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On SS-semipermutable Subgroups of Finite Groups ⋆

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Abstract. We introduce a new subgroup embedding property in a finite group called SS-semipermutability and investigate the influence of SS-semipermutable subgroups on the structure of finite groups. Our results unify and generalize some earlier results.

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Key words: S-semipermutable; SS-semipermutable; p-nilpotent; the generalized Fitting subgroup.

1. Introduction

Throughout the paper, all groups are finite. We use conventional notions and notation, as in Huppert [2]. G always denotes a group, $|G|$ is the order of G, $O_p(G)$ is the maximal normal p-subgroup of G, $O^p(G) = \langle g \in G \mid p \nmid o(g) \rangle$ and $\Phi(G)$ is the Frattini subgroup of G. Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation, provided that (i) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (ii) if $G/M \in \mathcal{F}$ and $G/N \in \mathcal{F}$, then $G/(M \cap N) \in \mathcal{F}$ for any normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation ([2, p. 713, Satz 8.6]).

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Two subgroups H and K of a group G are said to be permutable if $HK = KH$. A subgroup H of G is said to be S-permutable (or S-quasinormal, π-quasinormal) in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel [4] and was investigated by many authors. In 1996, Y. Wang [10] introduced c-normal subgroup which in fact is a special supplemented subgroup. A subgroup H of G is called c-normal in G if there is a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. Recently, A. N. Skiba in [8] introduced the following concept, which covers both S-permutability and c-normality:

Definition 1.1. Let H be a subgroup of G. H is called weakly S-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S-permutable in G.

From Q. Zhang and L. Wang [13], we know that a subgroup H of G is said to be S-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$. Here, we give a new concept which covers properly both S-semipermutability and Skiba's weakly S-permutability.

Definition 1.2. Let H be a subgroup of G. H is called SS-semipermutable in G if there exist a subnormal subgroup T of G and an S-semipermutable subgroup H_s of G contained in H such that $G = HT$ and $H \cap T \leq H_s$.

Remark 1.3. It is easy to see that weakly S-permutability (or S-semipermutability) implies SS-semipermutability. The converse does not hold in general.

Example 1.4. (a) Let $G = A_5$, the alternative group of degree 5. Then A_4 is SS-semipermutable in G, but not weakly S-permutable in G.

(b) Let $G = S_4$, the symmetric group of degree 4. Take $H = \langle (12) \rangle$. Then H is SS-semipermutable in G, but not S-semipermutable in G.

In the literature, authors usually put the assumptions on either the minimal subgroups (and cyclic subgroups of order 4 when $p = 2$) or the maximal subgroups of some kinds of subgroups of G when investigating the structure of G, such as [5, 9, 10, 13]. In the nice paper [8], Skiba provided a unified viewpoint for a series of similar problems.

Theorem 1.5. ([8], Theorem 1.3) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly S-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathcal{F}$.*

In the present article, Theorem 1.5 is extended as follows.

Theorem 1.6. (i.e., Theorem 3.5) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is SS-semipermutable, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathcal{F}$.*

The following result relating p-nilpotency of a group is the main step in the proof of Theorem 1.6.

Theorem 1.7. (i.e., Theorem 3.2) *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is SS-semipermutable in G, then G is p-nilpotent.*

2. Preliminaries

Lemma 2.1. *Suppose that H is an S-semipermutable subgroup of a group G and N is a normal subgroup of G. Then*

(a) *H is S-semipermutable in K whenever $H \leq K \leq G$;*

(b) *If H is a p-group for some prime $p \in \pi(G)$, then HN/N is S-semipermutable in G/N;*

(c) *If $H \leq O_p(G)$, then H is S-permutable in G.*

Proof. (a) is [13, Property 1], (b) is [13, Property 2], and (c) is [13, Lemma 3]. ■

Lemma 2.2. *Let U be an SS-semipermutable subgroup of a group G and N a normal subgroup of G. Then*

(a) *If $U \leq H \leq G$, then U is SS-semipermutable in H;*

(b) *Suppose that U is a p-group for some prime p. If $N \leq U$, then U/N is SS-semipermutable in G/N;*

(c) *Suppose U is a p-group for some prime p and N is a p'-subgroup, then UN/N is SS-semipermutable in G/N;*

(d) *Suppose U is a p-group for some prime p and U is not S-semipermutable in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = UM$;*

(e) *If $U \leq O_p(G)$ for some prime p, then U is weakly S-permutable in G.*

Proof. By the hypotheses, there are a subnormal subgroup K of G and an S-semipermutable subgroup U_s of G contained in U such that $G = UK$ and $U \cap K \leq U_s$.

(a) $H = H \cap UK = U(H \cap K)$ and $U \cap (H \cap K) = U \cap K \leq U_s$. By Lemma 2.1(a), U_s is S-semipermutable in H. Obviously, $H \cap K$ is subnormal in H. Hence U is SS-semipermutable in H.

(b) $G/N = UK/N = U/N \cdot NK/N$ and $(U/N) \cap (KN/N) = (U \cap KN)/N = (U \cap K)N/N \le U_sN/N$. By Lemma 2.1(b), U_sN/N is S-semipermutable in G/N. Obviously, KN/N is subnormal in G/N. Hence U/N is SS-semipermutable in G/N.

(c) Since $|G|_{p'} = |NK|_{p'} = |K|_{p'}$, we have that $|N \cap K|_{p'} = |N|_{p'} = |N|$ and so $N \le K$. It is easy to see that $G/N = UN/N \cdot KN/N = UN/N \cdot K/N$ and $(UN/N) \cap (K/N) = (UN \cap K)/N = (U \cap K)N/N \le U_sN/N$. By Lemma 2.1(b), U_sN/N is S-semipermutable in G/N. Obviously, KN/N is subnormal in G/N. Hence UN/N is SS-semipermutable in G/N.

(d) If $K = G$, then $U = U \cap K \le U_s \le U$, therefore, $U = U_s$ is S-semipermutable in G, contrary to the hypotheses. Consequently, K is a proper subgroup of G. Hence, G has a proper normal subgroup L such that $K \le L$. Since G/L is a p-group, G has a normal maximal subgroup M such that $|G : M| = p$ and $G = MU$.

(e) We can get that by Lemma 2.1(c). ■

Lemma 2.3. ([8], Lemma 2.11) *Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly S-permutable in G. Then some maximal subgroup of N is normal in G.*

Lemma 2.4. *Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is SS-semipermutable in G. Then some maximal subgroup of N is normal in G.*

Proof. By Lemmas 2.3 and 2.2(e). ■

Lemma 2.5. ([2], III, 5.2 and IV, 5.4) *Suppose G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then*

(a) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$;*

(b) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;*

(c) *The exponent of P is p or 4.*

Lemma 2.6. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every subgroup of prime order or order 4 (when P is a nonabelian 2-group) of P is SS-semipermutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. By Lemma 2.2(a), it is easy to see that G is a minimal non-p-nilpotent group. By Lemma 2.5, $G = P \rtimes Q$. Let $x \in P$. Then the order of x is p or 4. By the hypothesis, $\langle x \rangle$ is SS-semipermutable in G. Then there are a subnormal subgroup T of G and an S-semipermutable subgroup $\langle x \rangle_s$ of G contained in $\langle x \rangle$ such that $G = \langle x \rangle T$ and $\langle x \rangle \cap T \le \langle x \rangle_s$. Hence $P = P \cap G = P \cap \langle x \rangle T =$

$\langle x\rangle(P\cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P\cap T)\Phi(P)/\Phi(P)\trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, $P\cap T\leq \Phi(P)$ or $P=(P\cap T)\Phi(P)=P\cap T$. If $P\cap T\leq\Phi(P)$, then $\langle x\rangle=P\trianglelefteq G$. It follows that G is p-nilpotent, a contradiction. If $P=P\cap T$, then $T=G$ and so $\langle x\rangle=\langle x\rangle_s$ is S-semipermutable in G. We have $\langle x\rangle Q$ is a proper subgroup of G and so $\langle x\rangle Q=\langle x\rangle\times Q$. It follows that $G=P\times Q$, a contradiction. ∎

Lemma 2.7. ([1], A, 1.2) *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(a) $U\cap VW=(U\cap V)(U\cap W)$;

(b) $UV\cap UW=U(V\cap W)$.

Lemma 2.8. *Let G be a group, P a p-subgroup of G and Q a q-subgroup of G, where q, p are different primes dividing $|G|$. If L is a subnormal subgroup of G and $PQ=QP$, then $PQ\cap L=(P\cap L)(Q\cap L)$.*

Lemma 2.9. ([2], VI, 4.10) *Assume that A and B are two subgroups of a group G and $G\neq AB$. If $AB^g=B^gA$ holds for any $g\in G$, then either A or B is contained in a nontrivial normal subgroup of G.*

Lemma 2.10. ([3], X, 13) *Let G be a group and $N\trianglelefteq G$.*

(a) *If $N\trianglelefteq G$, then $F^*(N)\leq F^*(G)$;*

(b) *If $G\neq 1$, then $F^*(G)\neq 1$. In fact,*

$$F^*(G)/F(G)=Soc(F(G)C_G(F(G))/F(G));$$

(c) *$F^*(F^*(G))=F^*(G)\geq F(G)$. If $F^*(G)$ is solvable, then $F^*(G)=F(G)$.*

Lemma 2.11. ([11], Lemma 2.8) *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G=PM$, where p is a prime. Then $P\cap M$ is a normal subgroup of G.*

3. Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is SS-semipermutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G has a unique minimal normal subgroup N. Moreover G/N is p-nilpotent, and $\Phi(G)=1$.

Let N be a minimal normal subgroup of G. We shall prove that G/N satisfies the hypothesis of the theorem. Let M/N be a maximal subgroup of PN/N. It is easy to see $M = P_1N$ for some maximal subgroup P_1 of P. It follows that $P_1 \cap N = P \cap N$ is a Sylow p-subgroup of N. By the hypotheses, P_1 is SS-semipermutable in G. Then there are a subnormal subgroup T of G and an S-semipermutable subgroup $(P_1)_s$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_s$. Thus $G/N = M/N \cdot TN/N = P_1N/N \cdot TN/N$. It is easy to see that TN/N is subnormal in G/N. Since $(|N : P_1 \cap N|, |N : T \cap N|) = 1$, we have $(P_1 \cap N)(T \cap N) = N = N \cap G = N \cap P_1T$. By Lemma 2.7, $(P_1N) \cap (TN) = (P_1 \cap T)N$. It follows that $(P_1N/N) \cap (TN/N) = (P_1N \cap TN)/N = (P_1 \cap T)N/N \leq (P_1)_sN/N$. It follows from Lemma 2.1(b) that $(P_1)_sN/N$ is S-semipermutable in G/N. Hence M/N is SS-semipermutable in G/N. Therefore, G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ follow because the class of all p-nilpotent groups is a saturated formation.

(2) $O_{p'}(G) = 1$. If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by Step (1). Since $G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$ is p-nilpotent, we have G is p-nilpotent, a contradiction.

(3) $O_p(G) = 1$. If $O_p(G) \neq 1$, then $N \leq O_p(G)$. Since $N \nsubseteq \Phi(G) = 1$ by Step (1), G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Obviously, $G = O_p(G)M$ and so $O_p(G) \cap M$ is normal in G by Lemma 2.11. The uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Furthermore $P \cap M < P$. Thus there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Hence $P = NP_1$. By the hypothesis, P_1 is SS-semipermutable in G. Then there are a subnormal subgroup T of G and an S-semipermutable subgroup $(P_1)_s$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_s$. Since $|G : T|$ is a power of p and $T \lhd \lhd G$, we have $O^p(G) \leq T$. Since N is the unique minimal normal subgroup of G, $N \leq O^p(G)$. It follows that $P_1 \cap N = (P_1)_s \cap N$. For any Sylow q-subgroup G_q of G $(p \neq q)$, $(P_1)_s \cap N = (P_1)_sG_q \cap N \trianglelefteq (P_1)_sG_q$. Obviously, $P_1 \cap N \trianglelefteq P$. Therefore $P_1 \cap N$ is normal in G. By the minimality of N, we have $P_1 \cap N = N$ or $P_1 \cap N = 1$. If $P_1 \cap N = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. Thus $P_1 \cap N = 1$. Since $P_1 \cap N$ is a maximal subgroup of N, we have that N is of order p. Then G is p-nilpotent by Step (1), a contradiction.

(4) The final contradiction. By Steps (2) and (3), we have G is not solvable. Let L be a minimal subnormal subgroup of G. Then L is a non-abelian simple group. Let P_1 be a maximal subgroup of P, then there are a subnormal subgroup T of G and an S-semipermutable subgroup $(P_1)_s$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_s$. Thus for any Sylow q-subgroup G_q of G, we have $(P_1)_sG_q = G_q(P_1)_s (p \neq q)$. For any $x \in L$, $(P_1)_sG_q^x \cap L = ((P_1)_s \cap L)(G_q^x \cap L) = ((P_1)_s \cap L)(G_q \cap L)^x$ by Lemma 2.8. Obviously, $L \neq (G_q \cap L)^x((P_1)_s \cap L)$. By Lemma 2.9, we have that L is not a simple group, a contradiction. ■

Theorem 3.2. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is SS-semipermutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several Steps.

(1) $O_{p'}(G) = 1$. If $O_{p'}(G) \neq 1$, Lemma 2.2(c) guarantees that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

(2) $|D| > p$. By Lemma 2.6.

(3) $|P : D| > p$. By Theorem 3.1.

(4) P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is S-semipermutable in G.

Assume that $H \leq P$ such that $|H| = |D|$ and H is not S-semipermutable in G. By Lemma 2.2(d), we may assume G has a normal subgroup M such that $|G : M| = p$ and $G = HM$. Since $|P : D| > p$ by Step (3), M satisfies the hypotheses of the theorem. The choice of G yields that M is p-nilpotent. It is easy to see that G is p-nilpotent, contrary to the choice of G.

(5) If $N \leq P$ and N is minimal normal in G, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 2.4, N has a maximal subgroup which is normal in G, contrary to the minimality of N.

(6) Suppose that $N \leq P$ and N is minimal normal in G. Then G/N is p-nilpotent.

If $|N| < |D|$, G/N satisfies the hypotheses of the theorem by Lemma 2.1(b). Thus G/N is p-nilpotent by the minimal choice of G. So we may suppose that $|N| = |D|$ by Step (5). We will show that every cyclic subgroup of P/N of order p or order 4(when P/N is a non-abelian 2-group) is S-semipermutable in G/N. Let $K \leq P$ and $|K/N| = p$. By Step (2), N is non-cyclic, so are all subgroups containing N. Hence there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course, $|N| = |D| = |L|$. Since L is S-semipermutable in G by the hypotheses, $K/N = LN/N$ is S-semipermutable in G/N by Lemma 2.1(b). If $p = 2$ and P/N is non-abelian, take a cyclic subgroup X/N of P/N of order 4. Let K/N be maximal in X/N. Then K is maximal in X and $|K/N| = 2$. Since X is non-cyclic and X/N is cyclic, there is a maximal subgroup L of X such that N is not contained in L. Thus $X = LN$ and $|L| = |K| = 2|D|$. By the hypotheses, L is S-semipermutable in G. By Lemma 2.1(b), $X/N = LN/N$ is S-semipermutable in G/N. Hence G/N satisfies the hypotheses. By the minimal choice of G, G/N is p-nilpotent.

(7) $O_p(G) = 1$. Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By Step (6), G/N is p-nilpotent. It is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. Hence $O_p(G)$ is an elementary abelian p-group. On the other hand, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is p-nilpotent. Then G can be written as $G = N(M \cap P)M_{p'}$, where $M_{p'}$ is the normal p-

complement of M. Pick a maximal subgroup S of $M_p = P \cap M$. Then $NSM_{p'}$ is a subgroup of G with index p. Since p is the minimal prime in $\pi(G)$, we know that $NSM_{p'}$ is normal in G. Now by Step (3) and the induction, we have $NSM_{p'}$ is p-nilpotent. Therefore, G is p-nilpotent, a contradiction.

(8) The final contradiction. Let H be a subgroup of P with order $|D|$, and Q be a Sylow q-subgroup of G, where $q \neq p$. Let x be any element of G. Then by the hypotheses $HQ^x = Q^xH$. If $G \neq HQ$, then G is not simple by Lemma 2.9. Take a minimal normal subgroup L of G. Then $L < G$. If $|L|_p > |D|$, then L is p-nilpotent by the minimal choice of G. Let $L_{p'}$ be the normal p-complement of L. Since $L_{p'}$ *char* $L \trianglelefteq G$, we have $L_{p'} \trianglelefteq G$ and so $L_{p'} \leq O_{p'}(G) = 1$ by Step (1). It follows that L is a p-group. Then $L \leq O_p(G) = 1$ by Step (7), a contradiction. If $|L|_p \leq |D|$, take $P_* \geq L \cap P$ such that $|P_*| = p|D|$. Hence P_* is a Sylow p-subgroup of P_*L. Since every maximal subgroup of P_* is of order $|D|$, every maximal subgroup of P_* is S-semipermutable in G by hypotheses, thus in P_*L by Lemma 2.1. Now applying Theorem 3.1, we get P_*L is p-nilpotent. Therefore, L is p-nilpotent, we have the same contradiction as above. Now we assume that $G = HQ$. Then G is solvable by Burnside's theorem, contrary to Steps (1) and (7) too. ∎

Corollary 3.3. *Suppose that G is a group. If every non-cyclic Sylow subgroup P of G has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is SS-semipermutable in G, then G has a Sylow tower of supersolvable type.*

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is SS-semipermutable in G. Then $G \in \mathcal{F}$.*

Proof. Since P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is SS-semipermutable in G by hypotheses, thus in E by Lemma 2.2(a). Applying Corollary 3.3, we conclude that E has a Sylow tower of supersolvable type. Let q be the largest prime divisor of $|E|$ and Q a Sylow q-subgroup of E. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathcal{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \leq O_q(G)$, H is weakly S-permutable in G by Lemma 2.2(e). By [8, Theorem 1.3], we get $G \in \mathcal{F}$. ∎

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with* *order* $2|D|$ (*if* P *is a non-abelian* 2*-group and* $|P : D| > 2$) *is* SS*-semipermutable in* G*. Then* $G \in \mathcal{F}$*.*

Proof. We distinguish two cases:

Case 1. $\mathcal{F} = \mathcal{U}$. Let G be a minimal counterexample.

(1) Every proper normal subgroup N of G containing $F^*(E)$ (if it exists) is supersolvable.

If N is a proper normal subgroup of G containing $F^*(E)$, then $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 2.10, $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup P of $F^*(E \cap N) = F^*(E)$, P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is SS-semipermutable in G by hypotheses, thus in N by Lemma 2.2(a). So N and $N \cap H$ satisfy the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable.

(2) $E = G$. If $E < G$, then $E \in \mathcal{U}$ by Step (1). Hence $F^*(E) = F(E)$ by Lemma 2.10. It follows that every Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 2.2(e), every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is weakly S-permutable in G. Applying Theorem A for the special case $\mathcal{F} = \mathcal{U}$, $G \in \mathcal{U}$, a contradiction.

(3) $F^*(G) = F(G) < G$. If $F^*(G) = G$, then $G \in \mathcal{U}$ by Theorem 3.4, contrary to the choice of G. So $F^*(G) < G$. By Step (1), $F^*(G) \in \mathcal{U}$ and $F^*(G) = F(G)$ by Lemma 2.10.

(4) The final contradiction. Since $F^*(G) = F(G)$, each non-cyclic Sylow subgroup of $F^*(G)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is weakly S-permutable in G by Lemma 2.2(e). Applying Theorem A, $G \in \mathcal{U}$, a contradiction.

Case 2. $\mathcal{F} \neq \mathcal{U}$. By hypotheses, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is SS-semipermutable in G, thus in E by Lemma 2.2(a). Applying Case 1, $E \in \mathcal{U}$. Then $F^*(E) = F(E)$ by Lemma 2.10. It follows that each Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 2.2(e), each non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is weakly S-permutable in G. Applying Theorem A, $G \in \mathcal{F}$. ■

Corollary 3.6. *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$*. Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$*. If all maximal subgroups of any Sylow subgroup of* $F^*(H)$ *are either c-normal* ([12], *Theorem* 3.1) *or s-quasinormal* ([7], *Theorem* 3.4) *in* G*, then* $G \in \mathcal{F}$*.*

Corollary 3.7. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all cyclic subgroups of any Sylow subgroup of $F^*(H)$ of prime order or order 4 are either c-normal* ([12], *Theorem* 3.2) *or s-quasinormal* ([6], *Theorem* 3.3) *in G, then $G \in \mathcal{F}$.*

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