Proof. For any $X \in \Gamma(TM)$ and $\xi \in \Gamma Rad(TM)$, using (24), we have

$$\overline{\nabla}_X \phi\xi = \phi\overline{\nabla}_X \xi, \tag{54}$$

then using (17) and (19), we obtain

$$\nabla_X \xi + h(X, \xi) = -\phi(\nabla_X \phi\xi + h(X, \phi\xi)) - \epsilon g(A^*_\xi X, V)V. \tag{55}$$

Let $\xi \in \Gamma(D_1)$ then $\phi\xi \in \Gamma(D_1)$, again using (17) in (55), we obtain

$$\nabla_X \xi + h(X, \xi) = -\phi(-A^*_{\phi\xi} X + \nabla^{*t}_X \phi\xi + h(X, \phi\xi)) - \epsilon g(A^*_\xi X, V)V. \tag{56}$$

Equating tangential components of the above equation both sides, we get

$$\nabla_X \xi = f A^*_{\phi\xi} X - f \nabla^{*t}_X \phi\xi - Bh(X, \phi\xi) - \epsilon g(A^*_\xi X, V)V, \tag{57}$$

therefore $\nabla_X \xi \in \Gamma(RadTM)$ if and only if $Bh(X, \phi\xi) = 0$, $f A^*_{\phi\xi} X - f \nabla^{*t}_X \phi\xi \in \Gamma(RadTM)$ and $g(A^*_\xi X, V) = 0$ or if and only if

$$h(X, \phi\xi) \in \Gamma(L \perp S)^\perp, \quad A^*_{\phi\xi} X - \nabla^{*t}_X \phi\xi \in \Gamma(\phi D_2 \perp D_1), \tag{58}$$

and

$$A^*_\xi X \in \Gamma(\overline{D} \perp D_0 \perp \phi D_2). \tag{59}$$

Similarly, let $\xi \in \Gamma(D_2)$ then using (16) in (55) and then compare the tangential components of the resulting equation, we obtain

$$\nabla_X \xi = -f\nabla^*_X \phi\xi - fh^*(X, \phi\xi) - Bh(X, \phi\xi) - \epsilon g(A^*_\xi X, V)V, \tag{60}$$

therefore $\nabla_X \xi \in \Gamma(RadTM)$ if and only if $Bh(X, \phi\xi) = 0$, $f\nabla^*_X \phi\xi + fh^*(X, \phi\xi) \in \Gamma(RadTM)$ and $g(A^*_\xi X, V) = 0$ or if and only if

$$h(X, \phi\xi) \in \Gamma(L\perp S)^\perp, \quad \nabla^*_X \phi\xi + h^*(X, \phi\xi) \in \Gamma(\phi D_2 \perp D_1) \tag{61}$$

and

$$A^*_\xi X \in \Gamma(\overline{D}\perp D_0 \perp \phi D_2). \tag{62}$$

Hence from (58), (59), (61) and (62) the assertion follows. ■

4. Totally Contact Umbilical *GCR*-Lightlike Submanifolds

Definition 4.1. ([14]). If the second fundamental form h of a submanifold tangent to the characteristic vector field V, of an infinite Sasakian manifold $\overline{M}$ is of the form

$$h(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y, V) + \eta(Y)h(X, V), \tag{63}$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called a totally contact umbilical and totally contact geodesic if $\alpha = 0$.

The above definition also holds for a lightlike submanifold M. For a totally contact umbilical lightlike submanifold M, we have

$$h^l(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\alpha_L + \eta(X)h^l(Y, V) + \eta(Y)h^l(X, V), \tag{64}$$

$$h^s(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\alpha_S + \eta(X)h^s(Y, V) + \eta(Y)h^s(X, V), \tag{65}$$

where $\alpha_L \in \Gamma(ltr(TM)$ and $\alpha_S \in \Gamma(S(TM^\perp)$.

Lemma 4.2. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $\nabla_X X \in \Gamma(D \oplus \{V\})$, for any $X \in \Gamma(D \oplus \{V\})$.*

Proof. Since $\bar{D} = \phi(L\perp S)$ therefore $\nabla_X X \in \Gamma(D \oplus \{V\})$ if and only if

$$g(\nabla_X X, \phi\xi) = g(\nabla_X X, \phi W) = 0, \tag{66}$$

for any $\xi \in \Gamma(D_2)$ and $W \in \Gamma(S)$. Since M is a totally contact umbilical *GCR*-lightlike submanifold therefore using (7), (9), (24) and (64), we obtain

$$\begin{aligned} g(\nabla_X X, \phi\xi) &= \bar{g}(\bar{\nabla}_X X, \phi\xi) \\ &= -\bar{g}(\bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X, \xi) \\ &= -\bar{g}(\nabla_X \phi X + h^s(X, \phi X) + h^l(X, \phi X), \xi) \\ &\quad + \bar{g}(-g(X, X)V + \epsilon\eta(X)X, \xi) \\ &= -\bar{g}(h^l(X, \phi X), \xi) \\ &= -\bar{g}(X, \phi X)\bar{g}(\alpha_L, \xi) \end{aligned}$$

$$= 0. \tag{67}$$

Also

$$\begin{aligned} g(\nabla_X X, \phi W) &= \bar{g}(\bar{\nabla}_X X, \phi W) \\ &= -\bar{g}(\bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi)X, W) \\ &= -\bar{g}(\nabla_X \phi X + h^l(X, \phi X) + h^s(X, \phi X), W) \\ &\quad + \bar{g}(-g(X, X)V + \epsilon\eta(X)X, W) \\ &= -\bar{g}(h^s(X, \phi X), W) \\ &= -\bar{g}(X, \phi X)\bar{g}(\alpha_s, W) \\ &= 0. \end{aligned} \tag{68}$$

Hence using (67) and (68), the assertion follows. ■

Theorem 4.3. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $\alpha \in \Gamma(L \perp S)$.*

Proof. Using (33) for $X \in \Gamma(D_0)$, we obtain $h^s(X, fX) = \omega P_1 \nabla_X X + Ch^s(X, X)$, further using (65) we get $\alpha_S g(X, \phi X) = \omega P_1 \nabla_X X + g(X, X)C\alpha_S$. Using the above lemma, we get $g(X, X)C\alpha_S = 0$, then the non degeneracy of D_0 implies $C\alpha_S = 0$. Hence $\alpha_S \in \Gamma(S)$.

Similarly by using (32) and (64) we can prove $\alpha_L \in \Gamma(L)$. Hence $\alpha \in \Gamma(L \perp S)$. ■

Remark 4.4. Since $\alpha \in \Gamma(L \perp S)$ therefore for $X \in D_0$, using (63), we have $h(X, X) = g(X, X)\alpha$, this implies that $h(X, X) \in \Gamma(L \perp S)$.

Theorem 4.5. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $\alpha_L = 0$.*

Proof. Since M is a totally contact umbilical GCR-lightlike submanifold then, by direct calculations, using (9), (10) and (24) and then taking tangential parts of the resulting equation, we obtain

$$A_{\phi Z} Z + f\nabla_Z Z + Bh^l(Z, Z) + Bh^s(Z, Z) = g(Z, Z)V, \tag{69}$$

where $Z \in \phi(S)$. Hence for $\xi \in \Gamma(D_2)$, we obtain

$$\bar{g}(A_{\phi Z} Z, \phi\xi) + \bar{g}(h^l(Z, Z), \xi) = 0. \tag{70}$$

Using (13), we get

$$\bar{g}(h^s(Z, \phi\xi), \phi Z) + \bar{g}(h^l(Z, Z), \xi) = 0. \tag{71}$$

Thus, from (64) and (65) we derive $g(Z, Z)\bar{g}(\alpha_L, \xi) = 0$ then the non degeneracy of ϕS implies that $\alpha_L = 0$, which completes the proof. ■

Lemma 4.6. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $\nabla_X \phi X = \phi \nabla_X X$, for any $X \in \Gamma(D_0)$.*

Proof. For any $X \in \Gamma(D_0)$ using (35) and (37), we have $\omega \nabla_X X = h(X, fX) - Ch(X,X)$. Since M is totally contact umbilical therefore using (65), we have $\omega \nabla_X X = g(X, \phi X)\alpha_S - Ch(X,X)$, then using Remark 4.4, we get $\omega \nabla_X X = 0$. Hence

$$\nabla_X X \in \Gamma(D). \tag{72}$$

Let $X, Y \in \Gamma(D_0)$ then using (24), we have

$$g(\nabla_X \phi X, Y) = \bar{g}(\bar{\nabla}_X \phi X, Y) = \bar{g}(\phi \bar{\nabla}_X X, Y), \tag{73}$$

using (18) and (19), we further have

$$g(\nabla_X \phi X, Y) = -\bar{g}(\bar{\nabla}_X X, \phi Y) = -g(\nabla_X X, \phi Y) = g(\phi \nabla_X X, Y), \tag{74}$$

this implies $g(\nabla_X \phi X - \phi \nabla_X X, Y) = 0$, then the non degeneracy of the distribution D_0 gives the result. ■

Theorem 4.7. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $\alpha_S = 0$.*

Proof. For $W \in \Gamma(S(TM^{\perp}))$ and $X \in \Gamma(D_0)$, using (24), (63) and Lemma 4.6 we have

$$\begin{aligned}
\bar{g}(\phi \bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X \phi X + g(X,X)V - \epsilon \eta(X) X, \phi W) \\
&= \bar{g}(\bar{\nabla}_X \phi X, \phi W) \\
&= \bar{g}(\nabla_X \phi X, \phi W) + \bar{g}(h(X, \phi X), \phi W) \\
&= \bar{g}(\phi \nabla_X X, \phi W) + \bar{g}(X, \phi X) g(\alpha, \phi W) \\
&= \bar{g}(\nabla_X X, W) \\
&= 0.
\end{aligned} \tag{75}$$

Also using (65), we have

$$\begin{aligned}
\bar{g}(\phi \bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X X, W) - \eta(\phi W)\eta(\phi \bar{\nabla}_X X) \\
&= \bar{g}(\nabla_X X + h^s(X,X) + h^l(X,X), W) \\
&= \bar{g}(\nabla_X X, W) + \bar{g}(h^s(X,X), W) \\
&= g(X,X) g(\alpha_s, W).
\end{aligned} \tag{76}$$

Therefore using (75) and (76), we get $g(X,X)g(\alpha_S, W) = 0$, then the non degeneracy of D_0 and $S(TM^{\perp})$ implies that $\alpha_S = 0$. ■

Theorem 4.8. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then M is a totally contact geodesic GCR-lightlike submanifold.*

Proof. From Theorem 4.5 and Theorem 4.7 the result follows. ■

Theorem 4.9. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that $\bar{\nabla}_X V \in \Gamma(TM)$ then the induced connection ∇ is a metric connection on M.*

Proof. Using Theorem 4.5, we have $\alpha_L = 0$. Since $\bar{\nabla}_X V \in \Gamma(TM)$ this implies $h^l(X, V) = 0$ therefore from (64) we get

$$h^l = 0, \tag{77}$$

then using Theorem 2.2 in [7, page 159], the induced connection ∇ becomes a metric connection on M. ■

Theorem 4.10. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that $\bar{\nabla}_X V \in \Gamma(TM)$ then M is a totally geodesic GCR-lightlike submanifold.*

Proof. Using Theorem 4.7, we have $\alpha_S = 0$. Since $\bar{\nabla}_X V \in \Gamma(TM)$, this implies $h^s(X, V) = 0$. Therefore from (65), we obtain

$$h^s = 0. \tag{78}$$

Then using (77) and (78), the assertion follows. ■

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