Vietnam Journal of Mathematics **39**:2(2011) 135-144

Vietnam Journal of MATHEMATICS

Some Sharp Modified Simpson Type Inequalities and Applications

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Received November 18, 2010
Revised May 9, 2011

Abstract. Some sharp modified Simpson type inequalities have been proven. Applications in numerical integration are also considered.

2000 Mathematics Subject Classification: 26D15.

Key words: Sharp inequality; Modified Simpson's rule; Modified Simpson type inequality; Numerical integration.

1. Introduction

In [5], N. Ujević and A. J. Roberts have derived by finite differences a so called modified Simpson's rule as follows

$$\int_a^b f(x)\,dx = \frac{b-a}{30}[7f(a)+16f(\frac{a+b}{2})+7f(b)] - \frac{(b-a)^2}{60}[f'(b)-f'(a)] + R(f), \quad (1)$$

where $f \in C^{\infty}[a,b]$, and the error term is, to a leading order estimate,

$$R(f) \approx \frac{(b-a)^6}{604800}[f^{(5)}(b) - f^{(5)}(a)]. \quad (2)$$

Here we give a revised version for (2) since the expression in [3]-[5] contained a misprint.

A simple illustrative example has been given in [5] to show that the modified Simpson's rule (1) may be very effective.

On the other hand, for some integer $n > 1$, it is not difficult to find the identity

$$\begin{aligned}
&(-1)^n \int_a^b S_n(x) f^{(n)}(x)\,dx \\
&= \int_a^b f(x)\,dx - \frac{b-a}{30}[7f(a) + 16f(\frac{a+b}{2}) + 7f(b)] + \frac{(b-a)^2}{60}[f'(b) - f'(a)] \\
&\quad - \sum_{k=3}^{[\frac{n-1}{2}]} \frac{(k-1)(k-2)(b-a)^{2k+1}}{15(2k+1)!2^{2k-2}} f^{(2k)}(\frac{a+b}{2}),
\end{aligned} \tag{3}$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$, $f : [a,b] \to \mathbb{R}$ with $f^{(n-1)}$ absolutely continuous on $[a,b]$ and $S_n(x)$ is the kernel given by

$$S_n(x) = \begin{cases} \dfrac{(x-a)^n}{n!} - \dfrac{7(b-a)(x-a)^{n-1}}{30(n-1)!} + \dfrac{(b-a)^2(x-a)^{n-2}}{60(n-2)!} & \text{if } x \in \left[a, \dfrac{a+b}{2}\right], \\ \dfrac{(x-b)^n}{n!} + \dfrac{7(b-a)(x-b)^{n-1}}{30(n-1)!} + \dfrac{(b-a)^2(x-b)^{n-2}}{60(n-2)!} & \text{if } x \in \left(\dfrac{a+b}{2}, b\right]. \end{cases} \tag{4}$$

From [4] and [5], we see the following theorem holds.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n > 1$ and there exist constants γ_n, $\Gamma_n \in \mathbb{R}$ such that $\gamma_n \le f^{(n)}(x) \le \Gamma_n$ for a.e. $x \in [a,b]$. Then*

$$\begin{aligned}
&\Big| \int_a^b f(x)\,dx - \frac{b-a}{30}[7f(a) + 16f(\frac{a+b}{2}) + 7f(b)] + \frac{(b-a)^2}{60}[f'(b) - f'(a)] \\
&\qquad\qquad - \sum_{k=3}^{[\frac{n-1}{2}]} \frac{(k-1)(k-2)(b-a)^{2k+1}}{15(2k+1)!2^{2k-2}} f^{(2k)}(\frac{a+b}{2}) \Big| \\
&\le \frac{\Gamma_n - \gamma_n}{2} \times \begin{cases} \dfrac{19\sqrt{19}(b-a)^3}{10125}, & n = 2, \\ \dfrac{253(b-a)^4}{360000}, & n = 3, \\ \dfrac{(b-a)^5}{14580}, & n = 4, \\ \dfrac{(n-2)(n-4)(b-a)^{n+1}}{15(n+1)!2^n}, & n \ge 5 \text{ and odd.} \end{cases}
\end{aligned} \tag{5}$$

Remark 1.2. It is not difficult to find that the inequality (5) is sharp in the sense that we can choose a suitable $f : [a,b] \to \mathbb{R}$ to attain the equality in (5).

Indeed, for $n = 2$, we may find a function f such that f' is absolutely continuous on $[a, b]$ as

$$f'(x) = \begin{cases} \Gamma_2[x - \frac{(23+\sqrt{19})a+(7-\sqrt{19})b}{30}], & a \le x < \frac{(23+\sqrt{19})a+(7-\sqrt{19})b}{30}, \\ \gamma_2[x - \frac{(23+\sqrt{19})a+(7-\sqrt{19})b}{30}], & \\ & \frac{(23+\sqrt{19})a+(7-\sqrt{19})b}{30} \le x < \frac{(23-\sqrt{19})a+(7+\sqrt{19})b}{30}, \\ \Gamma_2[x - \frac{(23-\sqrt{19})a+(7+\sqrt{19})b}{30}] + \frac{\sqrt{19}(b-a)}{15}\gamma_2, & \\ & \frac{(23-\sqrt{19})a+(7+\sqrt{19})b}{30} \le x < \frac{(7+\sqrt{19})a+(23-\sqrt{19})b}{30}, \\ \gamma_2[x - \frac{(7+3\sqrt{19})a+(23-3\sqrt{19})b}{30}] + \frac{(8-\sqrt{19})(b-a)}{15}\Gamma_2, & \\ & \frac{(7+\sqrt{19})a+(23-\sqrt{19})b}{30} \le x < \frac{(7-\sqrt{19})a+(23+\sqrt{19})b}{30}, \\ \Gamma_2[x - \frac{(23-3\sqrt{19})a+(7+3\sqrt{19})b}{30}] + \frac{2\sqrt{19}(b-a)}{15}\gamma_2, & \frac{(7-\sqrt{19})a+(23+\sqrt{19})b}{30} \le x \le b, \end{cases}$$

and it follows that

$$f''(x) = \begin{cases} \Gamma_2, & a < x < \frac{(23+\sqrt{19})a+(7-\sqrt{19})b}{30}, \\ \gamma_2, & \frac{(23+\sqrt{19})a+(7-\sqrt{19})b}{30} < x < \frac{(23-\sqrt{19})a+(7+\sqrt{19})b}{30}, \\ \Gamma_2, & \frac{(23-\sqrt{19})a+(7+\sqrt{19})b}{30} < x < \frac{(7+\sqrt{19})a+(23-\sqrt{19})b}{30}, \\ \gamma_2, & \frac{(7+\sqrt{19})a+(23-\sqrt{19})b}{30} < x < \frac{(7-\sqrt{19})a+(23+\sqrt{19})b}{30}, \\ \Gamma_2, & \frac{(7-\sqrt{19})a+(23+\sqrt{19})b}{30} < x < b. \end{cases}$$

For $n = 3$, we may find a function f such that f'' is absolutely continuous on $[a, b]$ as

$$f''(x) = \begin{cases} \Gamma_3(x - \frac{4a+b}{5}), & a \le x < \frac{4a+b}{5}, \\ \gamma_3(x - \frac{4a+b}{5}), & \frac{4a+b}{5} \le x < \frac{a+b}{2}, \\ \Gamma_3(x - \frac{a+b}{2}) + \frac{3(b-a)}{10}\gamma_3, & \frac{a+b}{2} \le x < \frac{a+4b}{5}, \\ \gamma_3(x - \frac{a+b}{2}) + \frac{3(b-a)}{10}\Gamma_3, & \frac{a+4b}{5} \le x \le b, \end{cases}$$

and it follows that

$$f'''(x) = \begin{cases} \Gamma_3, & a < x < \frac{4a+b}{5}, \\ \gamma_3, & \frac{4a+b}{5} < x < \frac{a+b}{2}, \\ \Gamma_3, & \frac{a+b}{2} < x < \frac{a+4b}{5}, \\ \gamma_3, & \frac{a+4b}{5} < x < b. \end{cases}$$

For $n = 4$, we may find a function f such that f''' is absolutely continuous on $[a, b]$ as

$$f'''(x) = \begin{cases} \Gamma_4(x - \frac{2a+b}{3}), & a \le x < \frac{2a+b}{3}, \\ \gamma_4(x - \frac{2a+b}{3}), & \frac{2a+b}{3} \le x < \frac{a+2b}{3}, \\ \Gamma_4(x - \frac{a+2b}{3}) + \frac{b-a}{3}\gamma_4, & \frac{a+2b}{3} \le x \le b, \end{cases}$$

and it follows that

$$f^{(4)}(x) = \begin{cases} \Gamma_4, & a < x < \frac{2a+b}{3}, \\ \gamma_4, & \frac{2a+b}{3} < x < \frac{a+2b}{3}, \\ \Gamma_4, & \frac{a+2b}{3} < x < b. \end{cases}$$

For $n \ge 5$ and odd, we may find a function f such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ as

$$f^{(n-1)}(x) = \begin{cases} \Gamma_n(x - \frac{a+b}{2}), & a \le x < \frac{a+b}{2}, \\ \gamma_n(x - \frac{a+b}{2}), & \frac{a+b}{2} \le x \le b, \end{cases}$$

and it follows that

$$f^{(n)}(x) = \begin{cases} \Gamma_n, & a < x < \frac{a+b}{2}, \\ \gamma_n, & \frac{a+b}{2} < x < b. \end{cases}$$

Motivated by [1] and [2], in this paper, we will further derive some sharp modified Simpson type inequalities which are valid also for a larger class of functions, i.e., for the functions f for which $f^{(n)}$ is unbounded on (a, b) but $f^{(n)} \in L_2[a, b]$. Applications in numerical integration are also considered.

2. The Results

By elementary calculus, for some $n > 1$, it is not difficult to get

$$\int_a^b S_n(x)dx = \begin{cases} 0, & n \text{ odd}, \\ \frac{(n-2)(n-4)(b-a)^{n+1}}{15(n+1)!2^n}, & n \text{ even}, \end{cases} \tag{6}$$

and

$$\int_a^b S_n^2(x)dx = \frac{(4n^6 - 64n^5 + 403n^4 - 1260n^3 + 2021n^2 - 1500n + 360)(b-a)^{2n+1}}{225(2n-3)(4n^2-1)(n!)^2 2^{2n}}. \tag{7}$$

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ for some $n > 1$ and $f^{(n)} \in L_2[a, b]$ where n is an odd integer. Then we have*

$$\Big|\int_a^b f(x)dx - \frac{b-a}{30}[7f(a) + 16f(\frac{a+b}{2}) + 7f(b)] + \frac{(b-a)^2}{60}[f'(b) - f'(a)] - \sum_{k=3}^{[\frac{n-1}{2}]} \frac{(k-1)(k-2)(b-a)^{2k+1}}{15(2k+1)!2^{2k-2}} f^{(2k)}(\frac{a+b}{2})\Big| \le \frac{1}{15}\frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{4n^6 - 64n^5 + 403n^4 - 1260n^3 + 2021n^2 - 1500n + 360}{(2n-3)(4n^2-1)}}\sqrt{\sigma(f^{(n)})}, \tag{8}$$

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \frac{1}{b-a}\Big(\int_a^b f(t)\,dt\Big)^2 \tag{9}$$

and

$$\|f\|_2 := \Big[\int_a^b f^2(t)\,dt\Big]^{\frac{1}{2}}. \tag{10}$$

Inequality (8) is sharp in the sense that the constant $\frac{1}{15}$ cannot be replaced by a smaller one.

Proof. From (3), (6), (7) and (9), we can easily get

$$\begin{aligned}
&\Big|\int_a^b f(x)\,dx - \frac{b-a}{30}[7f(a) + 16f(\frac{a+b}{2}) + 7f(b)] + \frac{(b-a)^2}{60}[f'(b) - f'(a)] \\
&\quad - \sum_{k=3}^{[\frac{n-1}{2}]} \frac{(k-1)(k-2)(b-a)^{2k+1}}{15(2k+1)!2^{2k-2}} f^{(2k)}(\frac{a+b}{2})\Big| \\
&= \Big|\int_a^b S_n(x) f^{(n)}(x)dx\Big| \\
&= \Big|\int_a^b S_n(x)\Big[f^{(n)}(x) - \frac{1}{b-a}\int_a^b f^{(n)}(t)dt\Big]dx\Big| \\
&\le \Big(\int_a^b S_n^2(x)dx\Big)^{\frac{1}{2}}\Big(\int_a^b [f^{(n)}(x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}]^2 dx\Big)^{\frac{1}{2}}
\end{aligned}$$

$$= \Big[\frac{(4n^6-64n^5+403n^4-1260n^3+2021n^2-1500n+360)(b-a)^{2n+1}}{225(2n-3)(4n^2-1)(n!)^2 2^{2n}}\Big]^{\frac{1}{2}}$$
$$\times \Big\{\|f^{(n)}\|_2^2 - \frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]^2}{b-a}\Big\}^{\frac{1}{2}}$$
$$= \frac{1}{15}\frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{4n^6-64n^5+403n^4-1260n^3+2021n^2-1500n+360}{(2n-3)(4n^2-1)}}\sqrt{\sigma(f^{(n)})}.$$

We now suppose that (8) holds with a constant $C>0$ as

$$\Big|\int_a^b f(x)\,dx - \frac{b-a}{30}\big[7f(a)+16f(\frac{a+b}{2})+7f(b)\big] + \frac{(b-a)^2}{60}\big[f'(b)-f'(a)\big]$$
$$-\sum_{k=3}^{[\frac{n-1}{2}]}\frac{(k-1)(k-2)(b-a)^{2k+1}}{15(2k+1)!2^{2k-2}}f^{(2k)}(\frac{a+b}{2})\Big|$$
$$\le C\frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{4n^6-64n^5+403n^4-1260n^3+2021n^2-1500n+360}{(2n-3)(4n^2-1)}}\sqrt{\sigma(f^{(n)})}. \tag{11}$$

We may find a function $f:[a,b]\to\mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ as

$$f^{(n-1)}(x)=\begin{cases}\dfrac{(x-a)^{n+1}}{(n+1)!}-\dfrac{7(b-a)(x-a)^n}{30n!}+\dfrac{(b-a)^2(x-a)^{n-1}}{60(n-1)!} & \text{if } x\in[a,\frac{a+b}{2}],\\[2ex] \dfrac{(x-b)^{n+1}}{(n+1)!}+\dfrac{7(b-a)(x-b)^n}{30n!}+\dfrac{(b-a)^2(x-b)^{n-1}}{60(n-1)!} & \text{if } x\in(\frac{a+b}{2},b].\end{cases}$$

It follows that

$$f^{(n)}(x)=\begin{cases}\dfrac{(x-a)^n}{n!}-\dfrac{7(b-a)(x-a)^{n-1}}{30(n-1)!}+\dfrac{(b-a)^2(x-a)^{n-2}}{60(n-2)!} & \text{if } x\in[a,\dfrac{a+b}{2}],\\[2ex] \dfrac{(x-b)^n}{n!}+\dfrac{7(b-a)(x-b)^{n-1}}{30(n-1)!}+\dfrac{(b-a)^2(x-b)^{n-2}}{60(n-2)!} & \text{if } x\in(\dfrac{a+b}{2},b].\end{cases} \tag{12}$$

By (3), (4), (6), (7), (9), (10) and (12), it is not difficult to find that the left-hand side (L.H.S. for short) of the inequality (11) becomes

$$L.H.S.(11) = \frac{(4n^6-64n^5+403n^4-1260n^3+2021n^2-1500n+360)(b-a)^{2n+1}}{225(2n-3)(4n^2-1)(n!)^2 2^{2n}}, \tag{13}$$

and the right-hand side (R.H.S. for short) of the inequality (11) is

$$R.H.S.(11) = \frac{C(4n^6-64n^5+403n^4-1260n^3+2021n^2-1500n+360)(b-a)^{2n+1}}{15(2n-3)(4n^2-1)(n!)^2 2^{2n}}. \tag{14}$$

From (11), (13) and (14), we find that $C \geq \frac{1}{15}$ proving that the constant $\frac{1}{15}$ is the best possible in (8). ∎

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n > 1$ and $f^{(n)} \in L_2[a,b]$ where n is an even integer. Then we have*

$$\begin{aligned}
&\Big|\int_a^b f(x)\,dx - \frac{b-a}{30}\big[7f(a)+16f(\frac{a+b}{2})+7f(b)\big] + \frac{(b-a)^2}{60}\big[f'(b)-f'(a)\big] \\
&\quad - \sum_{k=3}^{[\frac{n-1}{2}]} \frac{(k-1)(k-2)(b-a)^{2k+1}}{15(2k+1)!2^{2k-2}} f^{(2k)}(\frac{a+b}{2}) \\
&\quad - \frac{(n-2)(n-4)(b-a)^n}{15(n+1)!2^n}\big[f^{(n-1)}(b)-f^{(n-1)}(a)\big]\Big| \\
&\leq \frac{1}{15}\frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!}\sqrt{\frac{4n^8-88n^7+387n^6-1076n^5+1269n^4-242n^3-199n^2-364n+168}{(2n-3)(4n^2-1)}} \\
&\quad \times \sqrt{\sigma(f^{(n)})}.
\end{aligned} \tag{15}$$

Inequality (15) is sharp in the sense that the constant $\frac{1}{15}$ cannot be replaced by a smaller one.

Proof. From (3), (6), (7) and (9),we can easily get

$$\begin{aligned}
&\Big|\int_a^b f(x)\,dx - \frac{b-a}{30}\big[7f(a)+16f(\frac{a+b}{2})+7f(b)\big] + \frac{(b-a)^2}{60}\big[f'(b)-f'(a)\big] \\
&\quad - \sum_{k=3}^{[\frac{n-1}{2}]} \frac{(k-1)(k-2)(b-a)^{2k+1}}{15(2k+1)!2^{2k-2}} f^{(2k)}(\frac{a+b}{2}) \\
&\quad - \frac{(n-2)(n-4)(b-a)^n}{15(n+1)!2^n}[f^{(n-1)}(b)-f^{(n-1)}(a)]\Big| \\
&= \Big|\int_a^b S_n(x)f^{(n)}(x)dx - \frac{1}{b-a}\int_a^b S_n(x)\,dx\int_a^b f^{(n)}(x)dx\Big| \\
&= \frac{1}{2(b-a)}\Big|\int_a^b\int_a^b [S_n(x)-S_n(t)][f^{(n)}(x)-f^{(n)}(t)]dxdt\Big| \\
&\leq \frac{1}{2(b-a)}\Big\{\int_a^b\int_a^b [S_n(x)-S_n(t)]^2dxdt\Big\}^{\frac{1}{2}}\Big\{\int_a^b\int_a^b [f^{(n)}(x)-f^{(n)}(t)]^2dxdt\Big\}^{\frac{1}{2}} \\
&= \Big\{\int_a^b S_n^2(x)\,dx - \frac{1}{b-a}[\int_a^b S_n(x)\,dx]^2\Big\}^{\frac{1}{2}}\Big\{\int_a^b [f^{(n)}(x)]^2dx - \frac{1}{b-a}[\int_a^b f^{(n)}(x)dx]^2\Big\}^{\frac{1}{2}} \\
&= \Big\{\frac{(4n^8-88n^7+387n^6-1076n^5+1269n^4-242n^3-199n^2-364n+168)(b-a)^{2n+1}}{225(2n-3)(4n^2-1)[(n+1)!]^2 2^{2n}}\Big\}^{\frac{1}{2}} \\
&\quad \times \Big\{\|f^{(n)}\|_2^2 - \frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]^2}{b-a}\Big\}^{\frac{1}{2}}
\end{aligned}$$

$$= \frac{1}{15}\frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!}\sqrt{\frac{4n^8-88n^7+387n^6-1076n^5+1269n^4-242n^3-199n^2-364n+168}{(2n-3)(4n^2-1)}} \times \sqrt{\sigma(f^{(n)})}.$$

We now suppose that (15) holds with a constant $C > 0$ as

$$\begin{aligned}
&\Big|\int_a^b f(x)\,dx - \frac{b-a}{30}[7f(a)+16f(\frac{a+b}{2})+7f(b)] + \frac{(b-a)^2}{60}[f'(b)-f'(a)] \\
&- \sum_{k=3}^{[\frac{n-1}{2}]} \frac{(k-1)(k-2)(b-a)^{2k+1}}{15(2k+1)!2^{2k-2}} f^{(2k)}(\frac{a+b}{2}) \\
&- \frac{(n-2)(n-4)(b-a)^n}{15(n+1)!2^n}[f^{(n-1)}(b)-f^{(n-1)}(a)]\Big| \\
\le\ & C\frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!}\sqrt{\frac{4n^8-88n^7+387n^6-1076n^5+1269n^4-242n^3-199n^2-364n+168}{(2n-3)(4n^2-1)}} \\
&\times \sqrt{\sigma(f^{(n)})}.
\end{aligned} \tag{16}$$

We may find a function $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ as

$$f^{(n-1)}(x) = \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!} - \frac{7(b-a)(x-a)^n}{30n!} + \frac{(b-a)^2(x-a)^{n-1}}{60(n-1)!} - \\ \quad -\frac{(n-2)(n-4)(b-a)^{n+1}}{15(n+1)!2^{n+1}} \text{ if } x \in \big[a, \frac{a+b}{2}\big], \\ \frac{(x-b)^{n+1}}{(n+1)!} + \frac{7(b-a)(x-b)^n}{30n!} + \frac{(b-a)^2(x-b)^{n-1}}{60(n-1)!} + \\ \quad +\frac{(n-2)(n-4)(b-a)^{n+1}}{15(n+1)!2^{n+1}} \text{ if } x \in \big(\frac{a+b}{2}, b\big]. \end{cases}$$

It follows that

$$f^{(n)}(x) = \begin{cases} \frac{(x-a)^n}{n!} - \frac{7(b-a)(x-a)^{n-1}}{30(n-1)!} + \frac{(b-a)^2(x-a)^{n-2}}{60(n-2)!} \text{ if } x \in \big[a, \frac{a+b}{2}\big], \\ \frac{(x-b)^n}{n!} + \frac{7(b-a)(x-b)^{n-1}}{30(n-1)!} + \frac{(b-a)^2(x-b)^{n-2}}{60(n-2)!} \text{ if } x \in \big(\frac{a+b}{2}, b\big]. \end{cases} \tag{17}$$

By (3), (4), (6), (7), (9), (10) and (17), it is not difficult to find that the left-hand side of the inequality (16) becomes

$$L.H.S.(16) = \frac{(4n^8-88n^7+387n^6-1076n^5+1269n^4-242n^3-199n^2-364n+168)(b-a)^{2n+1}}{225(2n-3)(4n^2-1)[(n+1)!]^2 2^{2n}}, \tag{18}$$

and the right-hand side of the inequality (16) is

$$R.H.S.(16)$$
$$= \frac{C(4n^8 - 88n^7 + 387n^6 - 1076n^5 + 1269n^4 - 242n^3 - 199n^2 - 364n + 168)(b-a)^{2n+1}}{15(2n-3)(4n^2-1)[(n+1)!]^2 2^{2n}}. \tag{19}$$

From (16), (18) and (19), we find that $C \geq \frac{1}{15}$, proving that the constant $\frac{1}{15}$ is the best possible in (15). ∎

Corollary 2.3. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_2[a,b]$ where $n = 2,3,4,5$. Then we have sharp inequalities*

$$\Big| \int_a^b f(x)\,dx - \frac{b-a}{30}[7f(a) + 16f(\frac{a+b}{2}) + 7f(b)] + \frac{(b-a)^2}{60}[f'(b) - f'(a)] \Big| \leq C_n (b-a)^{n+\frac{1}{2}} \sqrt{\sigma(f^{(n)})}, \tag{20}$$

where

$$C_2 = \frac{1}{60\sqrt{3}},\ C_3 = \frac{1}{120\sqrt{105}},\ C_4 = \frac{1}{1440\sqrt{70}},\ C_5 = \frac{\sqrt{13}}{17280\sqrt{385}}. \tag{21}$$

Proof. It is immediate from Theorem 2.1 and Theorem 2.2. ∎

3. Applications in Numerical Integration

We restrict further considerations to the applications of Corollary 2.3.

Theorem 3.1. *Let $\pi = \{x_0 = a < x_1 < \cdots < x_m = b\}$ be a given subdivision of the interval $[a,b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{m}$ and let the assumptions of Corollary 2.3 hold. Then for $n = 2,3,4,5$ we have*

$$\Big| \int_a^b f(t)\,dt - \frac{b-a}{30m} \sum_{i=0}^{m-1} [7f(x_i) + 16f(\frac{x_i + x_{i+1}}{2}) + 7f(x_{i+1})] + \frac{(b-a)^2}{60m^2}[f'(b) - f'(a)] \Big| \leq \frac{C_n (b-a)^{n+\frac{1}{2}}}{m^n} \sqrt{\sigma(f^{(n)})}, \tag{22}$$

where $C_n (n = 2,3,4,5)$ are given in (21).

Proof. From (20) and (21) we obtain

$$\Big| \int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{30}[7f(x_i) + 16f(\frac{x_i + x_{i+1}}{2}) + 7f(x_{i+1})]$$

$$+\frac{h^2}{60}[f'(x_{i+1})-f'(x_i)]\Big| \tag{23}$$
$$\le C_n h^{n+\frac{1}{2}}\Big[\int_{x_i}^{x_{i+i}}(f^{(n)}(t))^2dt-\frac{1}{h}(f'(x_{i+1})-f'(x_i))^2\Big]^{\frac{1}{2}},$$

where $C_n(n=2,3,4,5)$ are given in (21). By summing (23) over i from 0 to $m-1$ and using the generalized triangle inequality, we get

$$\Big|\int_a^b f(t)\,dt-\frac{h}{30}\sum_{i=0}^{m-1}\Big[f(x_i)+2f(\frac{x_i+x_{i+1}}{2})+f(x_{i+1})\Big]$$
$$+\frac{h^2}{60}[f'(b)-f'(a)]\Big| \tag{24}$$
$$\le C_n h^{n+\frac{1}{2}}\sum_{i=0}^{m-1}\Big[\int_{x_i}^{x_{i+i}}(f^{(n)}(t))^2\,dt-\frac{1}{h}(f'(x_{i+1})-f'(x_i))^2\Big]^{\frac{1}{2}}.$$

By using the Cauchy inequality twice, it is not difficult to obtain

$$\sum_{i=0}^{m-1}\Big[\int_{x_i}^{x_{i+1}}(f^{(n)}(t))^2\,dt-\frac{1}{h}(f'(x_{i+1})-f'(x_i))^2\Big]^{\frac{1}{2}}$$
$$\le\sqrt{m}\Big[\|f^{(n)}\|_2^2-\frac{m}{b-a}\sum_{i=0}^{m-1}(f'(x_{i+1})-f'(x_i))^2\Big]^{\frac{1}{2}} \tag{25}$$
$$\le\sqrt{m}\Big[\|f^{(n)}\|_2^2-\frac{(f'(b)-f'(a))^2}{b-a}\Big]^{\frac{1}{2}}=\sqrt{m\sigma(f^{(n)})}.$$

Consequently, the inequalities (22) with $C_n(n=2,3,4,5)$ given in (21) follow from (24) and (25). ■

Acknowledgment. The author wishes to thank the referee for his helpful comments and suggestions.

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