Vietnam Journal of Mathematics **39**:2(2011) 191-215

Vietnam Journal of MATHEMATICS

Generalized Quasi-Equilibrium Problems of Type II and Their Applications*

Nguyen Xuan Tan[1] **and Nguyen Thi Quynh Anh**[2]

[1] *Institute of Mathematics, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam.*

[2] *Department of basic sciences, Thai Nguyen university of information and communication Technology, Thai Nguyen, Vietnam.*

Received March 11, 2011
Revised May 18, 2011

Abstract. The generalized quasi-equilibrium problem of type II is formulated and some sufficient conditions on the existence of its solutions are shown. As special cases, we obtain several results on the existence of solutions to ideal quasivariational inclusion problems, quasivariational relation problems of type II, generalized quasi-KKM theorems, etc. As corollaries, we shall show several results on the existence of solutions to other problems in the vector optimization theory concerning multivalued mappings.

2000 Mathematics Subject Classification: 49J27, 49J53, 91B50, 90C48.

Key words: Generalized quasi-equilibrium problems of type II, upper and lower quasivariational inclusions, quasi-optimization problems, upper and lower $\mathcal{C}$-quasiconvex multivalued mappings, upper and lower $\mathcal{C}$- continuous multivalued mappings.

1. Introduction

Throughout this paper X, Z and Y are supposed to be real topological vector locally convex Hausdorff spaces, $D \subset X, K \subset Z$ are nonempty subsets. Given multivalued mappings $S : D\times K \to 2^D, T : D\times K \to 2^K; P_1 : D \to 2^D, P_2 : D \to 2^D, Q : K \times D \to 2^K$ and $F_1 : K \times D \times D \times D \to 2^Y, F : K \times D \times D \to 2^Y$,we are interested in the following problems:

* This work was supported by the NAFOSTED of Vietnam

(A) Find $(\bar{x}, \bar{y}) \in D \times K$ such that

1. $\bar{x} \in S(\bar{x}, \bar{y})$;
2. $\bar{y} \in T(\bar{x}, \bar{y})$;
3. $0 \in F_1(\bar{y}, \bar{x}, \bar{x}, z), \quad \text{for all} \quad z \in S(\bar{x}, \bar{y})$.

This problem is called a generalized quasi-equilibrium problem of type I and is denoted by $(GEP)_I$.

(B) Find $\bar{x} \in D$ such that

$$\bar{x} \in P_1(\bar{x})$$

and

$$0 \in F(y, \bar{x}, t), \text{ for all } \ t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This problem is called a generalized quasi-equilibrium problem of type II and is denoted by $(GEP)_{II}$.

In the above problems, the multivalued mappings S, T, P_1, P_2 and Q are constraints, F_1 and F are utility multivalued mappings that are often determined by equalities and inequalities, or by inclusions, not inclusions and intersections of other multivalued mappings, or by some relations in product spaces. The generalized quasi-equilibrium problems of type I are studied in [3]. In this paper we consider the existence of solutions to the second ones. The typical instances of generalized quasi-equilibrium problems of type II are the following:

(i) Quasi-equilibrium problem. Let $D, K, P_i, i = 1, 2, Q$ be as above. Let $\mathbb{R}$ be the space of real numbers with the subset of nonnegative numbers $\mathbb{R}_+$ and $\Phi : K \times D \times D \to \mathbb{R}$ be a function with $\Phi(y, x, x) = 0$, for all $y \in K, x \in D$. The generalized quasi-equilibrium problem $(GEP)_{II}$ is defined as follows: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in \Phi(y, \bar{x}, t) - \mathbb{R}_+, \text{ for all } t \in P_2(\bar{x}) \ \text{ and } y \in Q(\bar{x}, t).$$

This problem is known as a quasi-equilibrium problem: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$\Phi(y, \bar{x}, t) \geq 0, \text{ for all } \ t \in P_2(\bar{x}) \ \text{ and } \ y \in Q(\bar{x}, t).$$

This problem is studied by many authors, for examples, in [4], [6], [9], [13] and in the references therein.

(ii) Minty quasivariational problem. Let $< .,. >: X \times Z \to \mathbb{R}$ be a continuous bilinear function. We consider the following Minty quasivariational problem: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$< y, t - \bar{x} > \geq 0, \text{ for all } \ t \in P_2(\bar{x}) \ \text{ and } \ y \in Q(\bar{x}, t).$$

Setting $F(y, x, t) = < y, t - x > -\mathbb{R}_+, (GEP)_{II}$ reads as follows: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \bar{x}, t) \text{ ,for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

(iii) Ideal quasivariational inclusion problem of type II. Let D, K, Y, P_i, $i = 1, 2$, and Q be given as at the beginning of this section. Further, assume that $\mathcal{C} : K \times D \to 2^Y$ is a cone multivalued mapping (for any $(y, x) \in K \times D, \mathcal{C}(y, x)$ is a cone in Y) and G and H are multivalued mappings on $K \times D \times D$ with values in the space Y. We define the multivalued mappings $M : K \times D \to 2^X; F : K \times D \times D \to 2^Y$ by

$$M(y, x) = \{t \in D \mid G(y, x, t) \subseteq H(y, x, x) + \mathcal{C}(y, x)\},\ (y, x) \in K \times D$$

and

$$F(y, x, t) = t - M(y, x),\ (y, x, t) \in K \times D \times D.$$

Problem $(GEP)_{II}$ is formulated as follows: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \bar{x}, t),\ \text{ for all }\ t \in P_2(\bar{x})\ \text{ and }\ y \in Q(\bar{x}, t).$$

This shows

$$G(y, \bar{x}, t) \subseteq H(y, \bar{x}, \bar{x}) + \mathcal{C}(y, \bar{x}),\ \text{ for all }\ t \in P_2(\bar{x})\ \text{ and }\ y \in Q(\bar{x}, t).$$

This is an upper quasivariational inclusion problem studied in [9], [11], [12] and in the references therein.

(iv) Abstract quasivariational relation problem of type II. Let $D, K, P_1, i = 1, 2, Q$ be as above. Let $\mathcal{R}(y, x, t)$ be a relation linking $y \in K, x, t \in D$. We define the multivalued mappings $M : K \times D \to 2^X; F : K \times D \times D \to 2^Y$ by

$$M(y, x) = \{t \in D \mid \mathcal{R}(y, x, t) \text{ holds}\};$$

and

$$F(y, x, t) = t - M(y, x),\ (y, x, t) \in K \times D \times D.$$

Problem $(GEP)_{II}$ is formulated as follows: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \bar{x}, t), \text{ for all }\ t \in P_2(\bar{x})\ \text{ and }\ y \in Q(\bar{x}, t).$$

This becomes to find $\bar{x} \in D$ such that

$$\bar{x} \in P_1(\bar{x})$$

and

$$\mathcal{R}(y, \bar{x}, t) \text{ holds for all }\ t \in P_2(\bar{x})\ \text{ and }\ y \in Q(\bar{x}, t).$$

This is (VR) studied in [9].

(v) Differential inclusion. Let $D \subset C^1[a, b]$ be a nonempty set, where $C[a, b]$ and $C^1[a, b]$ are the spaces of continuous and continuously diffrentiable functions on the interval $[a, b]$, respectively. Let P_1, P_2 be given as above. Let Ω be a nonempty set and $U : D \times D \to 2^{\Omega}$ a multivalued mapping. Set $K = \Omega \times \mathbb{R}$ and $Q : D \times D \to 2^Y$ by $Q(x, t) = U(x, t) \times [a, b]$. Given a multivalued mapping $G : K \times D \times D \to 2^{C[a,b]}$. Problem of finding $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$x' \in G(y,\xi,\bar{x},t), \text{ for all } t \in P_2(\bar{x}) \text{ and } (y,\xi) \in Q(\bar{x},t),$$

studied in [5] becomes to find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y,\xi,\bar{x},t), \text{ for all } t \in P_2(\bar{x}) \text{ and } (y,\xi) \in Q(\bar{x},t),$$

where $F(y,\xi,x,t) = x' - G(y,\xi,x,t)$ and x' denotes the derivative of x.

2. Preliminaries and Definitions

Throughout this paper, as in the introduction, by X, Z and Y we denote real topological vector locally convex Hausdorff spaces. Given a subset $D \subseteq X$, we consider a multivalued mapping $F : D \to 2^Y$. The domain and the graph of F are denoted and defined by

$$\text{dom}F = \{x \in D |\ F(x) \neq \emptyset\},$$

$$\text{Gr}(F) = \{(x,y) \in D \times Y |\ y \in F(x)\},$$

respectively. We recall that F is said to be a closed mapping if the graph $Gr(F)$ of F is a closed subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $\text{cl}F(D)$ of its range $F(D)$ is a compact set in Y. A multivalued mapping $F : D \to 2^Y$ is said to be upper semicontinuous (lower semicontinuous), briefly: u.s.c (respectively, l.s.c) at $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$ (respectively, $F(\bar{x}) \cap U \neq \emptyset$), there exists an open set U of $\bar{x}$ that $F(x) \subseteq V$ (respectively, $F(x) \cap U \neq \emptyset$) for each $x \in U$ and F is said to be u.s.c (l.s.c) on D if it is u.s.c (respectively, l.s.c) at any point $x \in D$. These notions and definitions can be found in [1].

Further, let Y be a topological vector locally convex Hausdorff space with a cone C. We denote $l(C) = C \cap (-C)$. If $l(C) = \{0\}, C$ is said to be pointed. Let $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping. We recall the following definitions:

Definition 2.1. Let $F : K \times D \times D \to 2^Y$ be a multivalued mapping and, $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping.

(i) F is said to be upper (lower) $\mathcal{C}$-continuous at $(\bar{y},\bar{x},\bar{t}) \in \text{dom } F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $(\bar{y},\bar{x},\bar{t})$ such that:

$$F(y,x,t) \subseteq F(\bar{y},\bar{x},\bar{t})) + V + \mathcal{C}(\bar{y},\bar{x})$$

$$(\text{respectively}, F(\bar{y},\bar{x},\bar{t}) \subseteq F(y,x,t) + V - \mathcal{C}(\bar{y},\bar{x}))$$

holds for all $(y,x,t) \in U \cap \text{dom}F$;

(ii) If F is upper $\mathcal{C}$-continuous and lower $\mathcal{C}$-continuous at $(\bar{y},\bar{x},\bar{t})$ simultaneously, we say that it is $\mathcal{C}$-continuous at $(\bar{y},\bar{x},\bar{t})$;

(iii) If F is upper, lower,..., $\mathcal{C}$-continuous at any point of dom$F$, we say that it is upper, lower,..., $\mathcal{C}$-continuous on D;

(iv) In the case $\mathcal{C} = \{0\}$, a trivial cone in Y, we say that F is upper, lower continuous instead of upper, lower 0-continuous. And, F is continuous if it is upper and lower continuous simultaneously.

Definition 2.2. Let G be a multivalued mapping from D to 2^Y and C be a cone in Y. We say that:

(i) G is upper C-quasiconvex-like on D if for any $x_1, x_2 \in D,\ t \in [0,1]$, either

$$G(x_1) \subseteq G(tx_1 + (1-t)x_2) + C$$

or

$$G(x_2) \subseteq G(tx_1 + (1-t)x_2) + C$$

holds;

(ii) G is lower C-quasiconvex-like on D if for any $x_1, x_2 \in D,\ t \in [0,1]$, either

$$G(tx_1 + (1-t)x_2) \subseteq G(x_1) - C$$

or

$$G(tx_1 + (1-t)x_2) \subseteq G(x_2) - C$$

holds.

Definition 2.3. Let $F : K \times D \times D \to 2^Y, Q : D \times D \to 2^K$ be multivalued mappings. Let $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping. We say that

(i) F is diagonally upper $(Q, \mathcal{C})$-quasiconvex-like in the third variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x \in \text{co}\{x_1, ..., x_n\}$ there is an index $j \in \{1, ..., n\}$ such that

$$F(y, x, x_j) \subseteq F(y, x, x) + \mathcal{C}(y, x),\ \text{for all}\ \ y \in Q(x, x_j);$$

(ii) F is diagonally lower $(Q, \mathcal{C})$-quasiconvex-like in the third variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x \in \text{co}\{x_1, ..., x_n\}$ there is an index $j \in \{1, ..., n\}$ such that

$$F(y, x, x) \subseteq F(y, x, x_j) - \mathcal{C}(y, x),\ \text{for all}\ \ y \in Q(x, x_j).$$

Definition 2.4. Let $F : K \times D \times D \to 2^X, Q : D \times D \to 2^K$ be multivalued mappings. We say that F is Q-KKM if for any finite set $\{t_1, ..., t_n\} \subset D$ and $x \in \text{co}\{t_1, ..., t_n\}$, there is $t_j \in \{t_1, ..., t_n\}$ such that $0 \in F(y, x, t_j)$, for all $y \in Q(x, t_j)$.

Definition 2.5. Let $\mathcal{R}$ be a binary relation on $K \times D$. We say that $\mathcal{R}$ is closed if for any net (y_α, x_α) converging to (y, x) and $\mathcal{R}(y_\alpha, x_\alpha)$ holds for all α then $\mathcal{R}(y, x)$ holds.

Definition 2.6. Let $\mathcal{R}$ is a relation on $K \times D \times D$. We say that $\mathcal{R}$ is Q- KKM if for any finite set $\{t_1, ..., t_n\} \subset D$ and $x \in \text{co}\{t_1, ..., t_n\}$, there is a $t_j \in \{t_1, ..., t_n\}$ such that $\mathcal{R}(y, x, t_j)$ holds, for all $y \in Q(x, t_j)$.

Now, we give some necessary and sufficient conditions on the upper and the lower $\mathcal{C}$- continuities which we shall need in the next section.

Proposition 2.7. *Let $F : K \times D \times D \to 2^Y$ be a multivalued mapping and $\mathcal{C} : K \times D \to 2^Y$ be a cone upper continuous multivalued mapping with nonempty convex closed values.*

(1) *If F is upper $\mathcal{C}$-continuous at $(y_0, x_0, t_0) \in \text{dom}F$ with $F(y_0, x_0, t_0)) + \mathcal{C}(y_0, x_0)$ closed, then for any net $(y_\beta, x_\beta, t_\beta) \to (y_0, x_0, t_0)$, $v_\beta \in F(y_\beta, x_\beta, t_\beta) + \mathcal{C}(y_\beta, x_\beta), v_\beta \to v_0$ imply $v_0 \in F(y_0, x_0, t_0) + \mathcal{C}(y_0, x_0)$.*
Conversely, if F is compact and for any net $(y_\beta, x_\beta, t_\beta) \to (y_0, x_0, t_0)$, $v_\beta \in F(y_\beta, x_\beta, t_\beta) + \mathcal{C}(y_\beta, x_\beta), v_\beta \to v_0$ imply $v_0 \in F(y_0, x_0, t_0) + \mathcal{C}(y_0, x_0)$, then F is upper $\mathcal{C}$-continuous at (y_0, x_0, t_0).

(2) *If F is compact and lower $\mathcal{C}$-continuous at $(y_0, x_0, t_0) \in \text{dom}F$, then for any net $(y_\beta, x_\beta, t_\beta) \to (y_0, x_0, t_0), v_0 \in F(y_0, x_0, t_0) + \mathcal{C}(y_0, x_0)$, there is a net $\{v_\beta\}, v_\beta \in F(y_\beta, x_\beta, t_\beta)$, which has a convergent subnet $\{v_{\beta_\gamma}\}, v_{\beta_\gamma} - v_0 \to c \in \mathcal{C}(y_0, x_0)$(i.e $\quad v_{\beta_\gamma} \to v_0 + c \in v_0 + \mathcal{C}(y_0, x_0)$.*
Conversely, if $F(y_0, x_0, t_0)$ is compact and for any net $(y_\beta, x_\beta, t_\beta) \to (y_0, x_0, t_0)$, $v_0 \in F(y_0, x_0, t_0) + \mathcal{C}(y_0, x_0)$, there is a net $\{v_\beta\}, v_\beta \in F(y_\beta, x_\beta, t_\beta)$, which has a convergent subnet $\{v_{\beta_\gamma}\}, v_{\beta_\gamma} - v_0 \to c \in \mathcal{C}(y_0, x_0)$, then F is lower $\mathcal{C}$-continuous at (y_0, x_0, t_0).

Proof. We proceed the proof of this proposition exactly the one of Proposition 2.3 in [7]. ■

The proofs of the main results in our paper are based on the following theorem in [15].

Theorem 2.8. *Let D be a nonempty convex compact subset of X and $F : D \to 2^D$ be a multivalued mapping satisfying the following conditions:*

(1) *For all $x \in D, x \notin F(x)$ and $F(x)$ is convex;*

(2) *For all $y \in D, F^{-1}(y)$ is open in D.*
Then there exists $\bar{x} \in D$ such that $F(\bar{x}) = \emptyset$.

3. Main Results

Throughout this section, unless otherwise specified, by X, Z and Y we denote real topological vector locally convex Haussdorff spaces. Let $D \subseteq X, K \subseteq Z$ be nonempty subsets and $C \subseteq Y$ be a convex closed cone, $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping. Given multivalued mappings P_i, $i = 1, 2$, Q and F as in the introduction, we first prove the following theorem.

Theorem 3.1. *The following conditions are sufficient for $(GEP)_{II}$ to have a solution:*

(i) *D is a nonempty convex compact subset;*

(ii) *$P_1 : D \to 2^D$ is a multivalued mapping with a nonempty closed fixed point set $D_0 = \{x \in D | \ x \in P_1(x)\}$ in D;*

(iii) *$P_2 : D \to 2^D$ is a multivalued mapping with $P_2^{-1}(x)$ open and the convex hull $\mathrm{co}P_2(x)$ of $P_2(x)$ is contained in $P_1(x)$ for each $x \in D$;*

(iv) *For any fixed $t \in D$, the set*

$$B = \{x \in D | \ 0 \notin F(y, x, t), \ \textit{for some} \ \ y \in Q(x, t)\}$$

is open in D;

(v) *$F : K \times D \times D \to 2^Y$ is a $Q - KKM$ multivalued mapping.*

Proof. We define the multivalued mapping $M : D \to 2^D$ by

$$M(x) = \{t \in D | \, 0 \notin F(y, x, t), \ \text{for some } y \in Q(x, t)\}.$$

Observe that if for some $\bar{x} \in D, \bar{x} \in P_1(\bar{x})$, it gives $M(\bar{x}) \cap P_2(\bar{x}) = \emptyset$, then

$$0 \in F(y, \bar{x}, t), \ \text{for all} \ \ t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t)$$

and hence the proof of the theorem is completed. Thus, our aim is to show the existence of such a point $\bar{x}$. Indeed, by contrary, we assume that for any $x \in P_1(x)$, it implies that $M(x) \cap P_2(x) \neq \emptyset$. We consider the multivalued mapping $H : D \to 2^D$ defined by

$$H(x) = \begin{cases} co(M(x) \cap P_2(x)), & \text{if } x \in P_1(x), \\ P_2(x), & \text{otherwise.} \end{cases}$$

We show that H satisfies the hypotheses of Theorem 2.8 in Section 2. Indeed, since $H(x) \neq \emptyset$, we have $D = \bigcup_{x \in D} H^{-1}(x)$.

Moreover,

$$H^{-1}(x) = (coM)^{-1}(x) \cap (coP_2)^{-1}(x) \cup (P_2^{-1}(x) \setminus D_0),$$

where $D_0 = \{x \in D : x \in P_1(x)\}$ is a closed subset in D. Hence, $H^{-1}(x)$ is an open set in D, for every $x \in D$. Further, if there is a point $\bar{x} \in D$ such that $\bar{x} \in H(\bar{x}) = coM(\bar{x}) \cap coP_2(\bar{x})$, then one can find $t_1, ..., t_n \in M(\bar{x})$ such that $\bar{x} = \sum_1^n \alpha_i t_i, \alpha_i \geq 0, \sum_1^n \alpha_i = 1$. By the definition of M, we have

$$0 \notin F(y, x, t_i), \ \text{for some } y \in Q(x, t_i) \text{ for all } i = 1, ..., n.$$

On the other hand, since the multivalued mapping F is $Q-$KKM, it follows that there is an index $j = 1, ..., n$ such that

$$0 \in F(y, x, t_j), \text{ for all } y \in Q(x, t_j)$$

and we have a contradiction. Theorefore, we conclude that for any $x \in D, x \notin H(x)$. An application of Theorem 2.8 in Section 2 implies that there exists a point $\bar{x} \in D$ with $H(\bar{x}) = \emptyset$. If $\bar{x} \notin P_1(\bar{x})$, then $H(\bar{x}) = P_2(\bar{x}) = \emptyset$, which is impossible. Therefore, we conclude that $\bar{x} \in P_1(\bar{x})$ and $H(\bar{x}) = \text{co}M(\bar{x}) \cap \text{co}P_2(\bar{x}) = \emptyset$. Thus, we have a contradiction and the proof of the theorem is complete. ■

Theorem 3.2. *The following conditions are sufficient for $(GEP)_{II}$ to have a solution*

(i) *D is a nonempty convex compact subset;*

(ii) *$P_1 : D \to 2^D$ is a multivalued mapping with a nonempty closed fixed point set $D_0 = \{x \in X|\ x \in P_1(x)\}$ in D;*

(iii) *P_2 is lower semicontinuous with nonempty values and the convex hull co$P_2(x)$ of $P_2(x)$ is contained in $P_1(x)$ for each $x \in D$;*

(iv) *For any fixed $t \in D$, the set*

$$B = \{x \in D|\ 0 \notin F(y, x, t), \textit{ for some } y \in Q(x, t)\}$$

is open in D;

(v) *$F : K \times D \times D \to 2^Y$ is a $Q - KKM$ multivalued mapping.*

Proof. Let $\mathcal{U}$ be a basis of convex closed neighborhoods of the origin in the space X. For every $U \in \mathcal{U}$ we define the multivalued mappings $P_{1U}, P_{2U} : D \to 2^D$ by

$$P_{iU}(x) = (P_i(x) + U) \cap D, i = 1, 2, x \in D.$$

It is easy to prove that $P_{2U}^{-1}(t)$ is open in D for any $t \in D$ and the convex hull co$P_{2U}(x)$ of $P_{2U}(x)$ is contained in $P_{1U}(x)$ for every $x \in D$. Therefore, all the conditions of Theorem 3.1 for P_{1U}, P_{2U}, Q and F are satisfied and there exists $\bar{x}_U \in D$ such that

$$\bar{x}_U \in P_1(\bar{x}_U)$$

and

$$0 \in F(y, \bar{x}_U, t), \text{for all } t \in P_2(\bar{x}_U) \text{ and } y \in Q(\bar{x}_U, t).$$

By the compactness of D, without loss of generality, we may assume that $\bar{x}_U$ converges to $\bar{x}$ as U decreases. The closedness of P_1 yields $\bar{x} \in P_1(\bar{x})$. Take an arbitrary $t \in P_2(\bar{x})$. The set

$$B = \{x \in D|\ 0 \notin F(y, x, t), \text{ for some } \ y \in Q(x, t)\}$$

is open in D and so the set

$$A = \{x \in D|\ 0 \in F(y, x, t), \text{ for all } \ y \in Q(x, t)\}$$

is closed in D. Observe that $\bar{x}_U \in A$ and $\bar{x}_U$ converges to $\bar{x}$, we conclude $\bar{x} \in A$. This shows that $0 \in F(y, \bar{x}, t)$ for all $t \in P_2(\bar{x})$ and $y \in Q(\bar{x}, t)$.

The proof is complete. ■

Corollary 3.3. *Suppose that the following conditions hold*

(i) *D, K are nonempty convex compact sets;*

(ii) *P is a continuous multivalued mapping with nonempty closed convex values;*

(iii) *For any fixed $t \in D$, the set*

$$B = \{x \in D |\ 0 \notin F(y, x, t),\ \textit{for some}\ \ y \in Q(x, t)\}$$

is open in D;

(iv) *$F : K \times D \times D \to 2^Y$ is a $Q - KKM$ multivalued mapping.*
Then, the above problem has a solution.

Proof. The proof of this corollary follows immediately from Theorems 3.1 and 3.2 with $P = P_1 = P_2$. ■

Several applications of the above theorem to the solution existence of quasi-equilibrium, variational inclusion problems,..., can be shown in the following corollaries.

Corollary 3.4. *Let D, K, P_1, P_2 and Q be as in Theorem 3.1. Let $\Phi : K \times D \times D \to \mathbb{R}$ be a real diagonally $(Q, \mathbb{R}_+)$- quasiconvex-like in the third variable function with $\Phi(y, x, x) = 0$, for all $y \in K, x \in D$. In addition, assume that for any fixed $t \in D$ the function $\Phi(.,.,t) : K \times D \to \mathbb{R}$ is upper semicontinuous. Then there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and*

$$\Phi(y, \bar{x}, t) \geq 0\ \textit{for all}\ \ t \in P_2(\bar{x})\ \textit{and}\ y \in Q(\bar{x}, t).$$

Proof. Setting $F(y, x, t) = \Phi(y, x, t) - \mathbb{R}_+$ for any $(y, x, t) \in K \times D \times D$, we can see that for any fixed $t \in D$ the set

$$B = \{x \in D |\ 0 \notin F(y, x, t) \text{ for some } \ y \in Q(x, t)\}$$
$$= \{x \in D |\ \Phi(y, x, t) < 0\}$$

is open in D. Since Φ is diagonally upper $(Q, \mathbb{R}_+)$-quasiconvex-like in the third variable, then for any finite set $\{t_1, ..., t_n\} \subseteq D, x \in \text{co}\{t_1, ..., t_n\}$, there is an index $j \in \{1, ..., n\}$ such that

$$\Phi(y, x, t_j) \in \Phi(y, x, x) + \mathbb{R}_+,\ \text{for all}\ \ y \in Q(x, t_j).$$

This implies that $\Phi(y, x, t_j) \geq 0$ and so $0 \in F(y, x, t_j)$, for all $y \in Q(x, t_j)$. This shows that F is a $Q-$ KKM multivalued mapping from $K \times D \times D$ to 2^R. Therefore, P_1, P_2, Q and F satisfy all conditions in Theorem 3.1. This implies that there is a point $\bar{x} \in D$ such that

$$\bar{x} \in P_1(\bar{x})$$

and

$$0 \in F(y, \bar{x}, t), \text{for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This is equivalent to

$$\Phi(y, \bar{x}, t) \geq 0, \text{for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t),$$

and the proof is complete. ■

In the following corollary we assume that $\mathcal{C} : K \times D \to 2^Y$ is a given cone upper continuous multivalued mapping with convex closed values.

Corollary 3.5. *Let D, K, P_1, P_2 be as in Theorem 3.1 and $Q : D \times D \to 2^K$ be such that for any fixed $t \in D$, the multivalued mapping $Q(., t) : D \to 2^K$ is lsc. Let $G, H : K \times D \times D \to 2^Y$ be multivalued mappings with compact values and $G(y, x, x) \subseteq H(y, x, x) + \mathcal{C}(y, x)$, for any $(y, x) \in K \times D$. Let $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping with nonempty convex closed values. In addition, assume*

(i) *For any fixed $t \in D$, the multivalued mapping $G(., ., t) : K \times D \to 2^Y$ is lower $(-\mathcal{C})$-continuous and the multivalued mapping $N : K \times D \to 2^Y$, defined by $N(y, x) = H(y, x, x)$, is upper $\mathcal{C}$-continuous;*

(ii) *G is diagonally upper $(Q, \mathcal{C})$-quasiconvex-like in the third variable.*

Then, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$G(y, \bar{x}, t) \subseteq H(y, \bar{x}, \bar{x}) + \mathcal{C}(y, \bar{x}), \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

Proof. We define the multivalued mappings $M : K \times D \to 2^X, F : K \times D \times D \to 2^D$ by

$$M(y, x) = \{t \in D |\ G(y, x, t) \subseteq H(y, x, x) + \mathcal{C}(y, x)\},\ (y, x) \in K \times D$$

and

$$F(y, x, t) = t - M(y, x),\ (y, x, t) \in K \times D \times D.$$

For any fixed $t \in D$, we set

$$\begin{aligned} A &= \{x \in D |\ 0 \in F(y, x, t), \text{ for all } y \in Q(x, t)\} \\ &= \{x \in D |\ t \in M(y, x), \text{ for all } y \in Q(x, t)\} \\ &= \{x \in D |\ G(y, x, t) \subseteq H(y, x, x) + \mathcal{C}(y, x), \text{ for all } y \in Q(x, t)\}. \end{aligned}$$

We claim that this subset is closed in D. Indeed, assume that a net $\{x_\alpha\} \subset A$ and $x_\alpha \to x$. Take an arbitrary $y \in Q(x, t)$. Since $Q(., t)$ is a lower semicontiunous mapping and $x_\alpha \to x$, there exists a net $\{y_\alpha\}, y_\alpha \in Q(x_\alpha, t)$ such that $y_\alpha \to y$. For any neighborhood V of the origin in Y there is an index α_0 such that for all $\alpha \leq \alpha_0$ the following inclusions hold

$$\begin{aligned} G(y, x, t) &\subseteq G(y_\alpha, x_\alpha, t) + V + \mathcal{C}(y_\alpha, x_\alpha) \\ &\subseteq H(y_\alpha, x_\alpha, x_\alpha) + V + \mathcal{C}(y_\alpha, x_\alpha) \end{aligned}$$

$$\subseteq H(y,x,x) + 2V + \mathcal{C}(y,x).$$

This and the compact values of H imply that

$$G(y,x,t) \subseteq H(y,x,x) + \mathcal{C}(y,x),$$

and therefore $x \in A$. This implies that A is closed in D and the set

$$B = D \setminus A = \{x \in D | \ 0 \notin F(y,x,t), \text{ for some } \ y \in Q(x,t)\}$$

is open in D.

Further, since $G(y,x,x) \subseteq H(y,x,x) + \mathcal{C}(y,x)$, for any $(y,x) \in K \times D$ and G is diagonally upper $(Q,\mathcal{C})$-quasiconvex-like in the third variable we conclude that for any finite set $\{t_1,...,t_n\} \subseteq D, x \in \text{co}\{t_1,...,t_n\}$ there is an index $j \in \{1,...,n\}$ such that

$$G(y,x,t_j) \subseteq G(y,x,x) + \mathcal{C}(y,x) \subseteq H(y,x,x) + \mathcal{C}(y,x), \text{ for all } \ y \in Q(x,t_j).$$

It follows that $0 \in F(y,x,t_j)$ and then F is a $Q-$KKM multivalued mapping. By Theorem 3.1 we deduce that there is $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y,\bar{x},t), \text{ for all } \ t \in P_2(\bar{x}) \text{ and } \ y \in Q(\bar{x},t).$$

This is equivalent to

$$G(y,\bar{x},t) \subseteq H(y,\bar{x},\bar{x}) + \mathcal{C}(y,\bar{x}), \text{ for all } \ t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x},t).$$

■

Analogically, we obtain the following corollary.

Corollary 3.6. *Let D,K,P_1,P_2 and Q be as in Corollary 3.5. Let $G,H : K \times D \times D \to 2^Y$ be multivalued mappings with compact values and $H(y,x,x) \subseteq G(y,x,x) - \mathcal{C}(y,x)$, for any $(y,x) \in K \times D$. Let $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping with nonempty convex closed values. In addition, assume:*

(i) *For any fixed $t \in D$ the multivalued mapping $G(.,.,t) : K \times D \to 2^Y$ is upper $(-\mathcal{C})$-continuous and the multivalued mapping $N : K \times D \to 2^Y$ defined by $N(y,x) = H(y,x,x)$ is lower $\mathcal{C}$-continuous;*

(ii) *G is diagonally lower $(Q,\mathcal{C})$-quasiconvex-like in the third variable.*

Then there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$H(y,\bar{x},\bar{x}) \subseteq G(y,\bar{x},t) - \mathcal{C}(y,\bar{x}), \text{ for all } \ t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x},t).$$

Proof. The proof is similar as the previous one of Corollary 3.3. ■

Corollary 3.7. *Let D,K,P_1,P_2 and Q be as in Corollary 3.5. Let $\mathcal{R}$ be a relation linking $y \in K, x \in D, t \in D$. In addition, assume*

(i) *For any fixed* $t \in D$ *the relation* $\mathcal{R}(.,.,t)$ *linking elements* $y \in K, x \in D$ *is closed;*

(ii) $\mathcal{R}$ *is* Q*-KKM.*

Then there exists $\bar{x} \in D$ *such that* $\bar{x} \in P_1(\bar{x})$ *and* $\mathcal{R}(y, \bar{x}, t)$ *holds for all* $t \in P_2(\bar{x})$ *and* $y \in Q(\bar{x}, t)$.

Proof. We define the multivalued mappings $M : K \times D \to 2^X, F : K \times D \times D \to 2^D$ by

$$M(y, x) = \{t \in D |\ \mathcal{R}(y, x, t) \ \text{ holds}\}$$

and

$$F(y, x, t) = t - M(y, x),\ (y, x, t) \in K \times D \times D.$$

For any fixed $t \in D$, we set

$$\begin{aligned} A &= \{x \in D |\ \mathcal{R}(y, x, t) \text{ holds, for all } y \in Q(x, t)\} \\ &= \{x \in D |\ 0 \in F(y, x, t), \text{ for all } y \in Q(x, t)\}. \end{aligned}$$

By arguments as in the proof of Corollary 3.5, we conclude that this subset is closed in D, therefore, the set

$$B = D \setminus A = \{x \in D |\ 0 \notin F(y, x, t), \text{ for some } \ y \in Q(x, t)\}$$

is open in D.

Further, it is easy to check that $\mathcal{R}$ is Q-KKM, so is the multivalued mapping F. Therefore, to complete the proof of the corollary, it remains to apply Theorem 3.1 to deduce that there is $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and $\mathcal{R}(y, \bar{x}, t)$ holds for all $t \in P_2(\bar{x})$ and $y \in Q(\bar{x}, t)$. ■

4. Pareto and Weakly Quasi-Equilibrium Problems

In this section we shall apply the existence results of Section 3 to two problems of Pareto and weakly quasi-equilibrium problems that are extensively studied in the recent literature under different forms (see for examples, in [3], [5], [7], [8], [11], [12]...). Namely, we consider the problems of finding $\bar{x} \in D$ such that

$$\begin{aligned} &\bar{x} \in P(\bar{x}) \\ &G(y, \bar{x}, x) \not\subseteq -C(y, \bar{x}) \setminus \{0\}, \text{ for all } x \in P(\bar{x}) \text{ and for all } \ y \in Q(\bar{x}), \end{aligned}$$

and of finding $\bar{x} \in D$ such that

$$\begin{aligned} &\bar{x} \in P(\bar{x}) \\ &G(y, \bar{x}, x) \not\subseteq -intC(y, \bar{x}), \text{ for all } x \in P(\bar{x}) \text{ and for all } \ y \in Q(\bar{x}), \end{aligned}$$

where $P : D \to 2^D$ and $Q : D \to 2^K$ are multivalued mappings with nonempty values, $G : K \times D \times D \to 2^Y$ is a multivalued mapping with nonempty values and $C : K \times D \to 2^Y$ is a cone multivalued mappings with nonempty values .

To this end we need some concepts of continuity and convexity of multivalued mappings with respect to a cone-valued mapping (see, for examples, in [8], [11], [12], [14]).

Let $F, C : D \longrightarrow 2^Y$ be multivalued mappings.

1) F is upper (lower) C-hemicontinuous if for any $x, t \in D$, the following implication holds: $F(\alpha x+(1-\alpha)t)\cap C(\alpha x+(1-\alpha)t) \neq \emptyset$, for all $\alpha \in (0,1)$ implies that $F(t)\cap C(t) \neq \emptyset$ (respectivelly, $F(\alpha x+(1-\alpha)t) \not\subseteq -\text{int}C(\alpha x+(1-\alpha)t)$, for all $\alpha \in (0,1)$ implies that $F(t) \not\subseteq -\text{int}C(t)$).

F is said to be upper (lower) hemicontinuous if for any $x, t \in D$, the multivalued mapping $f : [0,1] \longrightarrow 2^Y$ defined by $f(\alpha) = F(\alpha x + (1-\alpha)t)$ is upper (respectively, lower) semicontinuous.

The concept of C-hemicontinuity was introduced by Bianchi and Pini [2] and then by Hadjisavvas [6] for single-valued mappings in the framework of variational inequality problems.

2) F is upper (respectively, lower) C-continuous at $\bar{x} \in D$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that: $F(x) \subseteq F(\bar{x}) + V + C(\bar{x})$ (respectively, $F(\bar{x}) \subseteq F(x) + V - C(\bar{x})$) holds for all $x \in U \cap \text{dom}F$. If F is a single-valued mapping, the concepts of upper C-continuity and lower C-continuity coincide, we say that it is C-continuous.

Let $F : D \times D \to 2^Y$ be a multivalued mapping.

3) F is diagonally upper (lower) $\mathcal{C}$-convex in the second variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x \in co\{x_1, ..., x_n\}, x = \sum_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0, \sum_{j=1}^{n} \alpha_j = 1$, such that

$$\sum_{j=1}^{n} \alpha_j F(x, x_j) \subseteq F(x, x) + \mathcal{C}(x),$$

$$(\text{ respectively, } F(x, x) \subseteq \sum_{j=1}^{n} \alpha_j F(x, x_j) - \mathcal{C}(x)).$$

4) F is diagonally upper (lower) $\mathcal{C}$-quasiconvex-like in the second variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x \in \text{co}\{x_1, ..., x_n\}, x = \sum_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0, \sum_{j=1}^{n} \alpha_j = 1$, there is an index i such that

$$F(x, x_i) \subseteq F(x, x) + \mathcal{C}(x),$$

$$(\text{ respectively, } F(x, x) \subseteq F(x, x_i) - \mathcal{C}(x)).$$

5) F is C- pseudomonotone if for any given $x, t \in D$

$$F(t, x) \not\subseteq -\text{int}C(t) \Rightarrow F(x, t) \subseteq -C(x).$$

6) F is C- strong pseudomonotone if for any given $x, t \in D$

$$F(t,x) \not\subseteq -(C(t)\backslash\{0\}) \Rightarrow F(x,t) \subseteq -C(x).$$

A sufficient condition for the upper C-hemicontinuity is given by upper semicontinuity of the mappings F and C.

Proposition 4.1. *If F and C are upper hemicontinuous with nonempty closed values and if for any $x \in D$, either $F(x)$ or $C(x)$ is a compact set, then F is upper C-hemicontinuous.*

Proof. For any fixed $x, t \in D$ the multivalued mappings $f, c : [0,1] \to 2^Y$ defined by $f(\alpha) = F(\alpha x + (1-\alpha)t)$ and $c(\alpha) = C(\alpha x + (1-\alpha)t)$, $\alpha \in [0,1]$ are upper semicontinuous at 0. Hence, for an arbitrary neighborhood V of the origin in Y there exists a neighborhood U of 0 in $[0,1]$ such that

$$F(\alpha x + (1-\alpha)t) \subseteq F(t) + V;$$

$$C(\alpha x + (1-\alpha)t) \subseteq C(t) + V, \text{ for all } \alpha \in U.$$

Therefore, if $F(\alpha x + (1-\alpha)t) \cap C(\alpha x + (1-\alpha)t) \neq \emptyset$ for all $\alpha \in (0,1)$, then $(F(t)+V) \cap (C(t)+V) \neq \emptyset$ for an arbitrary neighborhood V of the origin in Y. This implies $F(t) \cap (C(t) + 2V) \neq \emptyset$.The compactness of $F(t)$ (or, of $C(t)$) implies that $F(t) \cap C(t) \neq \emptyset$. Indeed, assume that for any V_β, we can take $a_\beta \in F(t) \cap (C(t) + 2V_\beta), a_\beta = b_\beta + v_\beta$ with $b_\beta \in C(t)$ and $v_\beta \in V_\beta$. Taking V_β such that $\cap_\beta V_\beta = \{0\}$, we can suppose that v_β tends to 0 as β tends to 0. Since $a_\beta \in F(t)$ and $F(t)$ is compact, without loss of generality, we may assume that a_β tends to $a \in F(t)$ as β tends to 0. Therefore, b_β also tends to a and then $a \in F(t) \cup C(t)$. The proof of the proposition is complete. ■

We need also the following lemmas, which generalize and improve Lemmas 2.3, 2.4 given in [6].

Lemma 4.2. *Let $F : K \times D \times D \to 2^Y$ be a multivalued mapping with nonempty values and $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping with $F(y,x,x) \cap C(y,x) \neq \emptyset$ for any $x \in D$ and $y \in K$. In addition, assume that*

(i) *For any fixed $x \in D, y \in K, F(y,.,x) : D \to 2^Y$ is upper $C(y,.)$-hemicontinuous;*

(ii) *For any fixed $y \in K, F(y,.,.)$ is $C(y,.)$-strong pseudomonotone;*

(iii) *For any fixed $y \in K, F(y,.,.)$ is diagonally lower $C(y,.)$-convex (or diagonally lower $C(y,.)$-quasiconvex-like) in the second variable.*
Then for any $t \in D, y \in K$ the following are equivalent:

1) $F(y,t,x) \not\subseteq -C(y,t)\backslash\{0\}$, *for all* $x \in D$;

2) $F(y,x,t) \subseteq -C(y,x)$, *for all* $x \in D$.

Proof. The fact 1)⇒ 2) directly follows from the definition of $C(y,.)$-strong pseudomonotonicity.

Now suppose that 2) holds, we then have

$$F(y, \alpha x + (1-\alpha)t, t) \subseteq -C(y, \alpha x + (1-\alpha)t), \text{ for all } x \in D.$$

We claim that for any $x \in D$

$$F(y, \alpha x + (1-\alpha)t, x) \cap C(y, \alpha x + (1-\alpha)t) \neq \emptyset, \text{ for all } \alpha \in (0,1].$$

Otherwise, we have

$$F(y, \alpha x + (1-\alpha)t, x) \cap C(y, \alpha x + (1-\alpha)t) = \emptyset, \text{ for some } \alpha \in (0,1].$$

This shows

$$F(y, \alpha x + (1-\alpha)t, x) \subseteq Y \backslash C(y, \alpha x + (1-\alpha)t).$$

Further, since $F(y,.,.)$ is diagonally lower $C(y,.)$- convex in the second variable, it follows that

$$F(y, \alpha x + (1-\alpha)t, \alpha x + (1-\alpha)t) \subseteq \alpha F(y, \alpha x + (1-\alpha)t, x)$$

$$+(1-\alpha)F(y, \alpha x + (1-\alpha)t, t) - C(y, \alpha x + (1-\alpha)t).$$

It follows that

$$\begin{aligned} F(y, \alpha x + (1-\alpha)t, \alpha x + (1-\alpha)t) &\subseteq Y \backslash C(y, \alpha x + (1-\alpha)t) - C(y, \alpha x + (1-\alpha) \\ &\subseteq Y \backslash C(y, \alpha x + (1-\alpha)t). \end{aligned}$$

We conclude

$$F(y, \alpha x + (1-\alpha)t, \alpha x + (1-\alpha)t) \cap C(y, \alpha x + (1-\alpha)t) = \emptyset,$$

and we have a contradiction. Therefore,

$$F(y, \alpha x + (1-\alpha)t, x) \cap C(y, \alpha x + (1-\alpha)t) \neq \emptyset,$$

for all $\alpha \in (0,1]$. And, the upper $C(y,.)$-hemicontinuity of $F(y,.,.)$ implies that there exists $v \in Y$ such that $v \in F(y,t,x) \cap C(y,t)$, for all $x \in D$. Remarking that $C(y,t) \cap (-C(y,t) \backslash \{0\}) = \emptyset$, we see that $v \notin -(C(y,t) \backslash \{0\})$. Thus, we obtain

$$F(y,t,x) \not\subseteq -C(y,t) \backslash \{0\}, \quad \text{for all } \ x \in D.$$

Next, we assume that $F(y,.,.)$ is diagonally lower $C(y,.)$- quasiconvex-like in the second variable. We have either

$$F(y, \alpha x + (1-\alpha)t, \alpha x + (1-\alpha)t) \subseteq F(y, \alpha x + (1-\alpha)t, x) - C(y, \alpha x + (1-\alpha)t)$$

or

$$F(y, \alpha x + (1-\alpha)t, \alpha x + (1-\alpha)t) \subseteq F(y, \alpha x + (1-\alpha)t, t) - C(y, \alpha x + (1-\alpha)t).$$

In both cases, we get

$$F(y, \alpha x + (1-\alpha)t, \alpha x + (1-\alpha)t) \subseteq Y \backslash C(y, \alpha x + (1-\alpha)t),$$

and also have a contradiction. Thus, we also get

$F(y, \alpha x + (1-\alpha)t, x) \cap C(y, \alpha x + (1-\alpha)t) \neq \emptyset$, for all $\alpha \in (0, 1]$.

By the similar arguments used in the above proof, we conclude that

$$F(y, t, x) \not\subseteq -C(y, t)\backslash\{0\}, \quad \text{for all } \ x \in D.$$

The proof is complete. ■

Lemma 4.3. *Let $F : K \times D \times D \to 2^Y$ be a multivalued mapping with nonempty values and $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping with $F(y, x, x) \not\subseteq -\text{int}C(y, x) \neq \emptyset$ for any $x \in D, y \in K$. In addition, assume that*

(i) *For any fixed $x \in D, y \in K$, $F(y, ., x) : D \to 2^Y$ is lower $C(y, .)$-hemicontinuous;*

(ii) *For any $y \in K$, $F(y, ., .)$ is $C(y, .)$- pseudomonotone;*

(iii) *For any $y \in K$, $F(y, ., .)$ is diagonally lower $C(y, .)$-convex (or diagonally lower $C(y, .)$-quasiconvex-like) in the second variable;*

Then for any $t \in D, y \in K$ the followings are equivalent:

1) $F(y, t, x) \not\subseteq -\text{int}C(y, t)$, *for all* $x \in D$;

2) $F(y, x, t) \subseteq -C(y, x)$, *for all* $x \in D$.

Proof. The proof of this lemma is similar as the one of Lemma 4.2. The fact 1) $\Rightarrow$ 2) directly follows from the definition of $C(y, .)$- pseudomonotonicity.

For the converse implication, asumme 2) holds, which implies

$$F(y, \alpha x + (1-\alpha)t, t) \subseteq -C(y, \alpha x + (1-\alpha)t), \text{ for all}\, x \in D \,\text{and}\, \alpha \in [0, 1].$$

We claim that for any $x \in D$

$$F(y, \alpha x + (1-\alpha)t, x) \not\subseteq -\text{int}C(y, \alpha x + (1-\alpha)t), \text{ for all}\, \alpha \in (0, 1].$$

Indeed, if not, there are some $x \in D$ and $\alpha \in [0, 1]$ such that

$$F(y, \alpha x + (1-\alpha)t, x) \subseteq -\text{int}C(y, \alpha x + (1-\alpha)t).$$

Further, since $F(y, ., .)$ is diagonally lower $C(y, .)$- convex in the second variable, it follows that

$$F(y, \alpha x + (1-\alpha)t, \alpha x + (1-\alpha)t)$$
$$\subseteq \alpha F(y, \alpha x + (1-\alpha)t, x) + (1-\alpha)F(y, \alpha x + (1-\alpha)t, t) - C(y, \alpha x + (1-\alpha)t)$$
$$\subseteq -\text{int}C(y, \alpha x + (1-\alpha)t) - C(y, \alpha x + (1-\alpha)t) \subseteq -\text{int}C(y, \alpha x + (1-\alpha)t).$$

This contradicts the fact $F(y, z, z) \not\subseteq -\text{int}C(y, z) \neq \emptyset$, for all $z \in D$.
The lower $C(y, .)$-hemicontinuity of $F(y, ., x)$ implies that

$$F(y,t,x) \not\subseteq -\text{int}C(y,t).$$

The proof is complete. ■

Now we are able to present sufficient conditions for existence of solutions of variational inclusion problems.

Corollary 4.4. *Assume that D, P are as in Corollary 3.3 and $Q : D \to 2^K$ is a lower semicontinuous mapping with nonempty values. In addition, suppose that $G : K \times D \times D \to 2^Y$ is a multivalued mapping with nonempty values and $\mathcal{C} : K \times D \to 2^Y$ is a cone multivalued mapping with $G(y,x,x) \cap C(y,x) \neq \emptyset$ for any $x \in D, y \in K$ satisfying the following conditions:*

(i) *For any fixed $x \in D, y \in K$ the set*

$$A = \{t \in D|\ G(y,x,t) \subseteq -C(y,x)\}$$

is closed in D.

(ii) *For any $y \in K$, $G(y,.,.)$ is $C(y,.)$-strongly pseudomonotone;*

(iii) *For any $y \in K$, $G(y,.,.)$ is diagonally lower $C(y,.)$-convex (or diagonally lower $C(y,.)$-quasiconvex-like) in the second variable.*
Then there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and

$$G(y,t,\bar{x}) \subseteq -C(y,t) \ for\ all\ t \in P(\bar{x})\ and\ y \in Q(\bar{x}).$$

Proof. We define $M : K \times D \to 2^D$ and $F : K \times D \times D \to 2^X$ by

$$M(y,t) = \{x \in D|\ G(y,t,x) \subseteq -C(y,t)\}, t \in D;$$

$$F(y,x,t) = x - M(y,t), (y,x,t) \in K \times D \times D.$$

For any fixed $t \in D, y \in K$ since the set A is closed, therefore, the set

$$B = \{x \in D|\ 0 \notin F(y,x,t)\} = Y \setminus A$$

is open in D. Let $\{t_1, ..., t_n\}$ be an arbitrary finite subset in D and $x = \sum_{i=1}^{n} \alpha_i t_i, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1$.We show that for a given $y \in K$ there is $i \in \{1, ..., n\}$ such that $0 \in F(y,x,t_i)$. Suppose $0 \notin F(y,x,t_i)$ for all $i = 1, ..., n$. This gives $G(y,t_i,x) \not\subseteq -C(y,t_i)$ for all $i = 1, ..., n$. The $C(y,.)$-strong pseudomonotonicity of $G(y,.,.)$ yields $G(y,x,t_i) \subseteq -C(y,x)\setminus\{0\}$ for all $i = 1, ..., n$. The diagonally lower $C(y,.)$-convexity in the second variable or, the diagonally lower $C(y,.)$-quasiconvexity-like in the second variable of $G(y,.,.)$ implies that $G(y,x,x) = G(y,x,\sum_{1}^{n} \alpha_i t_i) \subseteq -C(y,x)\setminus\{0\}$ and we have a contradiction with $G(y,x,x) \cap C(y,x) \neq \emptyset$. Therefore, there exists $j \in \{1, ..., n\}$ such that $0 \in F(y,x,t_j)$ and then $F(y,.,.)$ is a KKM multivalued mapping.

Lastly, we apply Corollary 3.3 with D, P, Q and F to conclude that there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and $0 \in F(y,\bar{x},t)$, for all $t \in P(\bar{x})$, $y \in Q(\bar{x})$.

This shows $\bar{x} \in M(y,t)$ for all $t \in P(\bar{x})$ and hence $G(y,t,\bar{x}) \subseteq -C(y,t)$ for all $t \in P(\bar{x})$ and $y \in Q(\bar{x})$.
The proof of the corollary is complete. ■

Corollary 4.5. *Assume that D, P and Q are as in Corollary 4.4. In addition, suppose that $G : K\times D\times D \to 2^Y$ is a multivalued mapping with nonempty values and $\mathcal{C} : K \times D \to 2^Y$ is a cone multivalued mapping with $G(y,x,x)\cap C(y,x) \neq \emptyset$ for any $x \in D, y \in K$ satisfying the following conditions:*

(i) *For any fixed $t \in D, y \in K, G(y,.,t) : D \to 2^Y$ is upper $C(y,.)$-hemicontinuous;*

(ii) *For any fixed $x \in D, y \in K$ the set*

$$A = \{t \in D|\ G(y,x,t) \subseteq -C(y,x)\}$$

is closed in D.

(iii) *For any fixed $y \in K$, $G(y,.,.)$ is $C(y,.)$-strongly pseudomonotone;*

(iv) *For any $y \in K$, $G(y,.,.)$ is diagonally lower $C(y,.)$-convex (or diagonally lower $C(y,.)$-quasiconvex-like) in the second variable.*

Then there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and

$$G(y,\bar{x},t) \nsubseteq -(C(y,\bar{x})\backslash\{0\}),\ \text{for all}\ \ t \in P(\bar{x}), y \in Q(\bar{x}).$$

Proof. We apply Corollary 4.4 to deduce that there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and $G(y,t,\bar{x}) \subseteq -C(y,t)$ for all $t \in P(\bar{x}), y \in Q(\bar{x})$. To complete the proof of this corollary, it remains to apply Lemma 4.2 with D replaced by $P(\bar{x})$. ■

By exploiting similar arguments used in the proofs of Corollaries 4.4 and 4.5, we obtain the following corollaries.

Corollary 4.6. *Assume that D, P and Q are as in Corollary 4.4. In addition, suppose that $G : K\times D\times D \to 2^Y$ is a multivalued mapping with nonempty values and $\mathcal{C} : K\times D \to 2^Y$ is a cone multivalued mapping with $G(y,x,x) \nsubseteq -intC(y,x)$ for any $x \in D, y \in K$ satisfying the following conditions:*

(i) *For any fixed $x \in D, y \in K$ the set*

$$A = \{t \in D|\, G(y,x,t) \subseteq -C(y,x)\}$$

is closed in D;

(ii) *For any $y \in K, G(y,.,.)$ is $C(y,.)$- pseudomonotone;*

(iii) *For any $y \in K$, $G(y,.,.)$ is diagonally lower $C(y,.)$-convex (or diagonally lower $C(y,.)$-quasiconvex-like) in the second variable.*
Then there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and

$$G(y,t,\bar{x}) \subseteq -C(y,t)\ \text{for all}\ t \in P(\bar{x})\ \text{and}\ y \in Q(\bar{x}).$$

Proof. Assume that D, P and Q are as in Corollary 4.4. In addition, suppose that $G : K \times D \times D \to 2^Y$ is a multivalued mapping with nonempty values and $\mathcal{C} : K \times D \to 2^Y$ is a cone multivalued mapping with $G(y, x, x) \nsubseteq -\text{int}C(y, x)$ for any $x \in D, y \in K$ satisfying the following conditions:

(i) For any fixed $t \in D, y \in K, G(y, ., t) : D \to 2^Y$ is lower $C(y, .)$-hemicontinuous;

(ii) For any fixed $x \in D, y \in K$ the set

$$A = \{t \in D |\ G(y, x, t) \subseteq -C(y, x)\}$$

is closed in D;

(iii) For any fixed $y \in K$, $G(y, ., .)$ is $C(y, .)$- pseudomonotone;

(iv) For any fixed $y \in K$, $G(y, ., .)$ is diagonally lower $C(y, .)$-convex (or diagonally lower $C(y, .)$-quasiconvex-like) in the second variable.
Then there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and

$$G(\bar{x}, t) \nsubseteq -\text{int}C(y, \bar{x}), \text{ for all } \ t \in P(\bar{x}) \text{ and } y \in Q(\bar{x}).$$

■

Remark 4.7. If $G : K \times D \times D \longrightarrow 2^Y$ is a multivalued mapping with nonempty compact values, for any fixed $x \in D, y \in K, G(y, x, .)) : D \to 2^Y$ is a lower $C(y, x)$-continuous multivalued mapping and $\mathcal{C} : K \times D \to 2^Y$ is a cone multivalued mapping with closed values, then for any fixed $x \in D, y \in K$, the set $A = \{t \in D |\ G(y, x, t) \subseteq -C(y, x)\}$ is closed in D.

Indeed, assume $t_\alpha \in A, t_\alpha \to t$, we have $G(y, x, t_\alpha) \subseteq -C(y, x)$. The lower $C(y, x)$-continuity of $G(y, x, .)$ in the second variable implies that for any neighborhood V of O in Y we have

$$G(y, x, t) \subseteq G(x, t_\alpha) + V - C(y, x).$$

This shows that $G(y, x, t) \subseteq V - C(y, x)$. Further, since $G(y, x, t)$ is a compact set and $C(y, x)$ is a closed set, then we deduce $G(y, x, t) \subseteq -C(y, x)$. So, A is closed in D.

Next, we consider a weak quasi-equilibrium problem with relaxed C-pseudomonotone conditions. Of course some price must be paid for it, namely the continuity of multivalued mappings are requested to be stronger.

Corollary 4.8. *Let D, P and Q be as in Corollary* 4.4. *Let $G : K \times D \times D \to 2^Y$ be a multivalued mapping with nonempty compact values such that for any fixed $t \in D, y \in K$, the multivalued mapping $G(y, ., t) : D \to 2^Y$ is upper $(-C(y, .)$-continuous. Let $C : K \times D \to 2^Y$ be a lower semicontinuous cone multivalued mapping. In addition, assume that:*

(i) *$G(y, x, x) \nsubseteq -\text{int}C(y, x)$ for any $x \in D, y \in K$;*

(ii) *For any fixed $y \in K$, $G(y, ., .)$ is diagonally lower $C(y, .)$-convex* (*or diagonally lower $C(y, .)$-quasiconvex-like*) *in the second variable.*

Then there exists $\bar{x} \in D$ *such that* $\bar{x} \in P(\bar{x})$ *and*

$$G(y, \bar{x}, t) \nsubseteq -\mathrm{int}C(y, \bar{x}), \text{ for all } t \in P(\bar{x}) \text{ and } y \in Q(\bar{x}).$$

Proof. We define the multivalued mappings $M : K \times D \to 2^D, F : K \times D \times D \to 2^D$ by

$$M(y, t) = \{x \in D|\ G(y, x, t) \nsubseteq -\mathrm{int}C(y, x)\},\ t \in D,$$

$$F(y, x, t) = x - M(y, t),\ (y, x, t) \in K \times D \times D.$$

For any fixed $t \in D, y \in K$, we claim that the set

$$B = \{x \in D|\ 0 \notin F(y, x, t)\} = \{x \in D|\ G(y, x, t) \subseteq -\mathrm{int}C(y, x)\}$$

is open in D. Indeed, take $x_0 \in B$. Since $G(y, x_0, t)$ is compact and $-\mathrm{int}C(y, x_0)$ is open, there exists a neighborhood V_0 of the origin in Y such that

$$G(y, x_0, t) + V_0 \subseteq -\mathrm{int}C(y, x_0).$$

The upper $-C(y, .)$-continuity of $G(y, ., t)$ and the lower semicontinuity of $C(y, .)$ imply that there exists a neighborhood U of x_0 in X such that

$$G(y, x, t) \subseteq G(y, x_0, t) + V_0 - C(y, x_0) \subseteq -\mathrm{int}C(y, x_0) \subseteq -\mathrm{int}C(y, x), \text{ for all } x \in U.$$

This shows that $U \subseteq B$ and hence B is an open set.

Let $\{t_1, ..., t_n\}$ be an arbitrary finite subset in D and $x = \sum_{i=1}^{n} \alpha_i t_i, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1$. We suppose that there is $y \in K, 0 \notin F(y, x, t_i)$ for all $i = 1, ..., n$. This gives $G(y, x, t_i) \subseteq -\mathrm{int}C(y, x)$ for all $i = 1, ..., n$. The diagonally lower $C(y, .)$-convexity in the second variable or, the diagonally lower $C(y, .)$-quasiconvexity-like in the second variable of $G(y, ., .)$ implies that $G(y, x, x) = G(y, x, \sum^n{}_1 \alpha_i t_i) \subseteq -\mathrm{int}C(y, x)$ and we have a contradiction with $G(y, x, x) \nsubseteq -\mathrm{int}C(y, x)$. Therefore, there exists $j = \{1, ..., n\}$ such that $0 \in F(y, x, t_j)$ and then $F(y, ., .)$ is a KKM multivalued mapping or, F is a Q-KKM multivalued mapping.

Lastly, we apply Theorem 3.1 (or Theorem 3.2) with $D, P = P_i, i = 1, 2, Q$ and F to conclude that there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and $0 \in F(y, \bar{x}, t)$, for all $t \in P(\bar{x})$ and $y \in Q(\bar{x})$. This shows $\bar{x} \in M(y, t)$ for all $t \in P(\bar{x})$ and hence $G(y, \bar{x}, t) \nsubseteq -\mathrm{int}C(y, \bar{x})$ for all $t \in P(\bar{x})$ and $y \in Q(\bar{x})$.

The proof of the corollary is complete. ■

Given D, K, P, Q as in Corollary 4.4, $G : K \times D \times D \longrightarrow 2^Y$ a multivalued mapping with nonempty values and $C : K \times D \longrightarrow 2^Y$ a cone multivalued mapping. In the rest of this section, we shall prove the existence of solutions for the following generalized Pareto quasi-equilibrium problem: Find $\bar{x} \in D$ such that

$$\bar{x} \in P(\bar{x})$$
$$\text{such that } G(y, \bar{x}, t) \nsubseteq -C(y, \bar{x}) \setminus \{0\}, \text{for all } t \in P(\bar{x}), y \in Q(\bar{x}).$$

First of all, we prove the following corollary.

Corollary 4.9. *Assume that D, K, P, Q are as in Corollary 4.4, $G : K \times D \times D \to 2^Y$ is a multivalued mapping with nonempty values and $\mathcal{C} : K \times D \to 2^Y$ is a cone multivalued mapping with $G(y, x, x) \cap C(y, x) \neq \emptyset$ for any $x \in D$. In addition, assume that the following conditions are satisfied:*

(i) *For any fixed $t \in D, y \in K$ the set*

$$A = \{x \in D| \ G(y, t, x) \subseteq -C(y, t)\}$$

is closed in D.

(ii) *$G(y, ., .)$ is $C(y, .)$-strongly pseudomonotone;*

(iii) *$G(y, ., .)$ is diagonally lower $C(y, .)$-convex (or diagonally lower $C(y, .)$-quasiconvex-like) in the third variable.*

Then there exists $\bar{x} \in D$ with $\bar{x} \in P(\bar{x})$ and

$$G(y, t, \bar{x}) \subseteq -C(y, t), \text{ for all } t \in P(\bar{x}) \text{ and} y \in Q(\bar{x}).$$

Proof. Define the multivalued mappings $M : K \times D \to 2^D$ and $\mathcal{F} : K \times D \times D \to 2^X$ by

$$M(y, t) = \{x \in D| \ G(y, t, x) \subseteq -C(y, t)\}, (y, t) \in K \times D,$$
$$\mathcal{F}(y, x, t) = x - M(y, t), (y, x, t) \in K \times D \times D.$$

For any fixed $t \in D, y \in K$ since the set A is closed, therefore, so is

$$B = \{x \in D| \ 0 \in \mathcal{F}(y, x, t)\}.$$

Let $\{t_1, ..., t_n\}$ be an arbitrary finite subset in D and $x = \sum_{i=1}^{n} \alpha_i t_i, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1$.We suppose that for any $y \in T(x), 0 \notin \mathcal{F}(y, x, t_i)$, for all $i = 1, ..., n$. This give $G(y, t_i, x) \nsubseteq -C(y, t_i)$, for all $i = 1, ..., n$. The $C(y, .)$-strong pseudomonotonicity of $G(y, ., .)$ yields $G(y, x, t_i) \subseteq -C(y, x)\setminus\{0\}$, for all $i = 1, ..., n$. The diagonally lower $C(y, .)$-convexity in the third variable or, the diagonally lower $C(y, .)$-quasiconvexity-like in the third variable of $G(y, ., .)$ implies that $G(y, x, x) = G(y, x, \sum_{1}^{n} \alpha_i t_i) \subseteq -C(y, x)\setminus\{0\}$ and we have a contradiction with $G(y, x, x) \cap C(y, x) \neq \emptyset$. Therefore, there exist $y \in Q(x)$ and an index $j \in \{1, ..., n\}$ such that $0 \in \mathcal{F}(y, x, t_j)$. This means that (iv) of Corollary 4.4 is satisfied.

According to Corollary 4.4, there exists $\bar{x} \in D$ with $\bar{x} \in P(\bar{x})$ such that $0 \in \mathcal{F}(y, \bar{x}, t)$ for all $t \in P(\bar{x})$ and $y \in Q(\bar{x})$. This shows that, there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and $G(\bar{y}, t, \bar{x}) \subseteq -C(\bar{y}, t)$, for all $t \in P(\bar{x})$ and $y \in Q(\bar{x})$.

The proof of the corollary is complete. ■

Combining Corollary 4.9 and Lemma 4.2, we have the following existence result for the above generalized Pareto quasi-equilibrium problem.

Corollary 4.10. *Let D, K, P, Q be as in Corollary 4.4, $G : K \times D \times D \to 2^Y$ be a multivalued mapping with nonempty values and $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping with $G(y, x, x) \cap C(y, x) \neq \emptyset$ for any $x \in D$ and $y \in K$.*

Assume that the following conditions are satisfied:

(i) *For any fixed $(y, t) \in K \times D, G(y, ., t) : D \longrightarrow 2^Y$ is upper $C(y, .)$-hemicontinuous;*

(ii) *For any fixed $t \in D, y \in K$ the set*

$$A = \{x \in D | \ G(y, t, x) \subseteq -C(y, t)\}$$

is closed in D.

(iii) *$G(y, ., .)$ is $C(y, .)$-strongly pseudomonotone;*

(iv) *$G(y, ., .)$ is diagonally lower $C(y, .)$-convex (or diagonally lower $C(y, .)$-quasiconvex-like) in the third variable.*

Then the above generalized Pareto quasi-equilibrium problem has a solution.

5. Vector Quasi-variational Inequality Problems

In this section, we apply the obtained results in Section 4 to generalized vector quasivariational inequality problems with multivalued mappings. It is well-known that the vector variational inequality theory initiated by Giannessi has emerged as a powerful tool for a wide class of vector optimization problems and has been extended and generalized in defferent directions by using innovative techniques and ideas, both for their own sake and for their applications.

Let $L(X, Y)$ be the set of all continuous linear mappings from X into Y and $< l, x >$ denote the value of l at x, where $l \in L(X, Y), x \in X$. It is clear that $< l, x > \in Y$. In addition, let K be a nonempty convex and compact subset of a locally convex Hausdorff space Z and let $D \subseteq X$ and $C : K \times D \to 2^Y$ be a cone multivalued mapping, $P : D \to 2^D$ and $Q : D \to 2^K, G : K \times D \to 2^{L(X,Y)}$ be multivalued mappings and $\theta : K \times D \times D \to X$ be a nonlinear mapping. In this section, we consider the Pareto and weakly generalized vector quasivariational inequality problems as follows:

1/ Find $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and $< G(y, \bar{x}), \theta(y, \bar{x}, t) > \not\subseteq -C(y, \bar{x}) \setminus \{0\}$, for all $t \in P(\bar{x})$, $y \in Q(\bar{x})$;

2/ Find $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and $< G(y, \bar{x}), \theta(y, \bar{x}, t) > \not\subseteq -\text{int}C(y, \bar{x})$, for all $t \in P(\bar{x})$, $y \in Q(\bar{x})$.

Definition 5.1. We say that

(i) For any fixed $y \in K, G(y,.) : D \to 2^{L(X,Y)}$ is $(C(y,.), \theta(y,.,.))$- pseudomonotone if for any given $x, t \in D$

$$< G(y,x), \theta(y,t,x) > \not\subseteq -\text{int}C(y,x) \Rightarrow < G(y,t), \theta(y,x,t) > \subseteq -C(y,t).$$

(ii) For any fixed $y \in K, G(y,.) : D \to 2^{L(X,Y)}$ is $(C(y,.), \theta(y,.,.))$- strongly pseudomonotone if for any given $x, t \in D$

$$< G(y,x), \theta(y,t,x) > \not\subseteq -(C(y,x) \setminus \{0\}) \Rightarrow < G(y,t), \theta(y,x,t) > \subseteq -C(y,t).$$

We can easily see that for any $y \in K, G(y,.)$ is $(C(y,.), \theta(y,.,.))$-(strongly) pseudomonotone if the multivalued mapping $F(y,.,.) : D \times D \to 2^Y$ defined by $F(y,x,t) =< G(y,x), \theta(y,x,t) >$ is $C(y,.)$-(strongly) pseudomonotone in the sense in Section 4.

We have the following corollaries which generalize and improve some results of Fang and Huang in [4] and Lemma 2.5 in [6].

Corollary 5.2. *Assume that D is a nonempty convex compact subset of X, $P : D \to 2^D$ is a continuous multivalued mapping with nonempty convex and closed values. In addition, suppose that $T : D \to 2^K$ is a lower semicontinuous multivalued mapping with nonempty values, $\theta : K \times X \times X \to X$ is a nonlinear mapping and $G : K \times D \to 2^{L(X,Y)}$ is a multivalued mapping and $\mathcal{C} : K \times D \to 2^Y$ is a cone multivalued mapping with $< G(y,x), \theta(y,x,x) > \cap C(y,x) \neq \emptyset$ for any $x \in D$ satisfying the following conditions:*

(i) *For any fixed $t \in D$, the multivalued mapping $< G(y,.), \theta(y,.,t) >: D \to 2^Y$ is upper C-hemicontinuous;*

(ii) *For any fixed $x \in D, y \in K$ the set*

$$A = \{t \in D | \ < G(y,x), \theta(y,x,t) > \subseteq -C(y,x)\}$$

is closed in D.

(iii) *For any $y \in K, G(y,.)$ is $(C(y,.), \theta(y,.,.))$-strongly pseudomonotone;*

(iv) *For any fixed $y \in K$, the multivalued mapping $F(y,.,.) : D \times D \to 2^Y$ defined by $F((y,x,t) =< G(y,x), \theta(y,x,t) >$ is diagonally lower $C(y,.)$-convex (or diagonally lower $C(y,.)$-quasiconvex-like) in the second variable.*

Then there exists $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and

$< G(y,\bar{x}), \theta(y,\bar{x},t) > \not\subseteq -(C(y,\bar{x}) \setminus \{0\})$, *for all $t \in P(\bar{x}), y \in Q(\bar{x})$.*

Proof. The proof of this corollary follows immediately from Corollary 4.5 by taking $F(y,x,t) =< Gy,x), \theta(y,x,t) >, (y,x,t) \in K \times D \times D$. ∎

Corollary 5.3. *Assume that D is a nonempty convex compact subset of X, $P : D \to 2^D$ is a continuous multivalued mapping with nonempty convex and closed values. In addition, suppose that $G : K \times D \to 2^{L(X,Y)}$ is a multivalued mapping with nonempty values, $\theta : K \times D \times D \to X$ is a nonlinear mapping* *and* $\mathcal{C} : K \times D \to 2^Y$ *is a cone multivalued mapping with* $< G(y,x), \theta(y,x,x) > \cap C(y,x) \neq \emptyset$ *for any* $x \in D, y \in K$ *satisfying the following conditions*

(i) *For any fixed* $t \in D, y \in K$ *the multivalued mapping* $< G(y,.), \theta(y,.,t) >: D \to 2^Y$ *is upper* C*-hemicontinuous;*

(ii) *For any fixed* $x \in D, y \in K$ *the set*

$$A = \{t \in D | \ < G(y,x), \theta(y,x,t) > \subseteq -C(y,x)\}$$

is closed in D*.*

(iii) *For any* $y \in K, G(y,.)$ *is* $(C(y,.), \theta(y,.,.))$*- pseudomonotone;*

(iv) *For any* $y \in K$*, the multivalued mapping* $F(y,.,.) : D \times D \to 2^Y$ *defined by* $F(y,x,t) =< G(y,x), \theta(y,x,t) >$ *is diagonally lower* $C(y,.)$*-convex in the second variable.*

Then, there exists $\bar{x} \in D$ *such that* $\bar{x} \in P(\bar{x})$ *and*

$$< G(y,\bar{x}), \theta(y,\bar{x},t) > \not\subseteq -\text{int}C(y,\bar{x}), \text{ for all } t \in P(\bar{x}), y \in Q(\bar{x}).$$

Proof. The proof of this corollary follows immediately from Corollary 4.7 by taking $F(y,x,t) =< G(y,x), \theta(y,x,t) >, (y,x,t) \in K \times D \times D$. ∎

Remark 5.4. 1/ If for any fixed $x \in D, y \in K$ the mapping $\theta(y,x,.) : D \to X$ is continuous, then condition (ii) of Corollaries 5.2 and 5.3 holds.

2/ If for any fixed $x \in D, y \in K$ the mapping $\theta(y,x,.) : D \to X$ is linear, then condition (iv) of Corollaries 5.2 and 5.3 holds.

3/ If $Y = X^*$ and for any fixed y, $G(.) : D \to X^*$ is a monotone hemicontinuous single-valued mapping and $P = D$ is a constant multivalued mapping, then Corollary 5.2 becomes: There exists $\bar{x} \in D$ such that

$$< G(\bar{x}), t - \bar{x} > \geq 0 \text{ (this is equivalent to} : < G(t), \bar{x} - t > \geq 0), \text{ for all } t \in D.$$

This is an original Stampacchia (and also Minty)variational inequality.

References

1. C. Berge, *Topological spaces*, Dover Publications, NY, 1997.
2. M. Bianchi and R. Pini, Coercivity conditions for equilibrium problems, *J. Optim. Theory Appl.*, **124**(2005), 79-92.
3. T.T.T. Duong and N.X. Tan, On the existence of solutions to generalized quasi-equilibrium problems of type II and related problems, *Adv. Nonlinear Var. Inequal.*, **13**(2010), 29-47.
4. K. Fan, *A minimax inequality and application, in Inequalities III*, O. Shisha (Ed), Acad. Press, New-York, 1972.
5. Y.P. Fang and N.J. Huang, Existence results for generalized implicit vector variational inequalities with multivalued mappings, *Indian J. Pure Appl. Math.*, **36**(2005), 629-640.

6. N. Hadjisavvas, Continuity and maximality properties of pseudomonotone operators, *J. Convex Anal.*, **10**(2003), 465-475.
7. L.J. Lin and N.X. Tan, On quasivariational inclusion problems of type I and related problems, *J. Global Optim.*, **39**(2007), 393-407.
8. D.T. Luc, Theory of vector optimization, *Lect. Notes in Eco. and Math. Systems*, Springer Verlag, Berlin, Germany, **319**(1989).
9. D.T. Luc and N.X. Tan, Existence conditions in variational inclusions with constraints, *Optimization*, **53**(2004), 505-515.
10. D.T. Luc, An abstract problem in variational analysis, *J. Optim. Theory Appl.*, **138**(2008), 65-76.
11. N.B. Minh and N.X. Tan, Some sufficient conditions for the existence of equilibrium points concerning multivalued mappings, *Vietnam J. Math.*, **28**(2000), 295-310.
12. N.B. Minh and N.X. Tan, On the existence of solutions of quasivariational inclusion problems of Stampacchia type, *Adv. Nonlinear Var. Inequal.*, **8**(2005), 1-16.
13. S. Park, Fixed points and quasi-equilibrium problems, *Nonlinear Oper. Theory. Math. and Com. Model.*, **32**(2000), 1297-1304.
14. N.X. Tan, On the existence of solutions of quasi-variational inclusion problems, *J. Optim. Theory Appl.*, **123**(2004), 619-638.
15. N.C. Yannelis and N.D. Prabhaker, Existence of maximal elements and equilibria in linear topological spaces, *J. Math. Eco.*, **12**(1983), 233-245.