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On the Stability and Boundedness of Solutions of a Class of Liénard Equations with Multiple Deviating Arguments

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Abstract. In this paper, we consider a class of Liénard type equations with multiple variable deviating arguments. By making use of the Lyapunov functional approach, under some suitable conditions, it is guaranteed that solutions of the equations considered are stable, bounded and uniformly bounded. We give three examples to illustrate the main results.

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1. Introduction

Liénard type equations appear in a number of physical models and are significant in describing fluid mechanical and nonlinear elastic mechanical phenomena. Hence, there has been a great interest of many researchers to study the dynamical behavior of all kinds of Liénard equations. Especially, several authors have contributed to the study of stability and boundedness of solutions for Liénard type equations. In particular, for detailed discussions, we refer the reader to the books or papers of Ahmad and Rama Mohana Rao [1], Burton [2], Burton and Hering [3], Èl'sgol'ts [4], Hale [5], Heidel [6], Kato [7, 8], Krasovskii [9], Liu and Huang [10, 11], Luk [12], Malyseva [13], Muresan [14], Mustafa and Tunç [15], Nápoles Valdés [16], Sugie [17], Sugie and Amano [18], Sugie et al. [19], Tunç

[20-27], C. Tunç and E. Tunç [28], Ye et al. [29], Yu and Xiao [30], Yu and Zhao [31], Yoshizawa [32], Zhang [33], Zhou and Xiang [34], Zhou and Jiang [35] and the references cited therein.

However, to the best of our knowledge, there is no published paper in the literature related to the stability and boundedness of solutions of Liénard equations with multiple constant or variable deviating arguments, except that of Tunç [25], Yu and Xiao [30] and Yu and Zhao [31], which are related to the stability and boundedness of the solutions. Thus, it is worth to continue the investigation of the topic for Liénard type equations with multiple variable deviating arguments.

In 2009, Yu and Xiao [30] considered the following Liénard type equation with multiple deviating arguments, $\tau_j(t), \quad (j=1,2,...,m)$:

$$x''(t)+f_1(x(t))(x'(t))^2+f_2(x(t))x'(t)+g_0(t,x(t))+\sum_{j=1}^{m} g_j(t,x(t-\tau_j(t)))=p(t), \tag{1}$$

where $f_1:\mathbb{R}\to\mathbb{R}$, $f_2:\mathbb{R}\to\mathbb{R}$, $g_0:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$, $g_1:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R},...,g_m:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $\mathbb{R}^+=[0,\infty)$, $\mathbb{R}=(-\infty,\infty)$, the delays $\tau_j(t)\geq 0$ $(j=1,2,...,m)$ and $p(t)$ are bounded and continuous functions on $\mathbb{R}^+$, and $\tau_j(t)$ are differentiable.

Define

$$a(x)=\exp(\int_0^x f_1(u)du), \quad \varphi(x)=\int_0^x a(u)[f_2(u)-a(u)]du, y=a(x)\frac{dx}{dt}+\varphi(x),$$

then Equation (1) can be transformed to the system as follows

$$\begin{aligned}\frac{dx(t)}{dt}&=\frac{1}{a(x(t))}[-\varphi(x(t))+y(t)],\\ \frac{dy(t)}{dt}&=-a(x(t))\{-y(t)-[g_0(t,x(t))-\varphi(x(t))]-\sum_{j=1}^{m} g_j(t,x(t-\tau_j(t)))+p(t)\}.\end{aligned} \tag{2}$$

By means of the above acceptations, Yu and Xiao [30] proved the following theorem.

Theorem 1.1. ([30]) *Assume that the following conditions hold:*

(C$_1$) *There exists a constant $\underline{d}>1$ such that*

$$\underline{d}\,|u|\leq sign(u)\varphi(u)=sign(u)\int_0^u a(s)[f_2(s)-a(s)]ds \text{ for all } u\in\mathbb{R};$$

(C$_2$) *For $j=0,1,2,...,m$, there exist non-negative constants h, L_j and q_j such that for all $u\in\mathbb{R}$,*

$$h = \max_{1 \le j \le m} \{\sup_{t \in \mathbb{R}} \tau_j(t)\} \ge 0, \sum_{j=1}^{m} L_j < 1,$$

$$|g_0(t,u) - \varphi(u)| = |g_0(u) - \int_0^u a(s)[f_2(s) - a(s)]\, ds| \le L_0\, |u| + q_0,$$

$$|g_j(t,u)| \le L_j\, |u| + q_j, j = 1, 2, ..., m;$$

(C$_3$) *$p(t)$ is bounded on $\mathbb{R}^+ = [0, \infty)$.*

Then solutions of Equation (1) *are uniformly bounded.*

In this paper, we consider the following Liénard type equation with multiple variable deviating arguments, $\tau_j(t)$, $(j = 1, 2, ..., m)$:

$$\begin{aligned} &x''(t) + f(t, x(t), x'(t))x'(t) + b_0(t)g_0(x(t)) + \sum_{j=1}^{m} b_j(t)g_j(x(t - \tau_j(t))) \\ &= e(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_m(t)), x'(t), x'(t - \tau_1(t)), ..., x'(t - \tau_m(t))), \end{aligned} \tag{3}$$

where $\tau_j(t)$ are deviating arguments with $0 \le \tau_j(t) \le \gamma_j$, $\tau_j'(t) \le \beta_j$, $0 < \beta_j < 1$, γ_j and β_j are some positive constants and γ_j to be chosen later; the primes in Equation (3) denote differentiation with respect to $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$; f, b_j, g_j, $(j = 0,\ 1,\ 2, ..., m)$, and e are continuous functions in their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^2$, $\mathbb{R}^+$, $\mathbb{R}$, $m-$ tuple $\mathbb{R}^+, ..., \mathbb{R}^+$, $m-$ tuple $\mathbb{R}, ..., \mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}^{2m+2}$, respectively, and also depend only on the arguments displayed explicitly with $g_j(0) = 0$, $(j = 0,\ 1, 2, ..., m)$. The continuity of these functions is a sufficient condition for existence of the solution of Equation (3) (see [4, p.14]). It is also assumed as basic that the functions f, g_j $(j = 0, 1, 2, ..., m)$, and e satisfy a Lipschitz condition in $x, x(t - \tau_1(t)), x(t - \tau_2(t)), ..., x'(t - \tau_m(t))$. By this assumption, the uniqueness of solutions of Equation (3) is guaranteed (see [4, p.15]). The derivatives $b_0'(t), b_j'(t), \dfrac{dg_j}{dx} \equiv g_j'(x)$, $(j = 1, 2, ..., m)$, exist and are continuous, and throughout the paper $x(t)$ and $x'(t)$ are abbreviated as x and x', respectively.

We write Equation (3) in system form as follows

$$\begin{aligned} x' &= y, \\ y' &= - f(t, x, y)y - \sum_{j=0}^{m} b_j(t)g_j(x) + \sum_{j=1}^{m} b_j(t) \int_{t-\tau_1(t)}^{t} g_j'(x(s))y(s)ds \\ &\quad + e(t, x, x(t - \tau_1(t)), ..., x(t - \tau_m(t)), y, y(t - \tau_1(t)), ..., y(t - \tau_m(t))). \end{aligned} \tag{4}$$

The aim of this paper is to give three results on the stability, boundedness and uniform boundedness of solutions of Equation (3), respectively. Motivated by Yu and Xiao [30], we first establish some sufficient conditions which guarantee stability of the zero solution of Equation (3), when $e(.) = 0$. Then we obtain some sufficient conditions, Theorem 2.3 and Theorem 2.6, which ensure the boundedness and uniform boundedness of all the solutions of Equation (3), when $e(.) \neq 0$.

It should be noted that Yu and Xiao [30] discussed uniform boundedness of the solutions of Equation (1). In addition to uniform boundedness of the solutions, we also discuss the stability and boundedness of the solutions of Equation (3). Besides, the author in [25] improved the result obtained by Yu and Zhao [31] for a different form of Equation (3). Our results will be different from that of Tunç [25], Yu and Xiao [30], Yu and Zhao [31], [1-24, 26-30, 32-35] and the references thereof. At the end, three examples are constructed to illustrate the theoretical analysis of this work.

Consider the non-autonomous delay differential system

$$\dot{x} = F(t, x_t), x_t = x(t+\theta), -r \leq \theta \leq 0, t \geq 0, \tag{5}$$

where $F : \mathbb{R}^+ \times C_H \to \mathbb{R}^n$ is a continuous mapping, $F(t, 0) = 0$, and we suppose that F takes closed bounded sets into bounded sets of $\mathbb{R}^n$. Here $(C, \|.\|)$ is the Banach space of continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0; C_H$ is the open H-ball in $C; C_H := \{\phi \in C([-r, 0], \mathbb{R}^n) : \|\phi\| < H\}$. Let S be the set of $\varphi \in C$ such that $\|\varphi\| \geq H$. We shall denote by $S^\bullet$ the set of all functions $\varphi \in C$ such that $|\varphi(0)| \geq H$, where H is large enough.

Theorem 1.2. ([2, 3]) *Suppose that there exists a Lyapunov functional $V(t, \phi)$ for* (5) *such that the following conditions are satisfied:*

(i) $W_1(|\phi(0)|) \leq V(t, \phi)$, (*where $W_1(r)$ is a wedge*) $V(t, 0) = 0$*;*

(ii) $\dot{V}(t, \phi) \leq 0$.

Then the zero solution of (5) is stable.

Theorem 1.3. ([32]) *Suppose that there exists a continuous Lyapunov functional $V(t, \varphi)$ defined for all $t \in \mathbb{R}^+, \varphi \in S^\bullet$ which satisfies the following conditions:*

(i) $a(|\varphi(0)|) \leq V(t, \varphi) \leq b_1(|\varphi(0)|) + b_2(\|\varphi\|)$, *where* $a(r), b_1(r), b_2(r) \in CI$, ($CI$ *denotes the families of continuous increasing functions*), *and are positive for $r > H$ and $a(r) - b_2(r) \to \infty$ as $r \to \infty$;*

(ii) $\dot{V}(t, \varphi) \leq 0$.

Then the solutions of (5) *are uniformly bounded.*

2. Main Results

Let $e(t, x, x(t-\tau_1(t)), ..., x(t-\tau_m(t)), y, y(t-\tau_1(t)), ..., y(t-\tau_m(t))) = 0$ and $\gamma = \max_{1 \leq j \leq m} \gamma_j$.

Our first main result is the following theorem.

Theorem 2.1. *In addition to the basic assumptions imposed on the functions* f, b_0, b_j, g_0 *and* g_j *appearing in Equation* (3), *we assume that there exist positive constants* a_1, α_j, ρ_j *and* L_j *such that the following conditions hold:*

$$b_0(t) \geq 1, b_0'(t) \leq 0, f(t, x, y) \geq a_1,$$

$$\rho_j \geq b_j(t) \geq 1, b_j'(t) \leq 0, \frac{g_0(x)}{x} \geq \alpha_0, \frac{g_j(x)}{x} \geq \alpha_j, (x \neq 0), \left|g_j'(x)\right| \leq L_1,$$

for all $j = 1, 2, ..., m, t \in \mathbb{R}^+, x, y \in \mathbb{R}$. *If*

$$\gamma < \frac{2a_1}{\sum\limits_{j=1}^{m} \frac{L_j \rho_j}{1-\beta_j} + \sum\limits_{j=1}^{m} L_j \rho_j},$$

then the zero solution of Equation (3) *is stable.*

Proof. Define the Lyapunov functional

$$V(t, x_t, y_t) = \sum_{j=0}^{m} b_j(t) \int_0^x g_j(s)ds + \frac{1}{2}y^2 + \sum_{j=1}^{m} \lambda_j \int_{-\tau_j(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds, \tag{6}$$

where $\lambda_1, \lambda_2, ..., \lambda_m$ are some positive constants to be chosen later.
Taking into account the assumptions

$$b_j(t) \geq 1$$

and

$$\frac{g_j(x)}{x} \geq \alpha_j, (x \neq 0),\ (j = 0, 1, ..., m),$$

we obtain

$$\begin{aligned} V(t, x_t, y_t) &= \frac{1}{2}y^2 + \sum_{j=0}^{m} b_j \int_0^x \frac{g_j(s)}{s} s ds + \sum_{j=1}^{m} \lambda_j \int_{-\tau_j(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds \\ &\geq \frac{1}{2}\sum_{j=0}^{m} \alpha_j x^2 + \frac{1}{2}y^2 + \sum_{j=1}^{m} \lambda_j \int_{-\tau_j(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds \end{aligned}$$

so that

$$V(t, x_t, y_t) \geq D_1(x^2 + y^2), \tag{7}$$

where $D_1 = \frac{1}{2}\min\{\sum\limits_{j=0}^{m} \alpha_j, 1\}$.

Thus, one can conclude that there exists a continuous function $\phi(s) \geq 0$ with $W_1(|\phi(0)|) \geq 0$ by choosing $W_1(|\phi(0)|) = D_1 \left|\phi(0)\right|^2$, the condition (i) in Theorem 1.2 is satisfied.

On the other hand, by an easy calculation, the time derivative of the functional $V(t, x_t, y_t)$ along the solutions of (4) gives

$$\begin{aligned} \frac{d}{dt}V(t,x_t,y_t) = &- f(t,x,y)y^2 \\ &+ y\sum_{j=1}^{m} b_j(t) \int_{t-\tau_j(t)}^{t} g_j'(x(s))y(s)ds + \sum_{j=0}^{m} b_j'(t)\int_0^x g_j(s)ds \\ &+ \sum_{j=1}^{m}\lambda_j\tau_j(t)y^2 - \sum_{j=1}^{m}\lambda_j\{1-\tau_j'(t)\}\int_{t-\tau_j(t)}^{t} y^2(s)ds. \end{aligned} \tag{8}$$

By using the assumptions of Theorem 2.1 and the inequality $2|uv| \le u^2 + v^2$, we get the following estimates for the terms included on the right hand side of (8):

$$- f(t,x,y)y^2 \le -a_1 y^2,$$

$$\sum_{j=0}^{m} b_j'(t)\int_0^x g_j(s)ds = \sum_{j=0}^{m} b_j'(t)\int_0^x \frac{g_j(s)}{s} s ds \le 0,$$

$$\begin{aligned} y\sum_{j=1}^{m} b_j(t)\int_{t-\tau_j(t)}^{t} g_j'(x(s))y(s)ds &\le \frac{1}{2}\sum_{j=1}^{m}(\rho_j L_j \tau_j(t))y^2 + \frac{1}{2}\sum_{j=1}^{m}(\rho_j L_j)\int_{t-\tau_j(t)}^{t} y^2(s)ds \\ &\le \frac{1}{2}\sum_{j=1}^{m}(\rho_j L_j\gamma_j)y^2 + \frac{1}{2}\sum_{j=1}^{m}\rho_j L_j\int_{t-\tau_j(t)}^{t} y^2(s)ds, \end{aligned}$$

$$-\sum_{j=1}^{m}\{\lambda_j(1-\tau_j'(t))\int_{t-\tau_j(t)}^{t} y^2(s)ds\} \le -\sum_{j=1}^{m}\{\lambda_j(1-\beta_j)\int_{t-\tau_j(t)}^{t} y^2(s)ds\},$$

$$\sum_{j=1}^{m}\lambda_j\tau_j(t)y^2 \le \sum_{j=1}^{m}\lambda_j\gamma_j y^2 \le \sum_{j=1}^{m}\gamma\lambda_j y^2 = (\gamma L)y^2,$$

where $\gamma = \max_{1\le j\le m}\gamma_j$ and $L = \sum_{j=1}^{m}\lambda_j$.

In view of the above discussion, we have

$$\frac{d}{dt}V(t,x_t,y_t) \le -\{a_1 - \gamma(L+2^{-1}N)\}y^2 - \sum_{j=1}^{m}\{\lambda_j(1-\beta_j) - \frac{1}{2}\rho_j L_j\}\int_{t-\tau_j(t)}^{t} y^2(s)ds,$$

where $N = \sum_{j=1}^{m}(\rho_j L_j)$.

Choose λ_j as follows

$$\lambda_j = \frac{L_j \rho_j}{2(1-\beta_j)}, \ (j = 1, 2, ..., m).$$

Hence, the above estimate implies

$$\begin{aligned} \frac{d}{dt}V(t, x_t, y_t) &\leq -\{a_1 - \gamma(L + 2^{-1}N)\}y^2 \\ &= -(a_1 - M\gamma)y^2, \end{aligned} \tag{9}$$

where $M = L + 2^{-1}N$, the constants L and N are defined above.
Thus, by virtue of (9) for a positive constant α, we obtain

$$\frac{d}{dt}V(t, x_t, y_t) \leq -\alpha y^2 \leq 0$$

provided that

$$\gamma < \frac{a_1}{M}.$$

As a result, subject to the above discussion, one can conclude that the zero solution of Equation (3) is stable. The proof of Theorem 2.1 is complete. ■

Example 2.2. Consider the nonlinear differential equation of second order with two deviating arguments:

$$\begin{aligned} x'' + \{t^2 + 3x^2 + 3(x')^2 + 7\}x' + \{(1 + \frac{1}{1+t^2})(x^3 + 2x)\} \\ + 2x(t - \tau_1(t)) + 2x(t - \tau_2(t)) = 0, \end{aligned}$$

which is a special case of Equation (3).

We write the above equation in system form as follows

$$\begin{aligned} x' &= y, \\ y' &= -(t^2 + 3x^2 + 3y^2 + 7)y - \{(1 + \frac{1}{1+t^2})(x^3 + 2x)\} \\ &\quad - 4x + 2\int_{t-\tau_1(t)}^{t} y(s)ds + 2\int_{t-\tau_2(t)}^{t} y(s)ds. \end{aligned}$$

Hence, we have

$$\begin{aligned} &f(t, x, y) = t^2 + 3x^2 + 3y^2 + 7 \geq 7 = a_1, \\ &b_0(t) = 1 + \frac{1}{1+t^2} \geq 1, \ b_0'(t) = -\frac{2t}{(1+t^2)^2} \leq 0, t \geq 0, \\ &g_0(x) = x^3 + 2x, g_0(0) = 0, \ \frac{g_0(x)}{x} = x^2 + 2 \geq 2 = \alpha_0, \end{aligned}$$

$$\begin{aligned}
&b_1(t) = 1,\ b_2(t) = 1,\\
&g_1(x) = 2x,\ g_1(0) = 0,\\
&\frac{g_1(x)}{x} \geq 2 = \alpha_1,\ g_1'(x) = 2, |g_1'(x)| = 2 = L_1,\\
&g_2(x) = 2x,\ g_2(0) = 0,\ \frac{g_2(x)}{x} \geq 2 = \alpha_2,\ g_2'(x) = 2,\ |g_2'(x)| = 2 = L_2.
\end{aligned}$$

If

$$\gamma < \frac{a_1}{M} = \frac{7(1-\beta_1)(1-\beta_2)}{2-(\beta_1+\beta_2)+2(1-\beta_1)(1-\beta_2)},\ 0 < \beta_1 < 1,\ 0 < \beta_2 < 1,$$

then all the assumptions of Theorem 2.1 hold. Therefore, the zero solution of the above equation is stable.

Let $e(t, x, x(t-\tau_1(t)), ..., x(t-\tau_m(t)), y, y(t-\tau_1(t)), ..., y(t-\tau_m(t))) \neq 0$. The second main result of this paper is the following theorem.

Theorem 2.3. *We assume that all the assumptions of Theorem* 2.1 *and the following assumptions*

$$|e(t, x, x(t-\tau_1(t)), ..., x(t-\tau_m(t)), y, y(t-\tau_1(t)), ..., y(t-\tau_m(t)))| \leq |p(t)|,$$

$$\int_0^t |p(s)|\, ds < \infty$$

hold. If

$$\gamma < \frac{2a_1}{\sum_{j=1}^{m} \frac{L_j \rho_j}{1-\beta_j} + \sum_{j=1}^{m} L_j \rho_j},$$

then for any solution x of Equation (3)*, there exists a positive constant K such that*

$$|x| \leq \sqrt{K}, |x'| \leq \sqrt{K}$$

for all $t \geq t_0 \geq 0$.

Proof. Taking into account the assumptions of Theorem 2.3, it is easy to see that

$$\begin{aligned}
&\frac{d}{dt}V(t, x_t, y_t)\\
\leq\ & -\alpha y^2 + ye(t, x, x(t-\tau_1(t)), ..., x(t-\tau_m(t)), y, y(t-\tau_1(t)), ..., y(t-\tau_m(t))).
\end{aligned}$$

By using the assumptions of Theorem 2.3 and the estimates $|y| < 1 + y^2$ and $y^2 \leq D_1^{-1} V(t, x_t, y_t)$, we have

$$\begin{aligned}
&\frac{d}{dt}V(t, x_t, y_t)\\
\leq\ & |y|\, |e(t, x, x(t-\tau_1(t)), ..., x(t-\tau_m(t)), y, y(t-\tau_1(t)), ..., y(t-\tau_m(t)))|
\end{aligned}$$

$$\le (1+y^2)\,|p(t)|$$
$$\le |p(t)| + D_1^{-1}V(t,x_t,y_t)\,|p(t)|\,. \tag{10}$$

Integrating both sides of (10) over $[0,t]$, using the assumption $\int_0^t |p(s)|\,ds < \infty$ and the Gronwall-Bellman inequality, (see Ahmad and Rao [1]), it follows that

$$V(t,x_t,y_t) \le \{V(0,x_0,y_0)+A\}\exp(D_1^{-1}A) < \infty, \tag{11}$$

where $A = \int_0^\infty |p(s)|\,ds$.

In view of (7) and (11), one can conclude that all solutions of Equation (3) are bounded. The proof of Theorem 2.3 is complete. ■

Example 2.4. Consider the differential equation of second order with two deviating arguments:

$$x'' + \{t^2 + 3x^2 + 3(x')^2 + 7\}x' + \{(1+\frac{1}{1+t^2})(x^3+2x)\}$$
$$+ 2x(t-\tau_1(t)) + 2x(t-\tau_2(t))$$
$$= \frac{2+\sin t+\cos t}{1+t^2+x^2+...+x^2(t-\tau_2(t))+x'^2+...+x'^2(t-\tau_2(t))},$$

which is a special case of Equation (3).

It follows that

$$|e(t,x,...,.x(t-\tau_2(t)),y,...,y(t-\tau_2(t)))| \le \frac{4}{1+t^2} = |p(t)|\,.$$

Hence, we get

$$\int_0^\infty |p(s)|\,ds = \int_0^\infty \frac{4}{1+s^2}ds = 2\pi < \infty.$$

In view of the discussion made above and that in Example 2.2, it can be seen that all the assumptions of Theorem 2.3 hold. Thus, we conclude that all solutions of the above equation are bounded.

Remark 2.5. To the best of our observations from the literature, we did not find any work on the stability and boundedness of the solutions of Liénard type equations with multiple variable deviating arguments, which uses Lyapunov's functional approach (see Krasovskii [9]), except that in [25]. That is to say that Theorems 2.1 and 2.3 raise two new results in the literature related to the stability and boundedness of solutions of Liénard type equations with multiple variable deviating arguments. This fact is a new improvement on the topic. It also follows that our assumptions are very elegant and understandable and their applicability can be easily verified.

The last main result of this paper is given by the following theorem.

Theorem 2.6. *In addition to the assumptions of Theorem* 2.3*, except*

$$\frac{g_j(x)}{x} \geq \alpha_j, \ (x \neq 0),$$

we suppose that

$$xg_j(x) > 0, (x \neq 0), \int_0^x g_j(s)ds \to +\infty \ \text{as} \ |x| \to \infty, \ (j = 0, 1, 2, ..., m).$$

If

$$\gamma < \frac{2a_1}{\sum_{j=1}^{m} \frac{L_j \rho_j}{1-\beta_j} + \sum_{j=1}^{m} L_j \rho_j},$$

then solutions of Equation (3) *are uniformly bounded.*

Proof. Define the Lyapunov functional

$$\begin{aligned} V_1(t, x_t, y_t) = {} & \exp(-2\int_0^t |p(s)|\, ds)\{\sum_{j=0}^{m} b_j(t) \int_0^x g_j(s)ds \\ & + \frac{1}{2}y^2 + 1 + \sum_{j=1}^{m} \lambda_j \int_{-\tau_j(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds\} \end{aligned}$$

so that

$$\begin{aligned} & \exp(-2\int_0^\infty |p(s)|\, ds)\{\sum_{j=0}^{m} b_j(t) \int_0^x g_j(s)ds + \frac{1}{2}y^2 + 1 + \sum_{j=1}^{m} \lambda_j \int_{-\tau_j(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds\} \\ & \leq V_1(t, x_t, y_t) \leq \{\sum_{j=0}^{m} b_j(t) \int_0^x g_j(s)ds + \frac{1}{2}y^2 + 1 + \sum_{j=1}^{m} \lambda_j \int_{-\tau_j(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds\}. \end{aligned}$$

Taking into account the assumption

$$\int_0^x g_j(s)ds \to +\infty \ \text{as} \ |x| \to \infty, \ (j = 0, \ 1, 2, ..., m),$$

and the last estimate, one can conclude that condition (i) of Theorem 1.3 holds.

By using the Lyapunov functional $V_1(t, x_t, y_t)$, the system (4) and the assumptions of Theorem 2.6, the time derivative of $V_1(t, x_t, y_t)$ can be estimated as follows:

$$\begin{aligned}
\frac{d}{dt}V_1(t,x_t,y_t) \leq & -2\,|p(t)|\exp(-2\int\limits_0^t |p(s)|\,ds)\{\sum_{j=0}^m b_j(t)\int\limits_0^x g_j(s)ds \\
& + \frac{1}{2}y^2 + 1 + \sum_{j=1}^m \lambda_j \int\limits_{-\tau_j(t)}^0 \int\limits_{t+s}^t y^2(\theta)d\theta ds\} \\
& - \alpha y^2 \exp(-2\int\limits_0^t |p(s)|\,ds) + \exp(-2\int\limits_0^t |p(s)|\,ds) \\
& \times\ ye(t,x,x(t-\tau_1(t)),...,x(t-\tau_m(t)),y,y(t-\tau_1(t)),...,y(t-\tau_m(t))) \\
\leq & -2\,|p(t)|\exp(-2\int\limits_0^t |p(s)|\,ds)\{\sum_{j=0}^m b_j(t)\int\limits_0^x g_j(s)ds + \frac{1}{2}y^2 + 1\} \\
& - \alpha y^2 \exp(-2\int\limits_0^t |p(s)|\,ds) + \exp(-2\int\limits_0^t |p(s)|\,ds)\,|p(t)| \\
& + y^2\,|p(t)|\exp(-2\int\limits_0^t |p(s)|\,ds) \\
\leq & -\alpha y^2 \exp(-2\int\limits_0^t |p(s)|\,ds) \leq 0.
\end{aligned}$$

In view of the discussion made in the proof of Theorem 2.6, one can conclude that all solutions of Equation (3) are uniformly bounded. The proof of Theorem 2.6 is complete. ■

Example 2.7. As a special case of Equation (3), we consider the nonlinear differential equation of second order with two deviating arguments:

$$\begin{aligned}
& x'' + \{t^2 + 3x^2 + 3(x')^2 + 7\}x' + \{(1 + \frac{1}{1+t^2})(x^3 + 2x)\} \\
& + 2x(t-\tau_1(t)) + 2x(t-\tau_2(t)) \\
= & \frac{1}{1 + t^2 + x^2 + ... + x^2(t-\tau_2(t)) + x'^2 + ... + x'^2(t-\tau_2(t))}.
\end{aligned}$$

We write the above equation in system form as follows

$$\begin{aligned}
x' &= y, \\
y' &= -(t^2 + 3x^2 + 3y^2 + 7)y - \{(1 + \frac{1}{1+t^2})(x^3 + 2x)\} \\
&\quad - 4x + 2\int_{t-\tau_1(t)}^t y(s)ds + 2\int_{t-\tau_2(t)}^t y(s)ds
\end{aligned}$$

$$+\frac{1}{1+t^2+x^2+...+x^2(t-\tau_2(t))+y^2+...+y^2(t-\tau_2(t))}.$$

Hence, it follows that

$$g_0(x)=x^3+2x,\ g_0(0)=0,$$
$$xg_0(x)=x^4+2x^2>0,\ (x\neq 0),$$
$$\int_0^x g_0(s)ds=\int_0^x (s^3+2s)ds=\frac{x^4}{4}+x^2\to+\infty \text{ as } |x|\to\infty,$$
$$g_1(x)=2x,\ g_1(0)=0,\ xg_1(x)=2x^2>0,\ (x\neq 0),$$
$$\int_0^x g_1(s)ds=\int_0^x 2sds=x^2\to+\infty \text{ as } |x|\to\infty,$$
$$g_2(x)=2x,\ g_2(0)=0,\ xg_2(x)=2x^2>0,\ (x\neq 0),$$
$$\int_0^x g_2(s)ds=\int_0^x 2sds=x^2\to+\infty \text{ as } |x|\to\infty.$$

In view of the discussion made above and that in Examples 2.2 and 2.4, it can be seen that all the assumptions of Theorem 2.6 hold. The above estimates show that all solutions of the equation considered are uniformly bounded.

Remark 2.8. When we take into account the assumptions of Theorem 1.1 and Theorem 2.6, it can be seen that our assumptions are completely different from those of Yu and Xiao [30]. The applicability of the assumptions of Theorem 2.6 to Liénard type equations with multiple variable deviating arguments can be easily observed. The procedure here used in the proof of Theorem 2.6 is completely different from that used in [30]. Theorem 2.6 also establishes a complementary result for the Liénard equation considered in [9].

3. Conclusion

Liénard type equations with multiple variable deviating arguments are considered. The stability, boundedness and uniform boundedness of solutions of these equations are investigated for some specific cases determined. In proving our results, we employ the Lyapunov functional approach by defining two new Lyapunov functionals. Three examples are also constructed to illustrate our theoretical findings.

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References

1. S. Ahmad, M. Rama Mohana Rao, *Theory of ordinary differential equations. With applications in biology and engineering*, Affiliated East-West Press Pvt. Ltd., New Delhi, 1999.
2. T.A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, 1985.
3. T.A. Burton, R.H. Hering, Liapunov theory for functional-differential equations, 20th Midwest ODE Meeting (Iowa City, IA, 1991), *Rocky Mountain J. Math*, **24** (1994), 3-17.
4. L.È. Èl'sgol'ts, *Introduction to the theory of differential equations with deviating arguments*, Translated from the Russian by Robert J. McLaughlin Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam, 1966.
5. J.K. Hale, Sufficient conditions for stability and instability of autonomous functional-differential equations. *J. Differential Equations*, **1** (1965) 452-482.
6. J.W. Heidel, Global asymptotic stability of a generalized Liénard equation, *SIAM J. Appl. Math.*, **19** (1970), 629-636.
7. J. Kato, On a boundedness condition for solutions of a generalized Liénard equation, *J. Differential Equations*, **65** (1986), 269-286.
8. J. Kato, A simple boundedness theorem for a Liénard equation with damping, *Ann. Polon. Math.*, **51** (1990), 183-188.
9. N.N. Krasovskii, *Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay*, Stanford, Calif.: Stanford University Press 1963.
10. B. Liu, L. Huang, Boundedness of solutions for a class of retarded Liénard equation, *J. Math. Anal. Appl.*, **286** (2003), 422-434.
11. B. Liu, L. Huang, Boundedness of solutions for a class of Liénard equations with a deviating argument, *Appl. Math. Lett.*, **21** (2008), 109-112.
12. W.S. Luk, Some results concerning the boundedness of solutions of Lienard equations with delay, *SIAM J.Appl. Math.*, **30** (1976), 768-774.
13. I.A. Malyseva, Boundedness of solutions of a Liénard differential equation, *Differetial'niye Uravneniya*, **15** (1979), 1420-1426.
14. M. Mureşan, Boundedness of solutions for Liénard type equations, *Mathematica*, **40**(63) (1998), 243-257.
15. O.G. Mustafa, C. Tunç, Asymptotically linear solutions of differential equations via Lyapunov functions, *Appl. Math. Comput.*, **215** (2009), 3076-3081.
16. Nápoles Valdés, E. Juan, Boundedness and global asymptotic stability of the forced Liénard equation, (Spanish) *Rev. Un. Mat. Argentina*, **41** (2000), 47-59.
17. J. Sugie, On the boundedness of solutions of the generalized Liénard equation without the signum condition, *Nonlinear Anal.*, **11** (1987), 1391-1397.
18. J. Sugie, Y. Amano, Global asymptotic stability of non-autonomous systems of Liénard type, *J. Math. Anal. Appl.*, **289** (2004), 673-690.
19. J. Sugie, D.L. Chen, H. Matsunaga, On global asymptotic stability of systems of Liénard type, *J. Math. Anal. Appl.*, **219** (1998), 140-164.
20. C. Tunç, Some new stability and boundedness results on the solutions of the nonlinear vector differential equations of second order, *Iran. J. Sci. Technol. Trans. A Sci.*, **30** (2006), 213-221.
21. C. Tunç, A new boundedness theorem for a class of second order differential equations, *Arab. J. Sci. Eng.Sect.A Sci.*, **33** (2008), 83-92.
22. C. Tunç, Some stability and boundedness results to nonlinear differential equations of Liénard type with finite delay, *J.Comput.Anal. Appl.*, **11** (2009), 711-727.
23. C. Tunç, Some new stability and boundedness results of solutions of Liénard type equations with a deviating argument, *Nonlinear Anal. Hybrid Syst.*, **4** (2010), 85-91.

24. C. Tunç, A note on boundedness of solutions to a class of non-autonomous differential equations of second order, *Appl. Anal. Discrete Math.*, **4** (2010), 361-372.
25. C. Tunc, New stability and boundedness results of Liénard type equations with multiple deviating arguments, *Izv. Nats. Akad. Nauk Armenii Mat.*, **45** (2010), 47-56.
26. C. Tunç, Boundedness results for solutions of certain nonlinear differential equations of second order, *J. Indones. Math. Soc.*, **16** (2010), 115-128.
27. C. Tunç, Stability and boundedness of solutions of non-autonomous differential equations of second order, *J. Comput. Anal. Appl.*, **13** (2011), 1067-1074.
28. C. Tunç, E. Tunç, On the asymptotic behavior of solutions of certain second-order differential equations, *J. Franklin Inst.*, **344** (2007), 391-398.
29. G.R. Ye, H.S. Ding, X.L. Wu, Uniform boundedness of solutions for a class of Liénard equations, *Electron. J. Differential Equations 2009*, no. 97, 5 pp.
30. Y. Yu, B. Xiao, Boundedness of solutions for a class of Liénard equation with multiple deviating arguments, *Vietnam J. Math.*, **37** (2009), 35-41.
31. Y. Yu, C. Zhao, Boundedness of solutions for a Liénard equation with multiple deviating arguments, *Electron. J. Differential Equations* 2009, no. 14, 5 pp.
32. T. Yoshizawa, Stability theory by Liapunov's second method, *Publications Math. Soc. Japan*, no. 9, The Mathematical Society of Japan, Tokyo 1966.
33. B. Zhang, Boundedness and stability of solutions of the retarded Liénard equation with negative damping, *Nonlinear Anal.*, **20**(1993), 303-313.
34. J. Zhou, L. Xiang, On the stability and boundedness of solutions for the retarded Liénard-type equation, *Ann. Differential Equations*, **15** (1999), 460-465.
35. X.F. Zhou, W. Jiang, Stability and boundedness of retarded Liénard-type equation, *Chinese Quart. J. Math.*, **18** (2003), 7-12.