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Improving the Solution of Nonlinear Volterra Integral Equations Using Rationalized Haar s-Functions

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Abstract. In this paper, we propose a new modification of rationalized Haar functions called rationalized Haar s-functions for the numerical solution of linear and nonlinear Volterra integral equations of the second kind. By selecting these functions and following the procedure of determining the wavelet expansion coefficients, the calculations are economized. This method converts the integral equation to a system of equations containing an identity matrix which simplifies the calculations. Finally by using numerical examples the accuracy of the solution is illustrated.

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1. Introduction

Integral equations play an important role in many fields of science and engineering and have been considered analytically and numerically until now ([1] and [2] or almost any other textbook on integral equations). In recent years, many kind of basic functions such as Fourier series [3], Chebyshev polynomials [4], crooked lines [5] and wavelets [6 − 15] have been used in numerical solutions of integral equations. The most attractive one specially for large scale problems is the wavelet, in which the kernel can be represented as a sparse matrix.

The orthogonal set of Haar wavelets is a group of square waves with magnitude $+2^{j/2}$,$-2^{j/2}$ and 0 [6], $j = 0, 1, 2, \ldots$. When Haar functions are applied in function approximation, expansion coefficients are multiplied by $2^{j/2}$ at the first step and then are divided to this number again. This process, which increases the size of calculations, and produces errors are probably due to the irrationality.

Lynch and Reis [7] have rationalized the Haar transform by deleting the irrational numbers and introducing the integral power of two. This modification results in what is called the rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar transform and can be efficiently implemented by using digital pipeline architecture [8]. The corresponding functions are known as RH functions. The RH functions can be composed to only three amplitudes $+1, -1$ and 0.

Generally, many authors have considered two groups of functions. The first group is called rationalized Haar wavelets and the second are rationalized Haar functions. In [9] and [10], the authors applied the rationalized Haar wavelets to solve differential equations. Also, Fredholm and Volterra integral equations of the second kind using rationalized Haar wavelets have been considered in [11], [12], [13] and [14]. In this paper, we apply another set of modified Haar functions called rationalized Haar s-functions for solving linear and nonlinear Volterra integral equations of the second kind. Hence the presented method is much easier and needs less calculations. The matrix of wavelet expansion coefficients related to this method is identity matrix, while it is diagonal in [13] and [14]. Therefore, we get an explicit formula which is one of the advantages of this method. Furthermore, the procedure used in this paper can also be applied to nonlinear Volterra integral equations.

2. Properties of Rationalized Haar s-functions

2.1. Rationalized Haar s-functions

Let φ and ψ be the Haar scaling function and the corresponding wavelet respectively, which are defined as

$$\varphi(x) = \begin{cases} 1, & x \in [0, 1), \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

$$\psi(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}), \\ -1, & x \in [\frac{1}{2}, 1), \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

The rationalized Haar wavelets are defined as

$$\psi_0(x) = \varphi(x), \tag{3}$$

$$\psi_1(x) = \psi(x), \tag{4}$$

$$\psi_n(x) = \psi_1(2^j x - k),\ n = 2^j + k, \tag{5}$$

where $j \geq 0$ and $0 \leq k < 2^j$. The orthogonality property is given by

$$\int_0^1 \psi_n(t)\psi_m(t)dt = \begin{cases} 2^{-j}, & n = m = 2^j + k, \\ 0, & n \neq m. \end{cases}$$

The set of $\{\psi_n\}_{n\in Z}$ has been used for function approximation and solving the integral equations in [11], [12], [13] and [14]. In this paper, we define rationalized Haar s-functions as

$$\varphi_{J,k}(x) = \varphi(2^J x - k),\ k = 0, 1, 2, \cdots, 2^J - 1. \tag{6}$$

Hence, for each J, $J = 0, 1, 2, \ldots$ the orthogonality property is given by

$$\int_0^1 \varphi_{J,k}(t)\varphi_{J,l}(t)dt = \begin{cases} 2^{-J}, & k = l, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

2.2. Function Approximation

Any function $f \in L^2[0, 1]$ may be approximated by rationalized Haar function series at resolution J as

$$f(x) = \sum_{k=0}^{2^J-1} a_k \varphi_{J,k}(x),\ x \in [0, 1], \tag{8}$$

or can also be written as

$$f(x) = \Phi^T(x)A, \tag{9}$$

where

$$a_k = 2^J \int_0^1 f(t)\varphi_{J,k}(t)dt,\ k = 0, 1, 2, \cdots, 2^J - 1,$$
$$\Phi(x) = [\varphi_{J,0}(x), \varphi_{J,1}(x), \cdots, \varphi_{J,2^J-1}(x)]^T,\ A = [a_0, a_1, \cdots, a_{2^J-1}]^T.$$

Here, we deal with the calculation of wavelet coefficient a_k for $f \in L^2[0, 1]$. Suppose the values of f are known at points $x_l = \dfrac{l}{2^{J+1}}, l = 1, 3, 5, \cdots, 2^{J+1} - 1.$

Substituting $x = x_l$ in Eq.(8), gives

$$f(l/2^{J+1}) = \sum_{k=0}^{2^J-1} a_k \varphi(\frac{l}{2} - k), \tag{10}$$

by substituting $l = 1, 3, 5, \cdots, 2^{J+1} - 1$ in Eq.(10) we finally get

$$F = IA, \tag{11}$$

where $F = [f(1/2^{J+1}), f(3/2^{J+1}), \cdots, f((2^{J+1}-1)/2^{J+1})]^T$ and I is the $2^J \times 2^J$ identity matrix.

Similar to Eq.(8) a function $k \in L^2([0,1] \times [0,1])$ may be approximated at resolution J as

$$k(x,t) = \Phi^T(x) S \Phi(t), \tag{12}$$

where S is a $2^J \times 2^J$ coefficient matrix. Imagining $k(x,t)$ as a one-dimensional function of variable x and t and invoking Eq.(10), we finally have

$$S = IKI = K, \tag{13}$$

where K is the $2^J \times 2^J$ kernel matrix with $K_{m,n} = k(\frac{2m-1}{2^{J+1}}, \frac{2n-1}{2^{J+1}})$.

2.3. Operation Matrix of Integration

The integration of the $\Phi(t)$ defined in Eq.(9) can be expanded into Haar series with matrix $P(x)$ as

$$\int_0^x \Phi(t)dt = P(x)\Phi(x), \tag{14}$$

where $P(x)$ is the $2^J \times 2^J$ upper-triangular operational matrix for integration and is given as

$$\mathbf{P}(\mathbf{x}) = \begin{pmatrix} x & \frac{1}{2^J} & \frac{1}{2^J} & \cdots & \frac{1}{2^J} \\ 0 & x - \frac{1}{2^J} & \frac{1}{2^J} & \cdots & \frac{1}{2^J} \\ 0 & 0 & x - \frac{2}{2^J} & \cdots & \frac{1}{2^J} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & & x - \frac{2^J-1}{2^J} \end{pmatrix}. \tag{15}$$

In other words, $P(x)$ is the operational matrix for integration with

$$P_{m,n}(x) = \begin{cases} \frac{1}{2^J}, & m < n, \\ x - \frac{m-1}{2^J}, & m = n, \\ 0, & m > n. \end{cases}$$

Let $M(x) = P(x)\Phi(x)$. According to the orthogonality of rationalized Haar functions, integration of the product of two functions gives

$$\int_0^x \Phi(x)\Phi^T(x)dx = \widetilde{M}(x), \tag{16}$$

where

$$\widetilde{M}(x) = diag\Big(M_{1,1}(x), M_{2,1}(x), \ldots, M_{2^J,1}(x)\Big). \tag{17}$$

3. Volterra Integral Equations of the Second Kind

3.1. Linear Integral Equations

Consider the Volterra linear integral equation of the second kind

$$f(x) - \int_0^x k(x,t)f(t)dt = g(x), \tag{18}$$

where $g(x) \in L^2[0,1]$, $k(x,t) \in L^2([0,1] \times [0,1])$ are known. The problem is to find an unknown function $f(x)$, satisfying Eq.(18). We approximate $f(x), g(x)$ and $k(x,t)$ as

$$f(x) = \Phi^T(x)A, \tag{19}$$

$$g(x) = \Phi^T(x)B, \tag{20}$$

$$k(x,t) = \Phi^T(x)S\Phi(t), \tag{21}$$

where $\Phi^T(x)$ is given by Eq.(9) and A is an unknown $2^J \times 1$ vector, S is a known $2^J \times 2^J$ matrix given by Eq.(12) and B is a known $2^J \times 1$ vector given by Eq.(9). It should be noted that in Volterra integral equations we have $t \leq x$. In other words, $k(x,t) = 0$, if $t > x$. Thus, K and hence S are lower-triangular matrices. By substituting Eqs.(19) – (21) in Eq.(18) we get

$$\Phi^T(x)A - \int_0^x \Phi^T(x)S\Phi(t)\Phi^T(t)Adt = \Phi^T(x)B,$$

$$\Phi^T(x)A - \Phi^T(x)S\Big(\int_0^x \Phi(t)\Phi^T(t)dt\Big)A = \Phi^T(x)B. \tag{22}$$

By using Eqs.(16) and (22) we obtain

$$\Phi^T(x)(I - S\widetilde{M}(x))A = \Phi^T(x)B, \tag{23}$$

and by substituting $x_l = \dfrac{l}{2^{J+1}}, l = 1, 3, 5, \cdots, 2^{J+1} - 1$ in Eq.(23) we finally get the following linear system

$$(I - 2^{-J}K')A = G, \tag{24}$$

where I is the $2^J \times 2^J$ identity matrix, $G = [g(1/2^{J+1}), g(3/2^{J+1}), \cdots, g((2^{J+1} - 1)/2^{J+1})]^T$ is a known matrix and K' is a known $2^J \times 2^J$ matrix with

$$K'_{m,n} = \begin{cases} K_{m,n}, & m \neq n, \\ \dfrac{1}{2}K_{m,n}, & m = n. \end{cases} \tag{25}$$

Eq.(24) is a system of linear equations and can be easily solved for the unknown vector A.

It should be noted that according Eq.(11), $F = A$ which means by getting matrix A from Eq.(24), matrix F will be resulted automatically. Substituting coefficients a_k in Eq.(8), approximation of unknown function f is acquired.

3.2. Nonlinear Integral Equations

Consider the nonlinear Volterra integral equation of the second kind

$$f(x) - \int_0^x k(x,t)h[f(t)]dt = g(x), \tag{26}$$

where $h \in L^2[0,1]$ is known. we can approximate $f(x), g(x)$ and $k(x,t)$ by the mentioned method in Section 2 again. Let we imagine $h(f)$ as a function of variable x. Then, $h(f)$ can be approximated as

$$h[f(x)] = h'(x) = \Phi^T(x)C, \tag{27}$$

where $C = [c_0, c_1, \cdots, c_{2^J-1}]^T$.
Similar to Eq.(11), we have

$$H(f) = IC, \tag{28}$$

where I is the $2^J \times 2^J$ identity matrix and $H(f) = [h'(1/2^{J+1}), h'(3/2^{J+1}), \cdots, h'((2^{J+1} - 1)/2^{J+1})]^T$. Substituting Eqs.(19)-(21) and (27) into Eq.(26), we get

$$\Phi^T(x)A - \int_0^x \Phi^T(x)S\Phi(t)\Phi^T(t)Cdt = \Phi^T(x)B,$$

and then

$$F - 2^{-J} K' H(f) = G. \tag{29}$$

Eq.(29) is a system of nonlinear equations and can be solved by many methods (e.g. Newton iteration method) for the unknown vector F.

4. Error Analysis

In this section, we show that for each J, the sequence $\{\varphi_{J,k}(x)\}_{k=0}^{2^J-1}$ is an orthogonal base in $L^2[0,1]$ and for each $f \in L^2[0,1]$ the series $\sum_{k=0}^{2^J-1} a_k \varphi_{J,k}(x)$ converges to f when $J \to \infty$.

It can be observed that for $J = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots, 2^J - 1$ we have

$$\mathrm{Supp}\varphi_{J,k} = \left[k2^{-J}, (k+1)2^{-J}\right).$$

The intervals $\left[k2^{-J}, (k+1)2^{-J}\right)$ form the family of dyadic intervals. For each J, any two dyadic intervals $\left[k2^{-J}, (k+1)2^{-J}\right)$ and $\left[k'2^{-J}, (k'+1)2^{-J}\right)$ do not overlap except the case that $k = k'$. This fact easily gives the following proposition.

Proposition 4.1. *For $J = 0, 1, 2, \ldots$, the sequence $\{\varphi_{J,k}(x)\}_{k=0}^{2^J-1}$ is orthogonal in $L^2[0,1]$.*

Proof. Let us consider the scalar products $\langle \varphi_{J,k}, \varphi_{J,k'} \rangle$. Then, we have

$$\langle \varphi_{J,k}, \varphi_{J,k'} \rangle = \int_0^1 \varphi_{J,k}(t)\varphi_{J,k'}(t)dt = \begin{cases} 2^{-J}, & k = k', \\ 0, & \text{otherwise}, \end{cases}$$

and the proposition is proved. ∎

In order to show that for each J, the sequence $\{\varphi_{J,k}(x)\}_{k=0}^{2^J-1}$ is an orthogonal base in $L^2[0,1]$, let us consider two families of closed subspaces of $L^2[0,1]$ as follows

$$S_J = \mathrm{Span}\{\varphi_{J,k}(x)\}_{k=0}^{2^J-1},$$

and

$$L_J = \left\{ f \in L^2[0,1] \mid f \text{ is constant on } \left[k2^{-J}, (k+1)2^{-J}\right),\ k = 0, 1, \ldots, 2^J - 1 \right\}.$$

Therefore, we have the following lemma.

Lemma 4.2. *For $J = 0, 1, 2, \ldots$, we have $S_J = L_J$.*

Proof. The proof followes directly from [16]. ∎

From Proposition 4.1 and Lemma 4.2 and the fact that $\bigcup_{J=0}^{\infty} L_J$ is dense in $L^2[0,1]$, we get immediately the following theorem.

Theorem 4.3. *for* $J = 0, 1, 2, \ldots$, *the sequence* $\{\varphi_{J,k}(x)\}_{k=0}^{2^J-1}$ *is an orthogonal base in* $L^2[0,1]$ *and for each* $f \in L^2[0,1]$ *the series* $\sum_{k=0}^{2^J-1} a_k \varphi_{J,k}(x)$ *converges to* f *when* $J \to \infty$.

Now, suppose that $f(x) \in L^2[0,1]$ and

$$\exists M > 0: \quad \forall x \in (0,1), \ |f'(x)| \leq M.$$

As we mentioned in Section 2.2, f can be approximated by rationalized Haar s-functions at resolution J as

$$\widetilde{f}(x) = \sum_{k=0}^{2^J-1} a_k \varphi_{J,k}(x).$$

Using the definition and properties of the Haar scaling function and the corresponding wavelet, we have

$$\operatorname{Span}\{\varphi_{J,k}\}_{k=0}^{2^J-1} = \operatorname{Span}\{\psi_n\}_{n=0}^{2^J-1}.$$

Therefore, the sequence $\{\psi_n(x)\}_{n=0}^{\infty}$ is an orthogonal base in $L^2[0,1]$ and for each $f \in L^2[0,1]$ the series $\sum_{n=0}^{\infty} b_n \psi_n(x)$ converges to f, where

$$b_n = 2^j \int_0^1 f(t)\psi_n(t)dt. \tag{30}$$

Here, $n = 2^j + k$, for $j = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^j - 1$. In other words, we have

$$\widetilde{f}(x) = \sum_{k=0}^{2^J-1} a_k \varphi_{J,k}(x) = \sum_{n=0}^{2^J-1} b_n \psi_n(x),$$

where $a_k = 2^J \int_0^1 f(t)\varphi_{J,k}(t)dt$ and $b_n = 2^j \int_0^1 f(t)\psi_n(t)dt$. Thus, we can compute the error analysis with respect to the sequence $\{\psi_n\}_{n=0}^{\infty}$, similar to the procedure which was done by Babolian and Shahsavaran [11], as indicated in the following. So, we have

$$f(x) - \widetilde{f}(x) = \sum_{n=2^J}^{\infty} b_n \psi_n(x).$$

Then

$$\| f(x) - \widetilde{f}(x) \|_E^2 = \int_0^1 (f(x) - \widetilde{f}(x))^2 dx = \sum_{n=2^J}^{\infty} \sum_{n'=2^J}^{\infty} b_n b_{n'} \int_0^1 \psi_n(x)\psi_{n'}(x)dx$$
$$= \sum_{j=J}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} 2^{-j} b_n^2.$$

Using Eq. (30) we get

$$b_n = 2^j \int_0^1 f(x)\psi(2^j x - k)dx = 2^j \left\{ \int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} f(x)dx - \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} f(x)dx \right\}.$$

Using the mean value theorem we have

$$\exists x_1, x_2;\ k2^{-j} \le x_1 < \left(k + \frac{1}{2}\right) 2^{-j} \text{ and } \left(k + \frac{1}{2}\right) 2^{-j} \le x_2 < (k+1)\, 2^{-j}$$

such that

$$b_n = 2^j \left\{ \left(\left(k + \frac{1}{2}\right) 2^{-j} - k2^{-j} \right) f(x_1) - \left((k+1)\, 2^{-j} - \left(k + \frac{1}{2}\right) 2^{-j} \right) f(x_2) \right\}$$
$$= \frac{1}{2} \left(f(x_1) - f(x_2) \right).$$

Using the mean value theorem, there is x_0 such that $x_1 < x_0 < x_2$ and

$$b_n^2 = 2^{-2} \left(f(x_1) - f(x_2) \right)^2 = 2^{-2} (x_2 - x_1)^2 f'^2(x_0) \le 2^{-2} 2^{-2j} M^2.$$

Thus

$$\| f(x) - \widetilde{f}(x) \|_E^2 = \sum_{j=J}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} 2^{-j} b_n^2 \le \sum_{j=J}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} 2^{-3j-2} M^2 \le \frac{M^2}{3} 2^{-2J}.$$

5. Numerical Examples

The following equations were solved by our method. In the following tables we applied the maximum of error

$$\varepsilon = \|f - \widetilde{f}\| = \max_{x \in [0,1]} |f(x) - \widetilde{f}(x)|,$$

where $\widetilde{f}$ is the approximate function of f.

Example 5.1. Consider the linear Volterra integral equation of the second kind

$$f(x) - \int_0^x (t - x) \cos(x - t) f(t) dt = \cos(x), \tag{31}$$

with an exact solution $f(x) = \frac{1}{3}(2 \cos \sqrt{3} x + 1)$. By using the method in Section 3, Eq.(31) is solved. The values of maximum error for $J = 3, 4, \cdots, 9$ are available in Table 1. This example is also presented in [14], but its results do not match with the exact solution and are not correct. In order to compare these two methods, we applied their method (based on Haar wavelet approximation) and computed the results. In Tables 2 and 3, the comparison between our method and the method presented in [14] are given for $J = 7$ and $J = 9$, respectively.

Tables 2 and 3 demonstrate that both methods have the same results, but our proposed method simplifies and economizes the computations.

Table 1. Maximum error quantity

J	ε
3	0.06280243438
4	0.03598072764
5	0.01157820970
6	0.00353411821
7	0.00319014802
8	0.00225506227
9	0.00071749827

Table 2. The comparison between our method and Haar wavelet method for $J = 7$

t	Our method	Haar Wavelet Method	Exact
0	0.9999923	0.9999923	1
0.1	0.9904934	0.9904934	0.9900249
0.2	0.9607112	0.9607112	0.9603984
0.3	0.9115649	0.9115649	0.9120068
0.4	0.8445713	0.8445713	0.8462984
0.5	0.7617983	0.7617983	0.7652395
0.6	0.6735894	0.6735894	0.6712558
0.7	0.5680065	0.5680065	0.5671597
0.8	0.4551805	0.4551805	0.4560663
0.9	0.3405935	0.3405935	0.3413002
1	0.2259475	0.2259475	0.2262956

Table 3. The comparison between our method and Haar wavelet method for $J = 9$

t	Our method	Haar Wavelet method	Exact
0	0.9999995	0.9999995	1
0.1	0.9900085	0.9900085	0.9900249
0.2	0.9603222	0.9603222	0.9603984
0.3	0.9121192	0.9121192	0.9120068
0.4	0.8467984	0.8467984	0.8462984
0.5	0.7650802	0.7650802	0.7652395
0.6	0.6710726	0.6710726	0.6712558
0.7	0.5675423	0.5675423	0.5671597
0.8	0.4562880	0.4562880	0.4560663
0.9	0.3413002	0.3413002	0.3413002
1	0.2265486	0.2265486	0.2262956

Example 5.2. Consider the nonlinear Volterra integral equation

$$f(x) - \int_0^x x(1 + \cot^2(2\pi t))f^2(t)dt = \sin(2\pi x) - x^2. \tag{32}$$

The exact solution is $f(x) = \sin(2\pi x)$. The Newton iteration method is used to solve Eq.(31). The initial value of F is given by $F_0 = [0, 0, \cdots, 0]^T_{2^J \times 1}$. After five iterations, the maximum errors for $J = 3, 4, \cdots, 9$ are given in Table 4. Furthermore, the computational results for J=7 and 9 with the exact solution are given in Table 5.

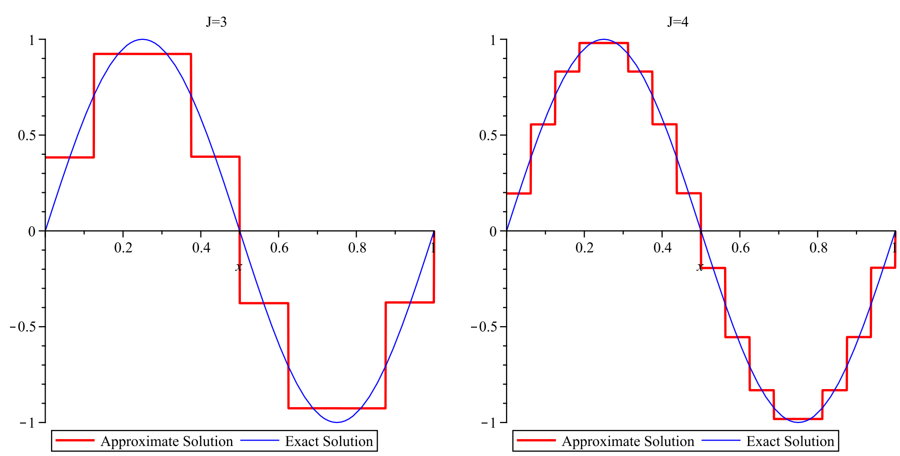
Table 4. Maximum error quantity

J	ε
3	0.32442327880
4	0.15153628180
5	0.07882303117
6	0.04906757784
7	0.01939712708
8	0.00862401817
9	0.00496240006

Table 5. Estimated and exact values of f(t) for Example 2

t	J=7	J=9	Exact
0	0.0182935	0.0048821	0
0.1	0.5758104	0.5897598	0.5877852
0.2	0.9495262	0.9514348	0.9510565
0.3	0.9495253	0.9514348	0.9510565
0.4	0.5758189	0.5897603	0.5877852
0.5	-0.0182935	-0.0048821	0
0.6	-0.5757943	-0.5897588	-0.5877852
0.7	-0.9495376	-0.9514356	-0.9510565
0.8	-0.9495389	-0.9514357	-0.9510565
0.9	-0.5757871	-0.5897584	-0.5877852
1	-0.0182867	-0.0048569	0

Figure 1 shows the comparison of exact solution and approximate solution of Example 2 for $J = 3, 4, \ldots, 8$. The convergence of the method can be observed.



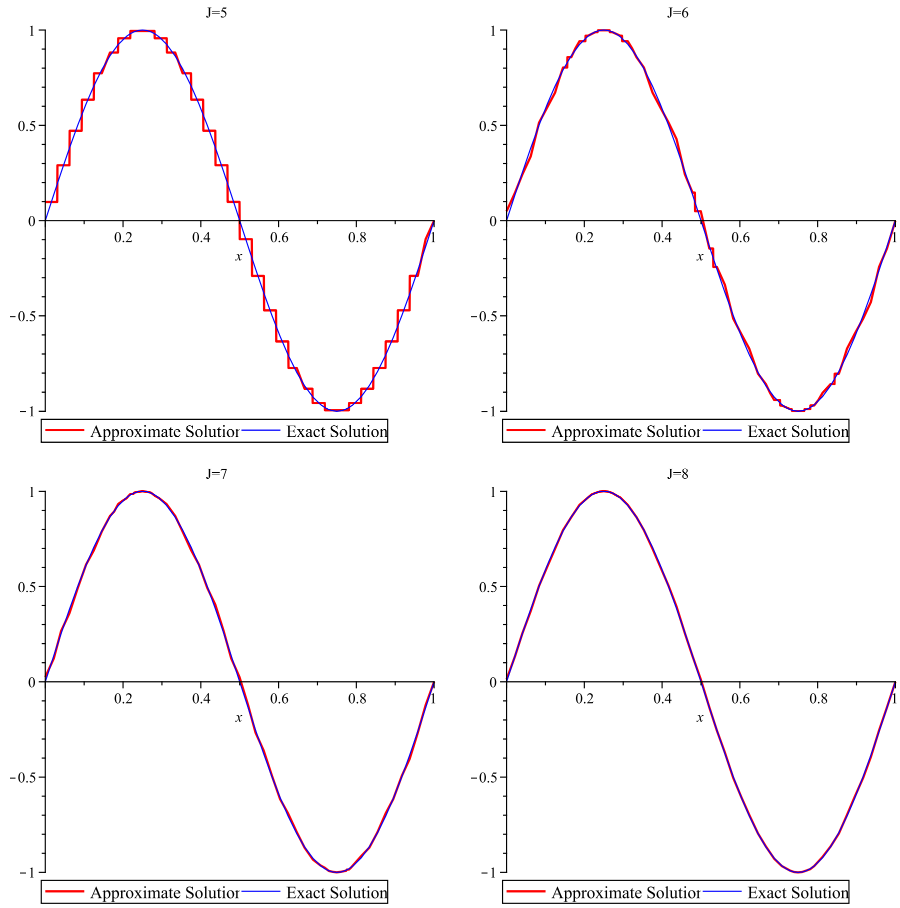


Fig. 1 Graphs of exact solution and approximate solution of Example 2 for $J = 3, 4, \ldots, 8$

6. Conclusion

In this paper we suggested a modification of rationalized Haar function for numerical solution of linear and nonlinear integral equations. This modification facilitates the solution of the system of linear and nonlinear equations. Speeds up the convergence towards the exact solutions are shown in the given examples.

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